SYMMETRIC EXTERIOR DIFFERENTIATION AND FLAT FORMS

VICTOR L. SHAPIRO

1. Introduction. Let ω be a continuous differential r-form defined in a bounded domain R of Euclidean n-space, E^n, where $n \geqq 1$ and $0 \leqq r \leqq n-1$. ω is called a flat form in R, **(3**, p. 263**)**, if there exists a constant N such that $|\int_{\partial\sigma}\omega| \leqq N\,|\sigma|$ for every $(r+1)$-simplex σ contained in R, where $|\sigma|$ designates the $(r+1)$-volume of σ. For $n = 1$ and ω a zero form, flatness is the same thing as the usual Lip 1 condition. As is well known, a necessary and sufficient condition that a continuous real-valued function of one variable $f(x)$ satisfy a Lip 1 condition over an interval (a, b) is that its upper and lower symmetric derivatives be bounded in (a, b) **(4**, pp. 22, 327**)**. We intend to prove the analogue of this result for r-forms. In particular, we shall prove the following theorem:

THEOREM. *Let ω be a continuous differential r-form defined in a bounded domain R of E^n. A necessary and sufficient condition that ω be flat in R is that its upper and lower symmetric exterior derivatives be bounded in R.*

For a theorem similar to the above, using a different notion of generalized exterior differentiation than that to be given here, we refer the reader to **(3**, p. 268, Theorem 9A**)**. The result given in the present paper is in a certain sense an improvement over the result in **(3)** where the hypothesis is a boundedness assumption for the integral over the boundaries of $(r+1)$-dimensional intervals parallel to a co-ordinate plane. Here, a much more restricted set of geometric figures is used, namely, r-spheres (and the form is assumed continuous).

2. Symmetric exterior differentiation. We shall suppose that E^n is endowed with the usual Cartesian co-ordinate system, and we shall define the symmetric exterior derivative of ω in terms of this co-ordinate system. Since flatness is defined independently of the co-ordinate system, it will follow from the above theorem that if the upper and lower symmetric exterior derivatives of ω are bounded in one Cartesian co-ordinate system of E^n, they are bounded in all Cartesian co-ordinate systems of E^n and possess a bound which is independent of the co-ordinate system.

Received October 26, 1960. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract no. AF 49(638)-988.

Letting $x = (x^1, \ldots, x^n)$ and $\mu = (\mu_1, \ldots, \mu_{r+1})$ with $\mu_1 < \ldots < \mu_{r+1}$, we define the upper symmetric exterior derivative of ω at the point x in R, $\bar{D}\omega(x)$, as follows:

First define $\bar{D}_\mu\omega(x)$ to be

$$\bar{D}_\mu\omega(x) = \limsup_{t\to 0} |B_\mu(x, t)|^{-1} \int_{S_\mu(x,t)} \omega,$$

where $S_\mu(x, t)$ is the r-sphere with centre x and radius t lying in the $(r + 1)$-plane parallel to the $x^{\mu_1} \ldots x^{\mu_{1+r}}$-plane, $B_\mu(x, t)$ is the closed $(r + 1)$-ball bounded by $S_\mu(x, t)$, $|B_\mu(x, t)|$ is the volume of $B_\mu(x, t)$ and $S_\mu(x, t)$ is oriented as usual with respect to the outer normal. Then, using the notation of **(3**, p, 64**)**, define $\bar{D}\omega$ (x) to be

$$\bar{D}\omega(x) = \sum_{\mu_1<\cdots<\mu_{r+1}} \bar{D}_\mu\omega(\mathrm{x})\mathrm{dx}^{\mu_1}\vee\ldots\vee\mathrm{dx}^{\mu_{r+1}}. \tag{1}$$

It is to be noticed that in case $\omega(x)$ is a zero form, $\bar{D}_{\mu_1}\omega(x)$ is the usual upper symmetric derivative of ω in the x^{μ_1}-direction.

Similarly, using lim inf in place of lim sup above, we define $\tilde{D}_\mu\omega(x)$, and then we define the lower symmetric exterior derivative of ω, $\tilde{D}\omega(x)$, to be the $(r + 1)$-form whose μth component is $\tilde{D}_\mu\omega(x)$.

