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JØRGENSEN'S INEQUALITY FOR QUATERNIONIC HYPERBOLIC SPACE WITH ELLIPTIC ELEMENTS

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Abstract

In this paper, we give an analogue of Jørgensen's inequality for nonelementary groups of isometries of quaternionic hyperbolic space generated by two elements, one of which is elliptic. As an application, we obtain an analogue of Jørgensen's inequality in the two-dimensional Möbius group of the above case.

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1. Introduction

Jørgensen's inequality [8] gives a necessary condition for a nonelementary two-generator subgroup of $\mathrm{PSL}(2, \mathbb{C})$ to be discrete. Viewing $\mathrm{PSL}(2, \mathbb{R})$ as the isometry group of complex hyperbolic 1-space, $H^1_{\mathbb{C}}$, one can seek to generalize Jørgensen's inequality to higher-dimensional complex hyperbolic isometries. There has been much research in this area.

Kamiya [9, 10] and Parker [14, 15] gave generalizations of Jørgensen's inequality to the two-generator subgroup of $\mathrm{PU}(n, 1)$ when one generator is a Heisenberg translation. By using the stable basin theorem, Basmajian and Miner [1] generalized Jørgensen's inequality to two-generator subgroups of $\mathrm{PU}(2, 1)$ when the generators are loxodromic or boundary elliptic. Several other inequalities are due to Jiang, Kamiya and Parker [6] using matrix methods rather than purely geometric methods. Jiang [7] and Kamiya [11] generalized Jørgensen's inequality to the two-generator subgroups of $\mathrm{PU}(2, 1)$ when one generator is a Heisenberg screw motion. A generalization also appears in [16] for the case when one generator is a regular elliptic element.

Following research on complex hyperbolic space, Kim and Parker opened up the study of quaternionic hyperbolic space in [13]. They proved some basic facts about the discreteness of two-generator subgroups, and the minimal volume of cusped

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quaternionic manifolds, and laid down some basic tools for the study of quaternionic hyperbolic space.

It is natural to ask whether theorems in complex hyperbolic space can be generalized to quaternionic hyperbolic space. In one attempt in this area Kim [12] found analogues in quaternionic hyperbolic space of results in [6, 7].

The purpose of this paper is to provide a condition for the nondiscreteness of two-generator subgroups of $\mathrm{Sp}(n, 1)$ with an elliptic element.

In order to state our theorem, we recall some facts about elliptic elements in $\mathrm{Sp}(n, 1)$. Every eigenvalue of an elliptic element $g \in \mathrm{Sp}(n, 1)$ has positive or negative type [4] and its eigenvalues fall into n similarity classes of positive type and one similarity class of negative type. Let Λ_i, $i = 1, \ldots, n$, be its positive classes of eigenvalues and Λ_{n+1} be its negative class. Then any element in Λ_i has norm 1 and the fixed point set $\mathrm{Fix}(g)$ of g in $H^n_{\mathbb{H}}$ contains only one fixed point if $\Lambda_{n+1} \neq \Lambda_i$, $i = 1, \ldots, n$, and is a totally geodesic submanifold which is equivalent to $H^m_{\mathbb{H}}$ or $H^m_{\mathbb{C}}$ (for some $m \leq n$) if Λ_{n+1} coincides with exactly m of the classes Λ_i, $i = 1, \ldots, n$. We call g a *regular elliptic* element if $\mathrm{Fix}(g)$ contains only one point; otherwise g is called a *boundary elliptic* element. We mention here that the definition of regular elliptic element is slightly different from Goldman's [5] which requires its eigenvalues to be distinct in the setting of complex numbers. If $g \in \mathrm{Sp}(n, 1)$ is elliptic, then g is conjugate to

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}), \tag{1.1}$$

where $\lambda_i \in \Lambda_i$, $i = 1, \ldots, n+1$. We define

$$\delta(g) = \max\{|\lambda_i - \lambda_{n+1}|^2 : i = 1, \ldots, n\}. \tag{1.2}$$

Since similarity classes Λ_i, $i = 1, \ldots, n+1$, of g are invariant under conjugation, we have the following proposition.

