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ABSTRACT

We prove packing stability for rational symplectic manifolds. This will rely on a general symplectic embedding result for ellipsoids which assumes only that there is no volume obstruction and that the domain is sufficiently thin relative to the target. We also obtain easily computable bounds for the Embedded Contact Homology capacities which are sufficient to imply the existence of some symplectic volume filling embeddings in dimension 4.

1. Introduction

We will study symplectic embeddings, that is, embeddings $f : (N, \sigma) \rightarrow (M, \omega)$ between symplectic manifolds such that $f^*\omega = \sigma$.

We will be particularly interested in symplectic ellipsoids. Let $E(a_1, \dots, a_n) \subset \mathbb{R}^{2n}$ be the open ellipsoid

$$E(a_1, \dots, a_n) = \left\{ \sum_{i=1}^n \frac{\pi(x_i^2 + y_i^2)}{a_i} < 1 \right\}.$$

Ellipsoids inherit a symplectic structure from the standard form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ on \mathbb{R}^{2n} . In our notation, the ball of capacity c is written $B^{2n}(c) = E(c, \dots, c)$ and we define $\lambda E(a_1, \dots, a_n) = E(\lambda a_1, \dots, \lambda a_n)$. Any given ellipsoid is symplectomorphic to the same ellipsoid with its factors permuted. Unless otherwise stated, when describing an ellipsoid we will list the factors in increasing order. Our main theorem is the following.

THEOREM 1.1. *There exists a constant $S(b_1, \dots, b_n)$ such that if $a_n/a_1 > S$ and $a_1 \cdots a_n \leq b_1 \cdots b_n$, there exists a symplectic embedding*

$$E(a_1, \dots, a_n) \longrightarrow E(b_1, \dots, b_n).$$

This will be established using a technique from Buse–Hind [BH11] which generates higher-dimensional embeddings from lower-dimensional ones, together with some new ellipsoid embeddings in dimension 4.

Remark 1.2. Some special cases of Theorem 1.1 are contained in our earlier paper [BH11] which avoided the use of Embedded Contact Homology (ECH). However, the paper [BH11] only claimed the existence of embeddings of arbitrarily large compact subsets of our ellipsoids. It was pointed out to us by E. Opshtein that in fact all of our embeddings exist between the open ellipsoids,

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see Proposition 3.4 and [Ops07, Lemma 2.1]. In the case of our four-dimensional embeddings we rely on work of McDuff, see [McD09, Corollary 1.6] and [McD91]. This says that the existence of embeddings of arbitrarily large compact subsets of a four-dimensional ellipsoid E into an ellipsoid E' implies that the full open ellipsoid E embeds into E' . The higher-dimensional analogue of this statement remains unknown.

Combining with work of Biran [Bir01] and Opshtein [Ops09], we also have a full filling result valid for general rational symplectic manifolds. Recall that the manifold (M, ω) is *rational* if $[\omega] \in H^2(M, \mathbb{Q}) \subset H^2(M, \mathbb{R})$.

THEOREM 1.3. *If (M^{2n}, ω) (or $(M^{2n}, \lambda\omega)$ for any $\lambda > 0$) is rational, then there exists a constant $S(M) > 0$ such that for all ellipsoids $E(a_1, \dots, a_n)$ with $a_n/a_1 > S(M)$ there exists a $\gamma > 0$ and a volume filling embedding $\gamma E(a_1, \dots, a_n) \rightarrow (M, \omega)$.*

We use the terms *volume filling* and *full filling* throughout to denote symplectic embeddings between manifolds of equal volume.