If $\bar{D}_\mu\omega(x)$ is finite and $\bar{D}_\mu\omega(x) = \tilde{D}_\mu\omega(x)$, we call the common value $D_\mu\omega(x)$. If $D_\mu\omega(x)$ exists for every $(r + 1)$-tuple μ, we say that the symmetric exterior derivative of ω, $D\omega(x)$, exists at the point x and we set $D\omega(x) = \bar{D}\omega(x) = \tilde{D}\omega(x)$. It is clear that if ω is in class C^1 in a neighbourhood of the point x, $D\omega(x)$ exists and equals $d\omega(x)$, the usual exterior derivative of ω at the point x.

We say that the upper and lower symmetric exterior derivatives of ω are bounded in R if there exists a number N such that $-N \leqq \tilde{D}_\mu\omega(x) \leqq \bar{D}_\mu\omega(x) \leqq N$ for all x in R and $(r + 1)$-tuples μ.

3. Proof of theorem. Since the proof of the necessity is fairly easy, we shall prove the sufficiency first. We shall suppose that ω is an r-form with $0 \leqq r \leqq n - 2$, the case $r = n - 1$ being covered essentially in **(2**, Theorem 2**)**. In particular, we state the following n-dimensional analogue of **(2**, Theorem 2**)** as a lemma, the proof being the same for n-dimensions as two dimensions:

LEMMA 1. *Let R be a bounded domain in E^n, $n \geqq 1$, and let η be a continuous differential $(n - 1)$-form defined in R. Suppose that*

(i) *$\bar{D}_{1\ldots n}\eta(x)$ and $\tilde{D}_{1\ldots n}\eta(x)$ are finite in R.*

(ii) *$\tilde{D}_{1\ldots n}\eta(x) \geqq L(x)$ where $L(x)$ is in L^1 on R.*

Then $D_{1\ldots n}\eta(x)$ exists almost everywhere in R, is in L^1 on every closed subdomain of R, and Stokes' Theorem holds with respect to η and $D\eta$ for every n-simplex and n-ball contained in the interior of R. In particular, for every $B_{1\ldots n}(x, t)$ contained in R,

$$\int_{S_{1\ldots n}(x,t)} \eta = \int_{B_{1\ldots n}(x,t)} D_{1\ldots n}\eta(y)\, dy. \tag{2}$$

With $\tilde{D}\omega(x)$ and $\bar{D}\omega(x)$ defined as above, see (1), and $-N \leqq \tilde{D}_\mu\omega(x) \leqq \bar{D}_\mu\omega(x) \leqq N$ for x in R and all $(r+1)$-tuples μ, we shall prove the theorem by showing that for every $(r+1)$-simplex σ contained in R,

$$\left|\int_{\partial\sigma}\omega\right| \leqq \binom{n}{r+1}N|\sigma|.$$

In order to do this, let σ be a fixed $(r+1)$-simplex contained in R, and let ζ_0 be the distance from σ to the boundary of R. Then with $U_\zeta(\sigma) = \{x, \operatorname{dist}(x, \sigma) < \zeta\}$ and $0 < \zeta_2 < \zeta_1 < \zeta_0$, construct a localizing function $\gamma(x)$ which is non-negative, in class C^∞, and takes the value one in $\bar{U}_{\zeta_2}(\sigma)$ and the value zero outside of $\bar{U}_{\zeta_1}(\sigma)$. Next, set $\omega'(x) = \gamma(x)\omega(x)$ for x in R. Then $\omega'(x)$ is a continuous differential r-form which is equal to $\omega(x)$ in $U_{\zeta_2}(\sigma)$. Consequently, in order to establish the theorem it is sufficient to show that

$$\left|\int_{\partial\sigma}\omega'\right| \leqq \binom{n}{r+1}N|\sigma|. \tag{3}$$

In order to establish (3), we need the following lemma:

LEMMA 2. *There exists a constant N_3 such that for x in $\bar{U}_{\zeta_1}(\sigma)$, $|\bar{D}_\mu\omega'(x)| \leqq N_3$ and $|\tilde{D}_\mu\omega'(x)| \leqq N_3$ for all $(r+1)$-tuples μ.*

To prove the lemma, we define N_1 and N_2 as follows:

$$N_1 = \sup_{x \text{ in } R}|\gamma(x)| + \sum_{j=1}^{n}\sup_{x \text{ in } R}|\partial\gamma(x)/\partial x^j|.$$

If $\omega(x)$ is a zero form, define

$$N_2 = \sup_{x \text{ in } \bar{U}_{\zeta_1}(\sigma)}|\omega(x)|.$$

If $\omega(x)$ is an r-form with $1 \leqq r \leqq n-2$ and $\omega_\lambda(x)$ is its λth component, set

$$N_2 = \sum_{\lambda_1<\ldots<\lambda_r}\sup_{x \text{ in } \bar{U}_{\zeta_1}(\sigma)}|\omega_\lambda(\mathrm{x})|.$$

We next set $N_3 = N_1N + (r+1)N_2N_1$ and shall establish the lemma by showing that for any fixed point x_0 in $\bar{U}_{\zeta_1}(\sigma)$ and any fixed $(r+1)$-tuple μ,

$$|\bar{D}_\mu\omega'(x_0)| \leqq N_3 \quad\text{and}\quad |\tilde{D}_\mu\omega'(x_0)| \leqq N_3. \tag{4}$$

Observing that $t|S_\mu(x_0, t)| = (r+1)|B_\mu(x_0, t)|$, we obtain that

$$\left|\int_{S_\mu(x_0,t)}(x^j - x_0^j)\omega_\lambda(x)dx^{\lambda_1} \vee \ldots \vee dx^{\lambda_r}\right| \leqq \tag{5}$$
$$(r+1)|B_\mu(x_0, t)|\sup_{x \text{ on } S_\mu(x_0,t)}|\omega_\lambda(x)|.$$

Next, we set

$$\gamma(x) = \gamma(x_0) + \sum_{j=1}^{n}\partial\gamma(x_0)/\partial x^j(x^j - x_0^j) + \gamma'(x)$$

where $\gamma'(x) = o(|x - x_0|)$ and observe that

$$(6) \quad \int_{S_\mu(x_0, t)} \omega'(x) = \gamma(x_0) \int_{S_\mu(x_0, t)} \omega(x) + \sum_{j=1}^{n} \partial\gamma(x_0)/\partial x^j \int_{S_\mu(x_0, t)} (x^j - x_0^j)\omega(x) + \int_{S_\mu(x_0, t)} \gamma'(x)\omega(x).$$

It follows from (5) and (6) that if ω is an r-form with $1 \leqq r \leqq n - 2$, then

$$(7) \quad |B_\mu(x_0, t)|^{-1} \left| \int_{S_\mu(x_0, t)} \omega' \right| \leqq N_1 |B_\mu(x_0, t)|^{-1} \left| \int_{S_\mu(x_0, t)} \omega \right| + [N_1(r + 1) + o(1)] \sum_{\lambda_1 < \ldots < \lambda_r} \sup_{x \text{ on } S_\mu(x_0, t)} |\omega_\lambda(x)|.$$

(4) follows immediately from (7) and the continuity of ω in case ω is not a zero form and an obvious modification of (7) in case it is a zero form, and the lemma is established.

To prove the theorem, with no loss in generality since we shall establish the result via Fourier analysis, we can assume that $\bar{R}$ is contained in the interior of the n-dimensional torus, $T = \{x, -\pi < x_j \leqq \pi, j = 1, \ldots, n\}$. We next define $\omega'(x)$ throughout T by setting it equal to zero in $T - R$, thus losing none of its continuity properties since it is already equal to zero in $R - \bar{U}_{\zeta_1}(\sigma)$. We can consequently consider ω' as a periodic continuous differential r-form defined throughout all of E^n.

