Canad. Math. Bull. Vol. **24** (1), 1981

NON-SINGULAR PERIODIC FLOWS ON T^3 AND PERIODIC HOMEOMORPHISMS OF T^2

BY
C. S. HARTZMAN*

A homeomorphism f of a space M is pointwise periodic if for each $m \in M$ there exists an integer k such that $f^k(m) = m$, where f^k is the kth iterate of f. Montgomery proves [5] that if M is a connected topological manifold, then f is periodic; i.e., there exists an integer n such that $f^n = id$. Noting this, Weaver [7] proves that if M is an orientable 2-manifold of class C^1, $U \subseteq M$ open and $C \subseteq U$ a compact connected set and if $g: U \to M$ of class C^1 is such that (i) $g(C) = C$ and (ii) whenever the derivative of g at points $x \in C$ has rank 2 it is orientation preserving, then $f = g|_C : C \to C$ periodic implies that all but a finite number of points of C have as least period the period of f. In particular, if M is a compact manifold, we may take $C = M$ in Weaver's result. This is the strongest result the author has seen for periodic maps of arbitrary period (not necessarily prime).

In this paper, the specific case $M = T^2$ and f a periodic homeomorphism isotopic to the identity is considered. An isotopy is a homotopy $H(x, t)$ which is a homeomorphism for each t. Specifically, it is proven that if $f: T^2 \to T^2$ is a periodic homeomorphism isotopic to the identity, then every point has the same period (Corollary 1.5).

By discussing periodic homeomorphisms of T^2 isotopic to the identity, we are in effect studying all non-singular periodic flows ϕ on T^3 (flows with all orbits closed and non-trivial). This is because every non-singular periodic C^1 flow on T^3 has a topological T^2 cross-section (Proposition 2.6) on which ϕ induces a homeomorphism f isotopic to the identity. The terminology here is made precise at end of this section. For certain cross-sections it seems likely that $f = id$. However, this is generally not true for all toral cross-sections and their induced maps f. Naugler [6] has shown that, even under conditions of differentiability, about the best that could be expected would be that f is isotopic to the identity.

The following notation is adhered to: $\phi: T^3 \times \mathbb{R} \to T^3$ is a C^1 flow on T^3 having only closed non-trivial orbits. It has been shown that if ϕ is a flow on a compact 3-manifold M with boundary having only closed non-trivial orbits, then the orbits are also the orbits of an S^1-action on M [1]. $\phi: T^3 \times S^1 \to T^3$

Received by the editors June 19, 1978 and in revised form March 17, 1980.
* The author acknowledges partial support of N.S.E.R.C. grant #A 8050.

will also be used to denote the corresponding S^1-action on T^3. Consider T^n as $\mathbb{R}^n/\mathbb{Z}^n$. The $\mathbb{R}$-action ϕ on T^3 then lifts to an $\mathbb{R}$-action $\Phi:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^3$ satisfying $\Phi(x+z, t)=\Phi(x, t)+z$ for all $x\in\mathbb{R}^3$, $z\in\mathbb{Z}^3$. Let $f:T^2\to T^2$ be a periodic homeomorphism of T^2 isotopic to the identity. f lifts to a map $F:\mathbb{R}^2\to\mathbb{R}^2$ of the form $F(x)=idx+p(x)$, where $p:\mathbb{R}^2\to\mathbb{R}^2$ is of period 1 in each component of x(i.e., see [2]). If $m\in T^2$ is a fixed point of f, it can be assumed that $z\in\mathbb{Z}^2$ represents that fixed point and further that $F(z)=z$. Hence $f^n=id$ on T^2 implies that $F^n=id$ on $\mathbb{R}^2$.