In the paper [CHLS07, §3], Cieliebak *et al.* define generalized symplectic capacities in terms of a fixed symplectic manifold (M, ω) . If M has dimension $2n$ then we get a capacity on the space of $2n$ -dimensional ellipsoids defined by

$$c^{(M,\omega)}E(a_1, \dots, a_n) := \inf\{c \mid E(a_1, \dots, a_n) \rightarrow (M, c\omega)\}.$$

Our result can be interpreted as saying that $c^{(M,\omega)}E(a_1, \dots, a_n)$ coincides with the normalized volume capacity

$$v^{(M,\omega)}E(a_1, \dots, a_n) := \sqrt[n]{\frac{\text{vol}(E(a_1, \dots, a_n))}{\text{vol}(M, \omega)}}$$

whenever M is rational and $a_n/a_1 > S(M)$, that is, whenever the ellipsoid is sufficiently thin.

The existence of an embedding between four-dimensional ellipsoids is determined by the ECH capacities of Hutchings, see [Hut11]. Namely, there is a sequence of numbers $\mathcal{N}(a, b)(k)$, $k \geq 0$, associated to a pair (a, b) and according to McDuff, [McD11], we have $E(a_1, a_2) \hookrightarrow E(b_1, b_2)$ if and only if $\mathcal{N}(a_1, a_2)(k) \leq \mathcal{N}(b_1, b_2)(k)$ for all $k \geq 0$. Although in principle this solves the problem of ellipsoid embeddings in dimension 4, it remains a difficult question to determine if a particular embedding exists since the $\mathcal{N}(a, b)(k)$ are given by quite complicated combinatorial formulas, [Hut11]. We will give quadratic bounds on a related piecewise linear function, see Proposition 2.5. One consequence is the following.

THEOREM 1.4. *Let $\beta \geq 1$ and $\alpha \geq (5\beta + 16)^2/16\beta$. Then there exists a full filling $E(1, \alpha) \rightarrow \sqrt{\alpha/\beta}E(1, \beta)$.*

Finally we discuss packing stability. The k th packing number of a compact, $2n$ -dimensional, symplectic manifold (M, ω) is

$$p_k(M, \omega) = \frac{\sup_c \text{Vol}(\bigsqcup_k B(c))}{\text{Vol}(M, \omega)},$$

where the supremum is taken over all c for which there exist a symplectic embedding of $\bigsqcup_k B(c)$, the disjoint union of k balls of capacity c , into (M, ω) . Naturally, $p_k(M, \omega) \leq 1$. The identity $p_k(M, \omega) = 1$ is equivalent to saying that (M, ω) admits a full filling by k identical balls, otherwise we say that there is a packing obstruction.

The symplectic manifold (M, ω) has *packing stability* if there exists an integer $N_{\text{stab}}(M, \omega)$ such that $p_i(M, \omega) = 1$ for all $i \geq N_{\text{stab}}(M, \omega)$.

THEOREM 1.5. *If (M^{2n}, ω) (or $(M^{2n}, \lambda\omega)$ for any $\lambda > 0$) is rational, then it has packing stability.*

This is due to Biran, [Bir99], in the case that M is a closed four-dimensional symplectic manifold with a rational cohomology class.

Bounds on $N_{\text{stab}}(M, \omega)$ can be derived from Theorem 1.1, although in specific examples we can obtain much sharper estimates.

Let $\mathbb{C}P^n$ be equipped with its Fubini–Study symplectic form, and denote by H_d^n a smooth hypersurface of degree d in $\mathbb{C}P^{n+1}$ with the induced symplectic form.

THEOREM 1.6. (i) $N_{\text{stab}}(\mathbb{C}P^n) \leq \lceil (8\frac{1}{36})^{n/2} \rceil$.

(ii) $N_{\text{stab}}(H_d^n) \leq \lceil (\frac{25}{16}d^{2/n} + 10d^{-(n-2)/n} + 16d^{-2(n-1)/n})^{n/2} \rceil$.

Of course, statement (i) here is just a refined estimate of statement (ii) valid for hyperplanes.