PROPOSITION 1.1. *If $g \in \mathrm{Sp}(n, 1)$ is elliptic, then $\delta(g)$ is invariant under conjugation.*

We now deduce a formula for $\delta(g)$ defined by (1.2) for an elliptic element $g \in \mathrm{Sp}(n, 1)$.

We use [17] as a reference for the properties of quaternions and matrices of quaternions (see Section 2 for an abbreviated description). Since each similarity class Λ_i has a unique complex number with nonnegative imaginary part, let

$$e^{\mathbf{i}\theta_i} \in \Lambda_i, \quad \text{for } i = 1, \ldots, n+1,$$

be such complex numbers; that is, $0 \leq \theta_i \leq \pi$ for each i. Let

$$\delta_{i,n+1} = \max\{|\lambda_i - \lambda_{n+1}|^2 : \lambda_i \in \Lambda_i,\ \lambda_{n+1} \in \Lambda_{n+1}\} \quad \text{for } i = 1, \ldots, n.$$

Then

$$\delta_{i,n+1} = \max_{u,w\in\mathbb{H}}\{|ue^{\mathbf{i}\theta_i}u^{-1} - we^{\mathbf{i}\theta_{n+1}}w^{-1}|^2\} = \max_{|w|=1}\{|e^{\mathbf{i}\theta_i} - we^{\mathbf{i}\theta_{n+1}}w^{-1}|^2\}.$$

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Let

$$T(w) = e^{\mathbf{i}\theta_i}\overline{w^{-1}}e^{-\mathbf{i}\theta_{n+1}}\bar{w} + we^{\mathbf{i}\theta_{n+1}}w^{-1}e^{-\mathbf{i}\theta_i}.$$

Then $|e^{\mathbf{i}\theta_i} - we^{\mathbf{i}\theta_{n+1}}w^{-1}|^2 = 2 - T(w)$ and

$$\delta_{i,n+1} = 2 - \min_{|w|=1} T(w).$$

We take the complex representation of the quaternion w with norm 1 to be

$$w = w_1 + w_2\mathbf{j} \quad \text{where } w_1, w_2 \in \mathbb{C}.$$

Then $w^{-1} = \bar{w} = \overline{w_1} - w_2\mathbf{j}$. Note that

$$z\,\mathbf{j} = \mathbf{j}\,\bar{z} \quad \text{for } z \in \mathbb{C}.$$

By direct computation

$$T(w) = 2(|w_1|^2\cos(\theta_i - \theta_{n+1}) + |w_2|^2\cos(\theta_i + \theta_{n+1})).$$

Hence

$$\min_{|w|=1} T(w) = \begin{cases} 2\cos(\theta_i + \theta_{n+1}) & \text{if } \cos(\theta_i - \theta_{n+1}) \geq \cos(\theta_i + \theta_{n+1}) \\ 2\cos(\theta_i - \theta_{n+1}) & \text{if } \cos(\theta_i - \theta_{n+1}) < \cos(\theta_i + \theta_{n+1}). \end{cases}$$

Therefore

$$\delta(g) = \max\left\{4\sin^2\frac{\theta_i \pm \theta_{n+1}}{2} : i = 1, \ldots, n\right\}. \tag{1.3}$$

The following is our main theorem.

THEOREM 1.2. *Let g and h be elements of* $\mathrm{Sp}(n, 1)$. *Suppose that g is an elliptic element with fixed point set* $\mathrm{Fix}(g) \in H^n_{\mathbb{H}}$. *If*

$$\inf_{q\in \mathrm{Fix}(g)} \cosh^2\frac{\rho(q, h(q))}{2}\delta(g) < 1, \tag{1.4}$$

then the group $\langle g, h\rangle$ generated by g and h is either elementary or not discrete.

Applying Theorem 1.2 to the subgroup $\mathrm{PU}(2, 1)$ of $\mathrm{Sp}(2, 1)$ with g a regular elliptic element, we obtain the following corollary.

COROLLARY 1.3 (See [16, Theorem 3.4]). *Let g and h be elements of* $\mathrm{PU}(2, 1)$ *such that g is a regular elliptic element with unique fixed point q. If*

$$\cosh^2\frac{\rho(q, h(q))}{2}\delta(g) < 1,$$

then the group $\langle g, h\rangle$ is either elementary or not discrete.