0.1. Definition. Given a non-singular flow ϕ on M^n, a topological cross-section is an embedded submanifold M^{n-1} such that (i) every trajectory through points of M^{n-1} intersects M^{n-1} for t both positive and negative, (ii) $\{\phi(M^{n-1}, t):t\in\mathbb{R}\}=M^n$, and (iii) there is a neighborhood U of M^{n-1} in M^n, an $\varepsilon>0$ and a homeomorphism h of U onto $M^{n-1}\times(-\varepsilon, \varepsilon)$ such that $u\in U$ is uniquely expressible as $\phi(m, t)$ for $m\in M^{n-1}$ and $h(\phi(m, t))=(m, t)$ for $\phi(m, t)\in U$.

If a non-singular flow ϕ of M^n has a cross-section, ϕ induces a map $f:M^{n-1}\to M^{n-1}$ which is a homeomorphism. It is defined by $f(m)=\phi(m, t(m))$, where $t(m)=\inf\{t>0:\phi(m, t)\in M^{n-1}\}$.

1. Periodic homeomorphisms of T^2

1.1. Lemma. *Let F be a homeomorphism of $\mathbb{R}^2$ isotopic to the identity such that $F:D^2\xrightarrow[onto]{}D^2$, where D^2 is a fixed disc. Then $F|_{D^2}$ is isotopic in D^2 to the identity on D^2.*

Proof. Let $G(x, t):\mathbb{R}^2\times I\to\mathbb{R}^2$ be the isotopy $id\sim F$. Consider the map $(G, id):D^2\times I\to\mathbb{R}^2\times I$ defined by $(x, t)\to(G(x, t), t)$. The image of $D^2\times I$ under (G, id) is homeomorphic to $D^2\times I$ since $G(x, t)$ is an isotopy. Furthermore, $D^2\times I$ and its image have the same base and top, $D^2\times\{0\}$ and $D^2\times\{1\}$ respectively and $[(G, id)](D^2\times I)$ is homeomorphic to $D^2\times I$. Let h be a homeomorphism $h:[(G, id)](D^2\times I)\to D^2\times I$ which is the identity on $D^2\times\{0\}\cup D^2\times\{1\}$. Let $p:D^2\times I\to D^2$ be the projection onto the first factor. The map $p\circ h\circ(G, id):D^2\times I\to D^2$ is the desired isotopy.

Note that $H(x, t)$ is invariant on ∂D^2, the boundary of D^2.

1.2. Lemma. *Let $F:D^2\to D^2$ be isotopic to the identity. Suppose that F has a fixed point p on ∂D^2. If $F^n=id$ for some n, then every point on ∂D^2 is a fixed point for F.*

Proof. Orient ∂D^2. This induces an order $<$ for points on $\gamma=\partial D^2-\{p\}$. Since F is isotopic to the identity on ∂D^2, it preserves the order on γ. Let $x\in\gamma$.

If $F(x)\neq x$, we may suppose that $x<F(x)$ and hence $x<F(x)<F^2(x)<\cdots<F^n(x)$, contradicting the fact that $F^n=id$. Q.E.D.

1.3. LEMMA. *Let f be a periodic map of T^2 (i.e., there exists n such that $f^n=id$) isotopic to the identity. If f has a fixed point p, then F has another fixed point.*

Proof. Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be the lift of f satisfying $F(z)=z$, $z\in\mathbb{Z}^2$ described in the introduction. Let $\gamma=\{(0,y):0\le y\le 1\}$ joining two representatives of the fixed point p. Consider the set of curves $F^k(\gamma)$, $k=0,1,\ldots,n-1$. Let Ω_0 denote the collection of components of the interior of the region bounded by the totality of curves $\{F^k(\gamma)\}$. Let Ω denote the closures of the elements Ω_0.