We next define the periodic bounded differential $(r + 1)$-form $\xi(x)$ with components $\xi_\mu(x)$ where

$$(8) \qquad \xi_\mu(x) = \limsup_{k \to \infty} |B_\mu(x, k^{-1})|^{-1} \int_{S_\mu(x, k^{-1})} \omega'.$$

Since ω' is a continuous periodic differential r-form, we see from Lemma 2 that $\xi_\mu(x)$ is a periodic bounded Borel function in E^n for every $(r + 1)$-tuple μ. Furthermore, since

$$\text{for } x \text{ in } U_{\zeta_2}(\sigma), \; \omega'(x) = \omega(x)$$

we see from the hypotheses of the theorem that

$$(9) \qquad |\xi_\mu(x)| \leqslant N \text{ for } x \text{ in } U_{\zeta_2}(\sigma) \text{ and every } (r + 1)\text{-tuple } \mu.$$

We next introduce the Fourier series

$$(10) \qquad \omega_\lambda'(x) \sim \sum a_m^\lambda e^{i(m,x)}$$

$$(11) \qquad \xi_\mu(x) \sim \sum b^\mu{}_m e^{i(m,x)}$$

where $a_m = (2\pi)^{-n}\int_T e^{-i(m,x)}\omega'_\lambda(x)dx$ and $b_m{}^\mu = (2\pi)^{-n}\int_T e^{-i(m,x)}\xi_\mu(x)dx$ with m designating an integral lattice point in E^n and (m, x) designating the usual scalar product. (If $\omega'(x)$ is a zero form, it is understood that $\omega_\lambda'(x) = \omega'(x)$. Otherwise, $\omega_\lambda'(x)$ represents the λth component of $\omega'(x)$.)

We next propose to show that

$$(12)\qquad b_m^{\mu_1\ldots\mu_{r+1}} = i\sum_{j=1}^{r+1}(-1)^{j-1}m^{\mu_j}a_m^{\mu_1\ldots\hat{\mu}_j\ldots\mu_{r+1}}$$

where $\mu_1\ldots\hat{\mu}_j\ldots\mu_{r+1}$ is an r-tuple with μ_j missing.

Let μ be fixed, and let $P_\mu(x_0)$ designate the plane parallel to the $x^{\mu_1}\ldots x^{\mu_{r+1}}$-plane going through the point x_0. Then by Lemma 2 and the fact that $\omega'(x)$ is zero in $T - \bar{U}_{\xi_1}(\sigma)$, we have that both $|\bar{D}_\mu\omega'(x)| \leqq N_3$ and $|\tilde{D}_\mu\omega'(x)| \leqq N_3$ for all x on $P_\mu(x_0)$. Observing that for x on $P_\mu(x_0)$, $S_\mu(x,t)$ and $B_\mu(x,t)$ are both point sets on $P_\mu(x_0)$, we see that Lemma 1 also applies to ω'. Consequently, for almost every x on $P_\mu(x_0)$, $D_\mu\omega'(x)$ exists. Furthermore, $D_\mu\omega'(x)$ is in L^1 on every bounded domain of $P_\mu(x_0)$, and for every x on $P_\mu(x_0)$,

$$\int_{S_\mu(x,t)}\omega' = \int_{B_\mu(x,t)} D_\mu\omega'(y)dy^{\mu_1}\ldots dy^{\mu_{r+1}}.$$

But from the definition of $D_\mu\omega'(x)$ and from (8), we see that wherever $D_\mu\omega'(x)$ exists, it must equal $\xi_\mu(x)$. We conclude that

$$(13)\qquad \int_{S_\mu(x,t)}\omega' = \int_{B_\mu(x,t)}\xi_\mu(y)dy^{\mu_1}\ldots dy^{\mu_{r+1}} \qquad \text{for } x \text{ on } P_\mu(x_0) \text{ and } t>0.$$