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Let

$$g = \begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{-\mathbf{i}\theta} \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{1.5}$$

As an application, by embedding $\mathrm{SL}(2, \mathbb{C})$ in $\mathrm{Sp}(1, 1)$ as in (3.8) of Section 3, we obtain the following theorem.

THEOREM 1.4. *Let g and h be elements of* $\mathrm{SL}(2, \mathbb{C})$ *given by* (1.5). *If* $\langle g, h\rangle$ *is discrete and nonelementary, then*

$$\inf_{|t|<1, t\in\mathbb{C}} 4f(t)\sin^2\theta \geq 1, \tag{1.6}$$

where

$$f(t) = \frac{|1-t^2|^2(|a|^2+|d|^2) + |1+t|^4|c|^2 + |1-t|^4|b|^2 + 2(\bar{t}-t)^2}{4(1-|t|^2)^2} + \frac{1}{2}. \tag{1.7}$$

Choosing $t = 0$ in the above theorem, we obtain the following corollary.

COROLLARY 1.5. *Let g and h be elements of* $\mathrm{SL}(2, \mathbb{C})$ *given by* (1.5). *If* $\langle g, h\rangle$ *is discrete and nonelementary, then*

$$\sin^2\theta(\|h\|^2 + 2) \geq 1, \tag{1.8}$$

where $\|h\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$.

REMARK 1.6. Jørgensen's inequality [8] gives that

$$|\mathrm{tr}(g)^2 - 4| + |\mathrm{tr}(ghg^{-1}h^{-1}) - 2| = 4\sin^2\theta(1 + |bc|) \geq 1$$

in the above case.

Let

$$h = \begin{pmatrix} -3/2 & 2\mathbf{i} \\ 2\mathbf{i} & 2 \end{pmatrix}.$$

Then $16\frac{1}{4} = (\|h\|^2 + 2) < 4(1 + |bc|) = 20$, which implies that Corollary 1.5 is better than Jørgensen's inequality in such a case. While the case

$$h = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix}$$

with

$$f(t) = \frac{7 + 14t_1^2t_2^2 + 2t_1^2 + 10t_2^2 + 7(t_1^4 + t_2^4)}{2(1 - t_1^2 - t_2^2)^2} + \frac{1}{2} > 4 > 1 + |bc| = 3$$

implies that Theorem 1.4 is weaker than Jørgensen's inequality in such a case.

It follows from the above comparison of Theorem 1.4 with Jørgensen's inequality, that neither theorem is a consequence of the other.

2. Preliminaries

In this section, we give some necessary background material on quaternionic hyperbolic geometry. More details can be found in [4, 5, 13].

Let $\mathbb{H}$ denote the division ring of real quaternions. Elements of $\mathbb{H}$ have the form $z = z_1 + z_2\mathbf{i} + z_3\mathbf{j} + z_4\mathbf{k} \in \mathbb{H}$ where $z_i \in \mathbb{R}$ and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Let $\overline{z} = z_1 - z_2\mathbf{i} - z_3\mathbf{j} - z_4\mathbf{k}$ be the *conjugate* of z, and let

$$|z| = \sqrt{\overline{z}z} = \sqrt{z_1^2 + z_2^2 + z_3^2 + z_4^2}$$

be the *modulus* of z. Define $\Re(z) = (z + \overline{z})/2$ to be the *real part* of z, and $\Im(z) = (z - \overline{z})/2$ to be the *imaginary part* of z. Also $z^{-1} = \overline{z}|z|^{-2}$ is the *inverse* of z. Observe that $\Re(wzw^{-1}) = \Re(z)$ and $|wzw^{-1}| = |z|$ for all z and w in $\mathbb{H}$. Two quaternions z and w are *similar* if there exists nonzero $q \in \mathbb{H}$ such that $z = qwq^{-1}$. The similarity class of z is the set $\{qzq^{-1} : q \in \mathbb{H} - \{0\}\}$.