Every element of Ω intersects γ. If not, then some element $\omega\in\Omega$ has no intersection with γ. Let $\beta=F^j(\gamma)$ be one of the curves having an arc forming part of the boundary of ω. Order the points on $F^j(\gamma)$ according to the order that their preimages occur on γ; i.e., $p_1<p_2$, for $p_1,p_2\in F^j(\gamma)$, if $y_1<y_2$ where $F^{-j}(p_i)=(0,y_i)$. Let $p_1=\inf F^j(\gamma)\cap\omega$ and $p_2=\sup F^j(\gamma)\cap\omega$. Since $\omega\cap\gamma=\phi$, p_1,p_2 do not lie on γ. But $F^j(0,0)=(0,0)\in\gamma$ and $F^j(0,1)=(0,1)\in\gamma$, hence, $q_1=\sup\{q\in F^j(\gamma)\cap\gamma:q<p_1\}$ and $q_2=\inf\{q\in F^j(\gamma)\cap\gamma:q>p_2\}$ exist and are on γ. $\gamma\cup\{q\in F^j(\gamma):q_1\le q\le q_2\}$ bounds a region ω_0 whose closure intersects γ. Furthermore, since $F^j(\gamma)$ has an arc $\{x:p_1\le x_1\le x\le x_2\le p_2, x\in F^j(\gamma)\}$ in ω, there is a neighborhood of any point x, $x_1<x<x_2$ contained in $\omega\cup\omega_0$. Hence $\omega_0\cup\omega$ properly contains ω and is all contained in a single element of Ω contradicting the fact that $\omega\in\Omega$.

The elements of Ω can now be ordered as follows. For each $\omega_\alpha\in\Omega$, let $y_\alpha=\sup\{y:(0,y)\in\omega_\alpha\cap\gamma\}$. Say $\omega_\alpha<\omega_\beta$ if $y_\alpha<y_\beta$. It is possible that not every point x of γ is not contained in some element of Ω. In that case, such points x are contained in intervals $l\subset\cap\{F^j(\gamma)\}_{j=1}^n$. Let Λ denote the collection of such line segments. For each $l_\alpha\in\Lambda$, define $y_\alpha=\sup\{y:(0,y)\in l_\alpha\}$. Let Π denote the ordered set of y_α as defined above. $F(y_\alpha)\in\Pi$ for all $y_\alpha\in\Pi$. Define a chain σ_α to be the union of elements of $\Omega\cup\Lambda$ such that $y_\beta\le y_\alpha$. Denote the set of such chains by Σ.

The proof may now be completed. There are two cases; (i) Σ contains more than one element and (ii) Σ contains exactly one element. Case (i): Consider an element $\sigma_\alpha\in\Sigma$ and let y_α be the largest element of Π contained in σ_α. Suppose $y_\beta=F(y_\alpha)<y_\alpha$. Then the chain σ_α is mapped onto the chain σ_β properly contained σ_α. Hence $y_\alpha>F(y_\alpha)>F^2(y_\alpha)>F^2(y_\alpha)>\cdots>F^n(y_\alpha)$, contradicting the fact that $F^n=id$. A similar contradiction holds if $F(y_\alpha)>y_\alpha$. Thus $F(y_\alpha)=y_\alpha$ proving the lemma in this case. Case (ii): if Σ has one element, it is either a line joining the fixed points $(0,0)$ and $(0,1)$ and hence is a line of fixed points by the now familiar argument or it is the sole element $\omega\in\Omega$. In this last eventuality we note that $\partial\Omega$ is non-self intersecting and does not intersect γ in points other than $(0,0)$ and $(0,1)$. Thus $\partial\Omega$ forms a closed non-self-intersecting

https://doi.org/10.4153/CMB-1981-003-0 Published online by Cambridge University Press

curve bonding a topological disc. An application of Lemma 1.2 yields the additional required fixed point. Q.E.D.

1.4. PROPOSITION. *If f is a periodic map of period n of T^2 isotopic to the identity, then the fixed point set of f is either empty or all of T^2.*

Proof. Suppose that the fixed point set is not empty and not all of T^2; i.e., the fixed point set $\mathcal{F}$ of the lift F is not all of $\mathbb{R}^2$. $\mathcal{F}$ is closed so that $\mathbb{R}^2$-$\mathcal{F}$ is open. Let U be a component of $\mathbb{R}^2$-$\mathcal{F}$ and let γ be a curve in U joining two boundary points of U (which are of course fixed points). If $\gamma \cap F(\gamma) \neq \phi$, then the construction of Lemma 3 yields a fixed point in U. If $\gamma \cap F(\gamma) = \phi$, γ and $F(\gamma)$ may go through different components but then, $\gamma \cup F(\gamma)$ is a closed curve bounding a topological disc and an application of Lemma 1.2 yields fixed points in U. In either case, there is a contradiction so that either $\mathcal{F} = \phi$ or $\mathcal{F} = T^2$. Q.E.D.