Now $\omega_\lambda'(x)$ and $\xi_\mu(x)$ are periodic functions on $P_\mu(x_0)$. Consequently, letting $T_\mu(x_0)$ be the $(r+1)$-dimensional torus obtained by intersecting $P_\mu(x_0)$ with T, that is, $T_\mu(x_0) = \{x; x \text{ on } P_\mu(x_0) \text{ and } -\pi < x^{\mu_j} \leqq \pi, j=1,\ldots,r+1\}$, we can expand $\omega_\lambda'(x)$ and $\xi_\mu(x)$ in a Fourier series on $P_\mu(x_0)$ as follows:

$$(14)\qquad \begin{aligned}
\omega_\lambda'(x) &\sim \sum a_{m^\mu}^\lambda(x_0^{\mu'})e^{i(m^\mu,x^\mu)}\\
\xi_\mu(x) &\sim \sum b_{m^\mu}^\mu(x_0^{\mu'})e^{i(m^\mu,x^\mu)}\\
a_{m^\mu}^\lambda(x_0^{\mu'}) &= (2\pi)^{-(r+1)}\int_{T_\mu(x_0)}\omega_\lambda'(x)e^{-i(m^\mu,x^\mu)}dx^\mu\\
b_{m^\mu}^\mu(x_0^{\mu'}) &= (2\pi)^{-(r+1)}\int_{T_\mu(x_0)}\xi_\mu(x)e^{-i(m^\mu,x^\mu)}dx^\mu
\end{aligned}$$

where μ' is the complementary $[n-(r+1)]$-tuple to μ with

$$\mu_1' < \ldots < \mu_{n-(r+1)}',\ x_0^{\mu'} = (x_0^{\mu'1},\ldots,x_0^{\mu'n-(r+1)}),$$

and

$$dx^\mu = dx^{\mu 1}\ldots dx^{\mu r+1}.$$

We next notice that both sides of (13) are periodic functions on $P_\mu(x_0)$ and that by (14)

$$(2\pi)^{-(r+1)}\int_{T_\mu(x_0)}e^{-i(m^\mu,x^\mu)}\left[\int_{S_\mu(x,t)}\omega_{\mu_1\ldots\hat{\mu}_j\ldots\mu_{r+1}}'(y)dy^{\mu_1}\vee\ldots\right.$$
$$\left.\hat{d}y^{\mu_j}\ldots\vee dy^{\mu_{r+1}}\right]dx^\mu = \int_{S_\mu[0^\mu(x_0),t]}e^{i(m^\mu,y^\mu)}dy^{\mu_1}\ldots\hat{d}y^{\mu_j}\vee\ldots\vee$$
$$dy^{\mu_{r+1}}a_{m^\mu}^{\mu_1\ldots\hat{\mu}_j\ldots\mu_{r+1}}(x_0^{\mu'})$$

where $0^\mu(x_0)$ is the point on $P_\mu(x_0)$ whose μ_k-co-ordinate is zero, $k = 1, \ldots, r+1$. But

$$\int_{S_\mu[0^\mu(x_0),t]} e^{i(m^\mu,y^\mu)} dy^{\mu_1} \vee \ldots \hat{dy}^{\mu_j} \ldots \vee dy^{\mu_{r+1}} = (-1)^{j-1} m^{\mu_j} i \int_{B_\mu[0^\mu(x_0),t]} e^{i(m^\mu,y^\mu)} dy^\mu.$$

Consequently, the m^μth Fourier coefficient of the left side of (13) is given by

$$(15)\qquad i \sum_{j=1}^{r+1} (-1)^{j-1} m^{\mu_j} a_{m^\mu}^{\mu_1 \ldots \hat{\mu}_j \ldots \mu_{r+1}}(x_0^{\mu'}) \int_{B_\mu[0^\mu(x_0),t]} e^{i(m^\mu,y^\mu)} dy^\mu.$$

Similar considerations show that the m^μth Fourier coefficient of the right side of (13) is given by

$$(16)\qquad b_{m^\mu}^\mu(x_0^{\mu'}) \int_{B_\mu[0^\mu(x_0,t)]} e^{i(m^\mu,y^\mu)} dy^\mu.$$