Let $\mathbb{H}^{n,1}$ be the vector space of dimension $n+1$ over $\mathbb{H}$ with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = \overline{w_1}z_1 + \cdots + \overline{w_n}z_n - \overline{w_{n+1}}z_{n+1},$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in V with entries $(z_1, \ldots, z_{n+1})$ and $(w_1, \ldots, w_{n+1})$ respectively, $\mathbf{w}^*$ denotes the conjugate transpose of $\mathbf{w}$ and J is the Hermitian matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a *unitary transformation* g to be an automorphism of $\mathbb{H}^{n,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}), g(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in V. We denote the group of all unitary transformations by $\mathrm{Sp}(n, 1)$.

Following [4, Section 2], let

$$V_0 = \Big\{\mathbf{z} \in V - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\Big\},$$
$$V_- = \Big\{\mathbf{z} \in V : \langle \mathbf{z}, \mathbf{z} \rangle < 0\Big\}.$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(n, 1)$. We define V^s to be $V^s = V_- \cup V_0$. Let $P : V^s \to P(V^s) \subset \mathbb{H}^n$ be the projection map defined by

$$P(z_1, \ldots, z_n, z_{n+1})^t = (z_1 z_{n+1}^{-1}, \ldots, z_n z_{n+1}^{-1})^t,$$

where $(\ )^t$ denotes the transpose.

We define $H_{\mathbb{H}}^n = P(V_-)$ and $\partial H_{\mathbb{H}}^n = P(V_0)$. The Bergman metric on $H_{\mathbb{H}}^n$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle} \quad \text{where } \mathbf{z} \in P^{-1}(z), \mathbf{w} \in P^{-1}(w). \tag{2.1}$$

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The holomorphic isometry group of $H^n_{\mathbb{H}}$ with respect to the Bergman metric is the projective unitary group $\mathrm{PSp}(n, 1)$ and acts on $P(\mathbb{H}^{n,1})$ by matrix multiplication.

If $g \in \mathrm{Sp}(n, 1)$, by definition, g preserves the Hermitian form. Hence

$$\mathbf{w}^* J\mathbf{z} = \langle \mathbf{z}, \mathbf{w}\rangle = \langle g\mathbf{z}, g\mathbf{w}\rangle = \mathbf{w}^* g^* J g\mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in V. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for V, we see that $J = g^* J g$. From this we find $g^{-1} = J^{-1} g^* J$. That is,

$$g^{-1} = \begin{pmatrix} A^* & -\beta^* \\ -\alpha^* & \overline{a_{n+1,n+1}} \end{pmatrix} \quad \text{for } g = (a_{ij})_{i,j=1,\dots,n+1} = \begin{pmatrix} A & \alpha \\ \beta & a_{n+1,n+1} \end{pmatrix}.$$

Using the identities $gg^{-1} = g^{-1}g = I$ we obtain:

$$AA^* - \alpha\alpha^* = I_n, \quad -A\beta^* + \alpha\overline{a_{n+1,n+1}} = 0, \quad -|\beta|^2 + |a_{n+1,n+1}|^2 = 1; \tag{2.2}$$

$$A^*A - \beta^*\beta = I_n, \quad A^*\alpha - \beta^* a_{n+1,n+1} = 0, \quad -|\alpha|^2 + |a_{n+1,n+1}|^2 = 1. \tag{2.3}$$

For a nontrivial element g of $\mathrm{Sp}(n, 1)$, we say that g is *parabolic* if it has exactly one fixed point and this lies on $\partial H^n_{\mathbb{H}}$, g is *loxodromic* if it has exactly two fixed points and they lie on $\partial H^n_{\mathbb{H}}$, and g is *elliptic* if it has a fixed point in $H^n_{\mathbb{H}}$. In particular, if g has fixed point $q_0 = (0, \dots, 0)^t \in H^n_{\mathbb{H}}$, then g has the form

$$g = \mathrm{diag}(A, a),$$

where $A \in \mathrm{U}(n; \mathbb{H})$ and $a \in \mathrm{U}(1; \mathbb{H})$.

A subgroup G of $\mathrm{Sp}(n, 1)$ is called *nonelementary* if it contains two nonelliptic elements of infinite order with distinct fixed points; otherwise G is called *elementary*.