1.5. COROLLARY. *If f is a periodic map of T^2 isotopic to the identity, then every point has the same least period.*

Proof. Let p be a point of least period k. let $g = f^k$. $g(p) = p$ and g is a periodic map of T^2 isotopic to the identity. Apply Proposition 1.4 to g. Q.E.D.

2. Periodic flows on T^3

2.1. DEFINITION. If ϕ is a flow on M, $\phi: M \times \mathbb{R} \to M$, the orbit space M/ϕ is the space obtained by identifying points $\phi(m, t)$ that lie on the same orbit and imposing the quotient topology.

2.2. DEFINITION. A flow ϕ on M will be said to have no separatrices if M/ϕ is Hausdorff.

2.3. DEFINITION. A flow ϕ on $\mathbb{R}^n$ will be called completely unstable if $\phi(x, t)$ is unbounded as $t \to \infty$ and $t \to -\infty$.

These notions are discussed in [4].

2.4. LEMMA. *Let ϕ be a completely unstable flow with no separatrices on $\mathbb{R}^n$, $n = 2$ or 3. If ϕ is differentiable, then ϕ differentiably equivalent to the flow of $\dot{x}_1 = 1$, $\dot{x}_i = 0$ $2 \le i \le n$. Differentiably equivalent means that there is a diffeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ mapping trajectories of one system onto trajectories of the other.*

2.5. LEMMA. *If $\phi: T^3 \times S^1 \to T^3$ is a non-singular S^1-action of T^3, then $\phi(p, S^1)$ is non-null-homotopic for all $p \in T^3$.*

Proof. If $\phi(p, S^1)$ is null-homotopic for some p, then $\phi(p, S^1)$ is null-homotopic for all p, since $p \times S^1$ represents the same homotopy generator of

$T^3 \times S^1$ whatever the p. $\phi(p, S^1)$ is null-homotopic on T^3 if and only if its lift to $\mathbb{R}^3$ is a closed curve, since $\Phi(x+z, t) = \Phi(x, t) + z$; i.e., $\Phi(x, t)$ is a closed curve for every x. Since the closed cube $C: 0 \leq x, y, z \leq 1$ projects onto T^3 and $\Phi(x+z, t) = \Phi(x, t) + z$ and Φ is a periodic flow, the orbits of Φ on the region generated by translating C along trajectories can be considered the orbits of an S^1-action in such a way that this can be extended throughout $\mathbb{R}^3$ by the periodicity relation Φ satisfies. Hence Φ may be regarded as a non-singular S^1-action, $\Phi: \mathbb{R}^3 \times S^1 \to \mathbb{R}^3$. But it is known that any S^1-action on $\mathbb{R}^3$ must have singular points [3, pg. 50 ex. 1]. This contradiction completes the proof. Q.E.D.

2.6. Proposition. *If ϕ is a non-singular periodic flow on T^3, then ϕ has a topological cross-section homeomorphic to T^2.*

Proof. Lift ϕ to $\mathbb{R}^3$, $\Phi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Φ is completely unstable. If not, there would be a bounded orbit γ and since γ must project onto a closed orbit it would itself be closed, thus projecting onto a null-homotopic orbit; a contradiction of Lemma 2.5. Also, Φ has no separatrices, a fact which follows immediately from the fact that the orbits of Φ project onto closed orbits on T^3. Hence, we may apply Lemma 2.4.