If m^μ is not the zero integral lattice point, then for every $t > 0$, the integral in (15) and (16) by (**1**, p. 177) is a non-zero multiple of $J_{(r+1)/2}[(m^\mu, m^\mu)^{\frac{1}{2}} t]$ where $J_{(r+1)/2}(t)$ is the Bessel function of the first kind of order $(r+1)/2$. Since this function has only a countable number of zeros, we conclude from (13), (15), and (16) that

$$(17)\qquad b_{m^\mu}^\mu(x_0^{\mu'}) = i \sum_{j=1}^{r+1} (-1)^{j-1} m^{\mu_j} a_{m^\mu}^{\mu_1 \ldots \hat{\mu}_j \ldots \mu_{r+1}}(x_0^{\mu'}).$$

Since the integral in (15) and (16) is clearly not equal to zero if m^μ is the zero integral lattice point, we see that (17) also holds in this case.

We are now in a position to establish (12). Let m be any integral lattice point in E^n. Designate by m^μ the ordered $(r+1)$-tuple corresponding to the μ-components of m and by $m^{\mu'}$ the ordered $[n-(r+1)]$-tuple corresponding to the μ'-components of m. Then from (10), (11), and (14) it follows that

$$(18)\qquad b_m^\mu = (2\pi)^{-[n-(r+1)]} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} b_{m^\mu}^\mu(x_0^{\mu'}) e^{-i(m^{\mu'},x_0^{\mu'})} dx_0^{\mu'_1} \ldots dx_0^{\mu'_{n-(r+1)}}$$

and

$$(19)\qquad a_m^\lambda = (2\pi)^{-[n-(r+1)]} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} a_{m^\mu}^\lambda(x_0^{\mu'}) e^{-i(m^{\mu'},x_0^{\mu'})} dx_0^{\mu'_1} \ldots dx_0^{\mu'_{n-(r+1)}}.$$

But then (12) follows immediately from (17), (18), and (19).

We say that $\xi_\mu(x)$ is mean-continuous at the point x_0, if

$$|B_{1\ldots n}(x_0, t)|^{-1} \int_{B_{1\ldots n}(x_0,t)} \xi_\mu(x) dx \to \xi_\mu(x_0) \quad \text{as} \quad t \to 0.$$

Let $Z_\mu = \{x, x \text{ in } E^n \text{ and } \xi_\mu \text{ is not mean continuous at } x\}$. Then, as is well known, Z_μ is of Lebesgue measure zero in E^n. Let $Z = \cup_\mu Z_\mu$ where the summation is taken over all ordered $(r+1)$-tuples.

Next we introduce the Abel means of the series (10) and (11),

$$\omega'_\lambda(x, t) = \sum a^\lambda_m e^{i(m,x)-|m|t}, \qquad t > 0,$$
$$\xi_\mu(x, t) = \sum b^\mu_m e^{i(m,x)-|m|t}, \qquad t > 0.$$

It follows from **(1)** that

(20) $$\xi_\mu(x, t) \to \xi_\mu(x) \quad \text{as} \quad t \to 0 \qquad \text{for } x \text{ not in } Z,$$

and furthermore since $\omega'(x)$ is continuous in E^n that

(21) $$\omega'_\lambda(x, t) \to \omega'_\lambda(x) \qquad \text{as } t \to 0 \qquad \text{uniformly for } x \text{ in } E^n.$$

Also, since by Lemma 2, $|\xi_\mu(x)| \leqq N_3$ for all x in E^n, we have from **(1)** that

(22) $$|\xi_\mu(x, t)| \leqq N_3 \text{ for } t > 0 \text{ and all } x \text{ in } E^n.$$

Furthermore, we observe that the functions $\omega_\lambda'(x, t)$ and $\xi_\mu(x, t)$ are in class C^∞ on E^n for all $(r + 1)$-tuples μ and r-tuples λ. Consequently, we obtain from (12) on setting

$$\omega'(x, t) = \sum_{\lambda_1<\ldots<\lambda_r} \omega'_\lambda(x, t)dx^{\lambda_1} \vee \ldots \vee dx^{\lambda_r}$$

and

$$\xi(x, t) = \sum_{\mu_1<\ldots<\mu_{r+1}} \xi_\mu(x, t)dx^{\mu_1} \vee \ldots \vee dx^{\mu_{r+1}}$$

that

(23) $$d\omega'(x, t) = \xi(x, t) \qquad \text{for} \quad t > 0.$$

We now prove the theorem, that is, we shall establish (3) for the $(r + 1)$-simplex σ which has been fixed throughout the discussion.