As in complex hyperbolic n-space, we have the following proposition classifying elementary subgroups of $\mathrm{Sp}(n, 1)$.

PROPOSITION 2.1 (See [2, Lemma 2.4]).

(i) *If G contains a parabolic element but no loxodromic element, then G is elementary if and only if it fixes a point in $\partial H^n_{\mathbb{H}}$.*

(ii) *If G contains a loxodromic element, then G is elementary if and only if it fixes a point in $\partial H^n_{\mathbb{C}}$ or a point-pair $\{x, y\} \subset \partial H^n_{\mathbb{H}}$.*

(iii) *G is purely elliptic, that is, each nontrivial element of G is elliptic, then G is elementary and fixes a point in $\overline{H^n_{\mathbb{H}}}$.*

3. Proofs

PROOF OF THEOREM 1.2. For any fixed point $q \in \mathrm{Fix}(g)$, since $\cosh^2(\rho(q, h(q))/2)\,\delta(g)$ is invariant under conjugation, we may assume that g is of the form (1.1) having fixed point $q = (0, \dots, 0)^t \in H^n_{\mathbb{H}}$ and

$$h = (a_{ij})_{i,j=1,\dots,n+1} = \begin{pmatrix} A & \alpha \\ \beta & a_{n+1,n+1} \end{pmatrix}.$$

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Then

$$\cosh^2 \frac{\rho(q, h(q))}{2} = |a_{n+1,n+1}|^2, \quad \delta(g) = \max\{|\lambda_i - \lambda_{n+1}|^2 : i = 1, \ldots, n\}.$$

In what follows, we will show that if

$$|a_{n+1,n+1}|^2 \delta(g) < 1, \tag{3.1}$$

then the group $\langle g, h\rangle$ is either elementary or not discrete.

Let $h_0 = h$ and $h_{k+1} = h_k g h_k^{-1}$. We write

$$h_k = (a_{ij}^{(k)})_{i,j=1,\ldots,n+1} = \begin{pmatrix} A^{(k)} & \alpha^{(k)} \\ \beta^{(k)} & a_{n+1,n+1}^{(k)} \end{pmatrix}.$$

Then

$$\begin{aligned} h_{k+1} &= \begin{pmatrix} A^{(k+1)} & \alpha^{(k+1)} \\ \beta^{(k+1)} & a_{n+1,n+1}^{(k+1)} \end{pmatrix} \\ &= \begin{pmatrix} A^{(k)} & \alpha^{(k)} \\ \beta^{(k)} & a_{n+1,n+1}^{(k)} \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & \lambda_{n+1} \end{pmatrix} \begin{pmatrix} (A^{(k)})^* & \dfrac{-(\beta^{(k)})^*}{a_{n+1,n+1}^{(k)}} \\ -(\alpha^{(k)})^* & \end{pmatrix}, \end{aligned}$$

where $L = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Therefore

$$a_{n+1,n+1}^{(k+1)} = a_{n+1,n+1}^{(k)} \lambda_{n+1} \overline{a_{n+1,n+1}^{(k)}} - \beta^{(k)} L (\beta^{(k)})^* \tag{3.2}$$

and

$$\begin{aligned} |a_{n+1,n+1}^{(k+1)}|^2 &= (a_{n+1,n+1}^{(k)} \lambda_{n+1} \overline{a_{n+1,n+1}^{(k)}} - \beta^{(k)} L (\beta^{(k)})^*) \\ &\quad \times (a_{n+1,n+1}^{(k)} \overline{\lambda_{n+1}} \overline{a_{n+1,n+1}^{(k)}} - \beta^{(k)} L^* (\beta^{(k)})^*) \\ &= |a_{n+1,n+1}^{(k)}|^4 + \beta^{(k)} L (\beta^{(k)})^* \beta^{(k)} L^* (\beta^{(k)})^* \\ &\quad - a_{n+1,n+1}^{(k)} \lambda_{n+1} \overline{a_{n+1,n+1}^{(k)}} \beta^{(k)} L^* (\beta^{(k)})^* \\ &\quad - \beta^{(k)} L (\beta^{(k)})^* a_{n+1,n+1}^{(k)} \overline{\lambda_{n+1}} \overline{a_{n+1,n+1}^{(k)}}. \end{aligned} \tag{3.3}$$