Taking the inverse image of the x_2x_3-plane under the differentiable equivalence of Lemma 2.4, we get a surface Σ in $\mathbb{R}^3$ transverse to the flow Φ, that every orbit of Φ intersects exactly once. Consider the unit square $D = \{(x, y, z): 0 \leq x, y \leq 1, z = 0\}$. Let $f: D \to \Sigma$ denote the map obtained by translating D to Σ along trajectories. Although trajectories may intersect D more than once, $f(D)$ is a topological square transverse to the trajectories through it, that intersects each trajectory exactly once. We translate $f(D)$ along trajectories so that its translate projects onto a topological torus $T^2 \subset T^3$ that is topologically transverse to ϕ.

Let $\tau_1, \tau_2, \tau_3, \tau_4$ be the unique times that it takes trajectories through $p_1 = (0, 0, 0)$, $p_2(1, 0, 0)$, $p_3 = (1, 1, 0)$, $p_4(0, 1, 0)$ respectively to reach Σ. Define $t_i = -\tau_i$. Thus $\Phi(f(p_i), t_i) = p_i$. Let

$$\gamma_1 = \{f(x, y, 0): 0 \leq x \leq 1, y = 0\}, \qquad \gamma_2 = \{f(x, y, 0): 0 \leq y \leq 1, x = 0\},$$
$$\beta_1 = \{f(x, y, 0): 0 \leq x \leq 1, y = 1\}, \qquad \beta_2 = \{f(x, y, 0): 0 \leq y \leq 1, x = 1\}.$$

There is a continuous map $t(q)$ $q \in \gamma_1 \cup \gamma_2$ that extends t_i, $i = 1, 2, 4$. Let $\alpha_1 = \{\Phi(q, t(q)): q \in \gamma_1 \cup \gamma_2\}$. Let α_2 be the curve congruent to α_1 going through the points p_2, p_3, p_4. Because $\Phi(x+z, t) = \Phi(x, t) + z$, and trajectories of Φ intersect α_1 exactly once, trajectories of Φ intersect α_2 exactly once. Hence $t(q)$ on $\gamma_1 \cup \gamma_2$ can be extended to a continuous function $t(q)$ on $\gamma_1 \cup \gamma_2 \cup \beta_1 \cup \beta_2$ such that $\{\Phi(q, t(q)): q \in \beta_1 \cup \beta_2\} = \alpha_2$. Now extend $t(q)$ over all of $f(D)$ as a continuous function. Because of the congruence of α_1 with α_2 the surface $S = \{\Phi(q, t(q)): q \in f(D)\}$ projects onto a torus $T^2 \subset T^3$ and the topological transversality to Φ of S insures the topological transversality of T^2 to ϕ. Q.E.D.

Bibliography

1. D. B. A. Epstein, *Periodic flows on 3-manifolds*, Ann. Math., **95** (1972), pp. 66–82.

2. C. S. Hartzman, *Diffeomorphisms of tori, their lifts and suspensions*, Rocky Mountain J., **6** #2 (1976), pp. 269–291.

3. Wu Yi Hsiang, *Cohomology theory of topological transformation groups*, Springer Verlag, New York, 1975.

4. L. Markus, *Parallel dynamical systems*, Topology, **8** (1969), pp. 47–57.

5. D. Montgomery, *Pointwise periodic homeomorphisms*, Amer. J. Math, **59** (1937), pp. 118–120.

6. D. R. Naugler, *Differential equations on manifolds with cross-section*, Ph.D. Dissertation, Dalhousie University, 1975.

7. Weaver, N., *Pointwise periodic homeomorphisms of continua*, Ann. Math., **95** (1972), pp. 83–85.

Dalhousie University
Halifax, N.S.

Note added in the proof. Subsequent to the acceptance of this manuscript, Professor David Naugler discovered that lemma 1.3 was known to L. E. J. Brouwer [*Aufzahlung der periodischen Transformationen des Torus*, Koninklijke Nederlandse Akademie van Wetenschappen, Proc. 21 (1919), pp. 1352–1356]. The result, however, appears to have been "lost" over the years (we have found no subsequent reference to it). Furthermore, complete proofs are not given by Brouwer and the proofs indicated take place on branched covering spaces for T^2 and are inherently more complicated than the one offered here.