Lower bounds for the stability numbers $N_{\text{stab}}(M, \omega)$ come from embedding obstructions. For example, a consequence of the first inequality in [Gro85, §0.3.B] is the estimate $p_i(\mathbb{C}P^n) \leq i/2^n$, and so $N_{\text{stab}}(\mathbb{C}P^n) \geq 2^n$. (A construction, given for example in [Tra95], shows that $\mathbb{C}P^n$ can be fully filled by 2^n identical balls. Taking any i of these shows that in fact $p_i(\mathbb{C}P^n) = i/2^n$, see also the discussion in [MP94], §1.6.B.) In Proposition 5.1 we are able to slightly improve the estimate of Theorem 1.6(i) in dimension 6. Putting the two bounds together gives the following.

COROLLARY 1.7. $8 \leq N_{\text{stab}}(\mathbb{C}P^3) \leq 21$.

One can also consider packing with other symplectic manifolds, typically replacing the ball by other open subsets of $(\mathbb{R}^{2n}, \omega_0)$. Fixing a symplectic (D, ω_0) of dimension $2n$ we define the k th packing number with respect to D of a $2n$ -dimensional symplectic manifold (M, ω) by

$$p_k^D(M, \omega) = \frac{\sup_c \text{Vol}(\bigsqcup_k(D, c\omega_0))}{\text{Vol}(M, \omega)},$$

where the supremum is taken over all c for which there exist a symplectic embedding of $\bigsqcup_k(D, c\omega_0)$ into (M, ω) . Similarly to the above, we say that (M, ω) has packing stability with respect to (D, ω_0) if $p_k^D(M, \omega) = 1$ for all k sufficiently large. Taking D to be an ellipsoid, we can generalize Theorem 1.5 as follows.

THEOREM 1.8. *If (M^{2n}, ω) (or $(M^{2n}, \lambda\omega)$ for any $\lambda > 0$) is rational, then it has packing stability with respect to any symplectic ellipsoid.*

We know of no manifolds which have packing stability with respect to other domains in \mathbb{R}^{2n} , for example polydisks.

Outline of the paper

In §2 we discuss the ECH capacities, deriving our quadratic bounds and in particular proving Theorem 1.4. Section 3 gives the proof of Theorem 1.1. In §4.1 we describe how to obtain Theorem 1.3 and packing stability, Theorems 1.5 and 1.8, from Theorem 1.1 together with a construction of Opshtein, [Ops07]. In §5 we compute some examples and prove Theorem 1.6.

2. Ellipsoid embeddings in dimension 4

In this section we consider the embedding capacity function

$$f_\beta(\alpha) := \inf\{c \mid E(1, \alpha) \longrightarrow cE(1, \beta)\}$$

which is a natural extension of the function f_1 considered by McDuff and Schlenk in the paper [MS12, Theorem 1.1.2].

We will show that $f_\beta(\alpha)$ coincides with the normalized volume capacity of the ellipsoid $E(1, \beta)$ for sufficiently large values of α .

2.1 Description of ECH and some new estimates

The key ingredients for our study are results of Hutchings [Hut11] and McDuff [McD11] on ECH which together give necessary and sufficient conditions for a four-dimensional ellipsoid embedding.

A four-dimensional Liouville domain is a compact exact symplectic manifold (X, ω) such that there exists a contact form on ∂X which is a primitive of $\omega|_{\partial X}$. Hutchings associates to such (X, ω) an increasing sequence of real numbers $c_k(X, \omega)$ for $k \geq 0$ called the ECH capacities. The term capacity is justified by the following.

THEOREM 2.1 (Hutchings [Hut11]). *Let (X, ω) and (X', ω') be two Liouville domains as above. If there exists a symplectic embedding $\phi : (X, \omega) \rightarrow (X', \omega')$ then $c_k(X, \omega) \leq c_k(X', \omega')$ for all integers $k \geq 0$.*

In the same paper, Hutchings proceeds to compute the capacities of ellipsoids $E(a, b)$ as given in the following proposition.