If there exists some k such that $\beta^{(k)} = 0$, then by (2.2) and (2.3),

$$|a_{n+1,n+1}^{(k)}| = 1 \quad \text{and} \quad \alpha^{(k)} = 0,$$

which implies that q is a fixed point of $h_k = h_{k-1} g h_{k-1}^{-1}$. We deduce that q is a fixed point of h_{k-1} and $|a_{n+1,n+1}| = 1$ by induction. Thus q is a fixed point of h, which implies that $\langle g, h\rangle$ is elementary.

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In what follows, we may assume that $\beta^{(k)} \neq 0$.

We first consider the case when all the elements of $\beta^{(k)}$ are nonzero, that is,

$$a^{(k)}_{n+1,i} \neq 0 \quad \text{for } i = 1, \ldots, n.$$

In this case, noting that

$$\beta^{(k)} L(\beta^{(k)})^* \beta^{(k)} L^* (\beta^{(k)})^* \leq |\beta^{(k)}|^4,$$

we have

$$\begin{aligned} |a^{(k+1)}_{n+1,n+1}|^2 \leq |a^{(k)}_{n+1,n+1}|^4 &+ |\beta^{(k)}|^4 \\ &- a^{(k)}_{n+1,n+1}\lambda_{n+1}\overline{a^{(k)}_{n+1,n+1}}\Big(\sum_{i=1}^{n} a^{(k)}_{n+1,i}\overline{\lambda_i}\overline{a^{(k)}_{n+1,i}}\Big) \\ &- \Big(\sum_{i=1}^{n} a^{(k)}_{n+1,i}\lambda_i\overline{a^{(k)}_{n+1,i}}\Big) a^{(k)}_{n+1,n+1}\overline{\lambda_{n+1}}\overline{a^{(k)}_{n+1,n+1}}. \end{aligned} \tag{3.4}$$

Let

$$u_i = \overline{a^{(k)}_{n+1,i}}^{\,-1}\lambda_i\overline{a^{(k)}_{n+1,i}}, \quad i = 1, \ldots, n+1.$$

Then $u_i \in \Lambda_i$, $i = 1, \ldots, n+1$, and

$$u_i\overline{u_{n+1}} + u_{n+1}\overline{u_i} = 2 - |u_i - u_{n+1}|^2.$$

We can rewrite (3.4) as

$$|a^{(k+1)}_{n+1,n+1}|^2 \leq |a^{(k)}_{n+1,n+1}|^4 + |\beta^{(k)}|^4 - \sum_{i=1}^{n} |a^{(k)}_{n+1,n+1}|^2 |a^{(k)}_{n+1,i}|^2 (2 - |u_i - u_{n+1}|^2). \tag{3.5}$$

Noting that $-|\beta^{(k)}|^2 + |a^{(k)}_{n+1,n+1}|^2 = 1$, by (3.5) we have

$$|a^{(k+1)}_{n+1,n+1}|^2 - 1 \leq |a^{(k)}_{n+1,n+1}|^2 \sum_{i=1}^{n} |a^{(k)}_{n+1,i}|^2 |u_i - u_{n+1}|^2.$$

Therefore

$$|a^{(k+1)}_{n+1,n+1}|^2 - 1 \leq (|a^{(k)}_{n+1,n+1}|^2 - 1)|a^{(k)}_{n+1,n+1}|^2 \delta(g). \tag{3.6}$$

We remark that the case $\beta^{(k)} \neq 0$ with some $a^{(k)}_{n+1,t} = 0$ for $t \in \{1, \ldots, n\}$ also leads to the above inequality.