Let P_σ represent the $(r + 1)$-dimensional plane in E_n containing σ. Then one of two cases arises. Either $\sigma \cap Z$ is a set of $(r + 1)$-dimensional Lebesgue measure zero on P_σ or it is not. Let us suppose that the latter case prevails, for the reasoning we are about to use will hold in an obvious manner in the former case.

By Fubini's theorem there exists a sequence of points in E^n, $\{x_j\}_{j=1}^{\infty}$, with $x_j \to 0$ as $j \to \infty$ such that on the translated simplices, $\sigma + x_j$, $(\sigma + x_j) \cap Z$ is of $(r + 1)$-dimensional Lebesgue measure zero on $P_{\sigma+x_j}$. Furthermore, on each such simplex $\sigma + x_j$, we have by (23) that for $t > 0$,

(24) $$\int_{\partial(\sigma+x_j)} \omega'(x, t) = \int_{\sigma+x_j} \xi(x, t).$$

But then by the Lebesgue dominated convergence theorem, (20), (21), and (22), we conclude from (24) that

(25) $$\int_{\partial(\sigma+x_j)} \omega' = \int_{\sigma+x_j} \xi.$$

Since $x_j \to 0$ as $j \to \infty$, we see that for j sufficiently large, $\sigma + x_j$ lies in $U_{\zeta_2}(\sigma)$. Therefore, it follows from (9) and (25) that for j sufficiently large,

$$\left|\int_{\partial(\sigma+x_j)} \omega'\right| \leqq \binom{n}{r+1} N|\sigma|. \tag{26}$$

But since ω' is a continuous differential r-form,

$$\int_{\partial(\sigma+x_j)} \omega' \to \int_{\partial\sigma} \omega' \quad \text{as} \quad j \to \infty. \tag{27}$$

(3) follows immediately from (26) and (27), and the proof to the sufficiency condition of the theorem is complete.

To establish the necessity condition of the theorem, we need only show that the condition, $|\int_{\partial\sigma}\omega| \leqq N\,|\sigma|$ for every $(r+1)$-simplex in R implies that the following fact holds for a fixed μ, x, and t with $B_\mu(x, t)$ contained in R:

$$\left|\int_{S_\mu(x,t)} \omega\right| \leqq N|B_\mu(x, t)|. \tag{28}$$

To establish (28) we use the following well-known fact:

There exists a sequence of simplicial $(r+1)$-chains $\{A^j\}_{j=1}^{\infty}$ such that

(i) $\quad A^j = \sum_{k=1}^{K^j} \sigma_k^j,\ \sigma_k^j$ non-overlapping for $k = 1, \ldots, K^j$.

(ii) $\quad \sigma_k^j \subset B_\mu(x, t)$

(iii) $\quad \int_{\partial A^j} \omega = \sum_{k=1}^{K^j} \int_{\partial\sigma_k{}^j} \omega$

(iv) $\quad \int_{\partial A^j} \omega \to \int_{S\mu(x,t)} \omega \qquad$ as $j \to \infty$.

(v) $\quad |A^j| \to |B_\mu(x, t)| \qquad$ as $\to \infty$.

Since

$$|A_j| = \sum_{k=1}^{K_j} |\sigma_k^j|,$$

it follows from (ii) and (iii) that $|\int_{\partial A^j}\omega| \leqq N\,|A_j|$. But then (28) follows immediately from (iv) and (v), and the proof for the theorem is complete.

References

1. S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc., *40* (1936), 175–207.
2. V. L. Shapiro, *The divergence theorem for discontinuous vector fields*, Ann. Math., *68* (1958), 604–624.
3. H. Whitney, *Geometric integration theory* (Princeton, 1957).
4. A. Zygmund, *Trigonometric series* (Cambridge, 1959), I.

University of Oregon