PROPOSITION 2.2. *Given $0 < a \leq b$ consider the sequence $\mathcal{N}(a, b)(k)$ obtained by arranging in increasing order, with repetitions, all the numbers of the type $a\ell + bp$ with ℓ, p any nonnegative integers. Then $c_k(E(a, b)) = \mathcal{N}(a, b)(k - 1)$.*

Following this, McDuff showed that the necessary condition for ellipsoid embeddings coming from Theorem 2.1 is also sufficient.

THEOREM 2.3 (McDuff [McD11]). *There exists a symplectic embedding $\mathring{E}(a, b) \rightarrow E(a', b')$ if and only if $\mathcal{N}(a, b)(k) \leq \mathcal{N}(a', b')(k)$ for all natural numbers k .*

Comparing Theorems 2.1 and 2.3 we remark that $\mathring{E}(a, b) \rightarrow E(a', b')$ if and only if $E(a, b) \rightarrow \mathring{E}(\lambda a', \lambda b')$ for all $\lambda > 1$, see [McD11, Remark 1.3].

We will refer to $\mathcal{N}(a, b)(x)$ as being the piecewise linear function built by joining by line segments the points of $\mathcal{N}(a, b)(k)$. It is clearly sufficient to compare two such piecewise functions and moreover, if one defines, for any $y \geq 0$

$$R_{a,b}(y) = \sup\{k \mid \mathcal{N}(a, b)(k) \leq y\} \tag{1}$$

then we have the following corollary.

COROLLARY 2.4 (See Hutchings [Hut11], Bauer [Bau11]). *$E(a, b) \hookrightarrow E(a', b')$ if and only if*

$$R_{a,b}(y) \geq R_{a',b'}(y) \tag{2}$$

for all $y > 0$.

To prove Theorem 1.4 by using this corollary, we will first need the following estimates.

PROPOSITION 2.5. *For any $y > 0$, and $a < b$ we have that*

$$\frac{y^2}{2ab} + \frac{y}{2a} \leq R_{a,b}(y) \leq \frac{y^2}{2ab} + \frac{y}{2a} + \frac{y}{b} + \frac{b}{8a} + 1. \tag{3}$$

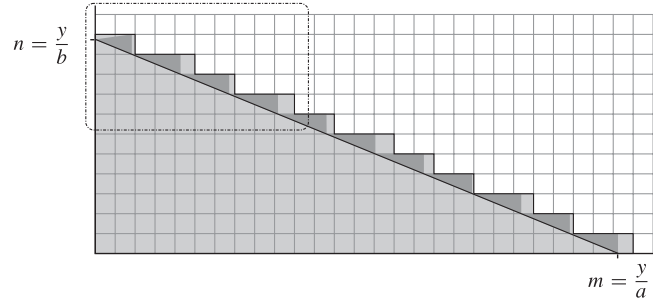


FIGURE 1. Triangles T_k and the region S .

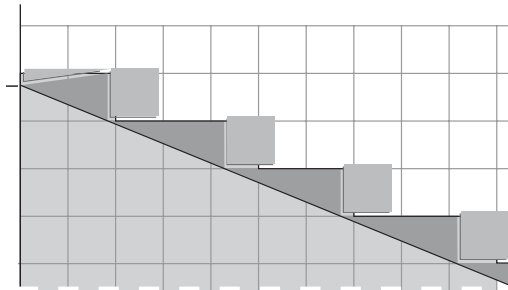


FIGURE 2. Added squares covering S .

Proof. As pointed out in [Hut11], $R_{a,b}(y)$ is interpreted geometrically as the number of non-negative integer vectors (m, n) in the closed triangle bounded by $m = 0$, $n = 0$, and the diagonal $am + bn = y$. Equivalently, this represents the area of the union of unit squares with lower-left corners (m, n) such that $am + bn \leq y$. The triangular area $y^2/2ab$ under the diagonal is then an obvious lower estimate for $R_{a,b}(y)$, but it omits the sizable area of the staircase-shaped region S above the diagonal. Let $\{y/b\} = y/b - \lfloor y/b \rfloor$ denote the fractional part of y/b .