Noting (3.1), we obtain by induction

$$|a^{(k+1)}_{n+1,n+1}| < |a^{(k)}_{n+1,n+1}|$$

and

$$|a^{(k+1)}_{n+1,n+1}|^2 - 1 < (|a_{n+1,n+1}|^2 - 1)(|a_{n+1,n+1}|^2 \delta(g))^{k+1}. \tag{3.7}$$

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Thus $|a^{(k)}_{n+1,n+1}| \to 1$ and $\{h_k\}$ is a sequence with distinct elements. By (2.2) and (2.3),

$$\beta^{(k)} \to 0, \quad \alpha^{(k)} \to 0$$

and

$$A^{(k)}(A^{(k)})^* \to I_n.$$

By passing to its subsequence, we may assume that

$$A^{(k_t)} \to A_\infty, \quad a^{(k_t)}_{n+1,n+1} \to a_\infty.$$

Thus h_{k+1} converges to

$$h_\infty = \begin{pmatrix} A_\infty & 0 \\ 0 & a_\infty \end{pmatrix} \in \mathrm{Sp}(n, 1),$$

which implies that $\langle h, g\rangle$ is not discrete. This concludes the proof. □

PROOF OF THEOREM 1.4. As in [3, 4], we can regard Sp(1, 1) as the isometries of hyperbolic 4-space H^4, whose model is the unit ball in the quaternions $\mathbb{H}$. SL(2, $\mathbb{C}$), the isometries of hyperbolic 3-space H^3, can be embedded as a subgroup of Sp(1, 1) as follows:

$$f \in \mathrm{SL}(2, \mathbb{C}) \hookrightarrow TfT^{-1} \in \mathrm{Sp}(1, 1), \tag{3.8}$$

where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix}.$$

Let g and h be as in (1.5) and $\hat{g} = TgT^{-1}$, $\hat{h} = ThT^{-1}$. Then

$$\hat{g} = \begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{-\mathbf{i}\theta} \end{pmatrix}, \quad \hat{h} = \frac{1}{2} \begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \mathbf{j} \\ \mathbf{j} & 1 \end{pmatrix} \in \mathrm{Sp}(1, 1). \tag{3.9}$$

That $\hat{g}$ and $\hat{h}$ belong to Sp(1, 1) can be verified directly from [3, Lemma 1.1]. Applying the formula (1.3) to $\hat{g}$ in which $\theta_1 = \theta_2 = \theta$, we have

$$\delta(\hat{g}) = 4 \sin^2 \theta.$$

It is easy to show that the fixed point set of $\hat{g}$ is $\{t\mathbf{j} \mid t \in \mathbb{C} \text{ with } |t| < 1\}$.

Let $\mathbf{z} = \binom{t\mathbf{j}}{1}$ and $\mathbf{w} = \hat{h}\mathbf{z}$. Then

$$\langle \mathbf{w}, \mathbf{z}\rangle = \frac{1}{2}(-t\mathbf{j}, 1) \begin{pmatrix} 1 & -\mathbf{j} \\ \mathbf{j} & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \mathbf{j} \\ \mathbf{j} & 1 \end{pmatrix} \begin{pmatrix} t\mathbf{j} \\ 1 \end{pmatrix}$$

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and

$$|\langle \mathbf{w}, \mathbf{z}\rangle|^2 = \frac{1}{4}((1-t)\mathbf{j}, -1-t)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \times \begin{pmatrix} |t|^2+1+\bar{t}+t & (1+t-\bar{t}-|t|^2)\mathbf{j} \\ (-1-t+\bar{t}+|t|^2)\mathbf{j} & |t|^2+1-\bar{t}-t \end{pmatrix} \\ \times \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}\begin{pmatrix} (t-1)\mathbf{j} \\ -1-\bar{t} \end{pmatrix}.$$

Since $\langle \mathbf{z}, \mathbf{z}\rangle = \langle \mathbf{w}, \mathbf{w}\rangle = |t|^2 - 1$, by direct computation, we have

$$\cosh^2 \frac{\rho(t\mathbf{j}, \hat{h}(t\mathbf{j}))}{2} = \frac{\langle \mathbf{z}, \mathbf{w}\rangle \langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle \langle \mathbf{w}, \mathbf{w}\rangle} = f(t),$$

where $f(t)$ is given by (1.7).

Applying Theorem 1.2 to $\hat{g}$, $\hat{h} \in \mathrm{Sp}(1, 1)$ given by (3.9) concludes the proof. □

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