To estimate the area of S , let $p_0 = (0, y/b)$, and let $p_1, \dots, p_{\lfloor y/b \rfloor}$ be the points of integer height along the diagonal, indexed from left to right. For every k with $1 \leq k \leq \lfloor y/b \rfloor$, let T_k be the triangle formed by p_{k-1} , p_k , and the point one unit above p_k . Every triangle $T_k \subset S$ rests atop the diagonal, these are the darkest triangles in Figure 1. They all have (vertical) base of length 1, and their (horizontal) heights add up to y/a ; thus, the total area of the triangles is $y/2a$, and

$$\frac{y^2}{2ab} + \frac{y}{2a} \leq R_{a,b}(y).$$

For the upper estimate, add unit squares to the right of $T_1, \dots, T_{\lfloor y/b \rfloor}$, see Figure 2. These have total area $\lfloor y/b \rfloor$, and cover the remainder of S , except (sometimes) for a thin triangle above T_1 sitting at the top left of Figure 2. The (vertical) base of this triangle has length $1 - \{y/b\}$. Its (horizontal) height is the first coordinate of p_1 , that is, $(b/a)\{y/b\}$. Therefore we get

$$R_{a,b}(y) \leq \frac{y^2}{2ab} + \frac{y}{2a} + \left\lceil \frac{y}{b} \right\rceil + \frac{b}{2a} \{y/b\} (1 - \{y/b\}) \leq \frac{y^2}{2ab} + \frac{y}{2a} + \frac{y}{b} + \frac{b}{8a} + 1,$$

where for the final inequality we note that $\{y/b\}(1 - \{y/b\}) \leq \frac{1}{4}$ since $0 \leq \{y/b\} < 1$. □

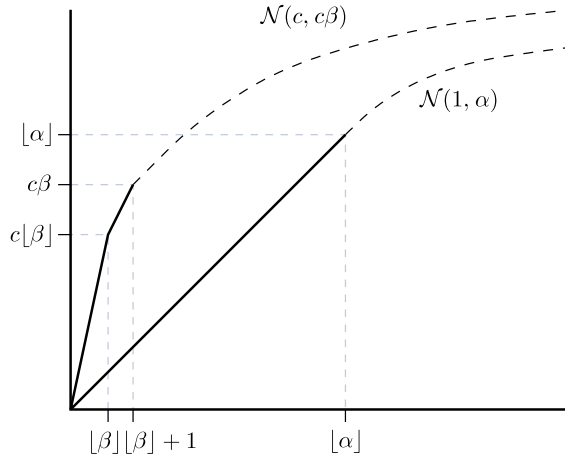


FIGURE 3. Graphs of the ECH capacity functions.

2.2 A volume filling ellipsoid embedding in dimension 4

We can now give a proof of Theorem 1.4 as follows.

Proof. Let us call $c := \sqrt{\alpha/\beta} > 1$ by hypothesis. Assuming $\alpha \geq (5\beta + 16)^2/16\beta$ and $\beta \geq 1$ it is required to show that we have a volume filling embedding $E(1, \alpha) \hookrightarrow E(c, c\beta)$. By Corollary 2.4 this is equivalent to showing

$$R_{1,\alpha}(y) \geq R_{c,c\beta}(y) \tag{4}$$

for all $y > 0$.

As $\alpha > \beta$ it is easy to see that for $0 \leq y < c\lfloor\beta\rfloor < \alpha$ we have $R_{1,\alpha}(y) = \lfloor y \rfloor + 1$ and $R_{c,c\beta}(y) = \lfloor y/c \rfloor + 1$. Therefore the inequality (4) holds for values of $y \leq c\lfloor\beta\rfloor$. Our hypothesis also implies that $\lfloor\beta\rfloor + 1 \leq \lfloor\alpha\rfloor$ and so the graphs of $N(1, \alpha)(x)$ and $N(c, c\beta)(x)$ are as shown in Figure 3 for small values of x . Hence we can also observe directly that inequality (4) holds in the range $c\lfloor\beta\rfloor \leq y \leq c\beta$.

It remains to show that (4) holds for the remaining values $y \geq c\beta$. Using the inequality (3) from Proposition 2.5 twice, once for $R_{1,\alpha}(y)$ and again for $R_{c,c\beta}(y)$, it is sufficient to show

$$\frac{y^2}{2\alpha} + \frac{y}{2} \geq \frac{y^2}{2c^2\beta} + \frac{y}{2c} + \frac{y}{c\beta} + \frac{c\beta}{8c} + 1 \tag{5}$$

for $y \geq c\beta$. Using the fact that $\alpha = c^2\beta$, this reduces to showing that

$$\frac{y}{2} \left(\frac{c\beta - \beta - 2}{c\beta} \right) \geq \frac{\beta}{8} + 1 \tag{6}$$

for all $y \geq c\beta$. But this is equivalent to our hypothesis $\alpha \geq (5\beta + 16)^2/16\beta$. □

3. Volume filling embeddings

THEOREM 3.1. *Define a constant S by $S^{1/(n-1)} = 6^n 20^{(n-2)/2} \max_k 20^{k(k-1)/2} (b_1 \cdots b_n)/b_k^n$. Then if $a_n/a_1 > S$ and $a_1 \cdots a_n = b_1 \cdots b_n$, there exists a volume preserving embedding*

$$E(a_1, \dots, a_n) \longrightarrow E(b_1, \dots, b_n).$$

Proof. We will apply a series of volume preserving embeddings to the ellipsoid $E(a_1, \dots, a_n)$. The factors of our ellipsoids will always be in increasing order. First recall our key tools.

Let $m(x) = (5x + 16)^2/16x$. Then m is decreasing on the interval $\{1 \leq x < 16/5\}$, increasing on $\{x > 16/5\}$ and $m(16/5) = 20$. Theorem 1.4 gives the following.

THEOREM 3.2. *If $b/a > m(d/c)$ and $ab = cd$, then $E(a, b) \longrightarrow E(c, d)$.*

COROLLARY 3.3. *Let $b/a > 36$ and $6a < d < \sqrt{ab}$. Then $E(a, b) \longrightarrow E(d, ab/d)$.*

Proof. The upper bound on d guarantees that the factors of our image ellipsoid are in increasing order. Therefore by Theorem 3.2 we just need to check that $b/a > m(ab/d^2)$. We compute

$$\frac{b}{a} - m\left(\frac{ab}{d^2}\right) = \frac{b}{a} - \frac{25 ab}{16 d^2} - 10 - \frac{16d^2}{ab} \geq \frac{b}{a} - \frac{25}{16} \frac{b}{36a} - 26 \geq 36 - \frac{25}{16} - 26 > 0$$

as required. □

We will also utilize the following.

PROPOSITION 3.4 ([Ops07, Lemma 2.1], [BH11, Proposition 2.1]). *If $E(a, b) \longrightarrow E(c, d)$ then*

$$E(a, b, a_3, \dots, a_n) \longrightarrow E(c, d, a_3, \dots, a_n)$$

for any a_3, \dots, a_n .

Note here that as the order of the factors is irrelevant, analogous statements hold for all other pairs of factors.

As mentioned in Remark 1.2 the paper [BH11] only established embeddings between arbitrarily large compact subsets of the ellipsoids. We thank E. Opshtein for remarking that the stronger version of Proposition 3.4 in fact follows from [Ops07] as we explain now. Proposition 3.4 is clearly a consequence of the following.

PROPOSITION 3.5. *Suppose that*

$$\bigsqcup_i E(a_{i1}, \dots, a_{in}) \longrightarrow E(b_1, \dots, b_n)$$

is a symplectic embedding of a disjoint union of ellipsoids. Then for any $c > 0$ there exists a symplectic embedding

$$\bigsqcup_i E(a_{i1}, \dots, a_{in}, c) \longrightarrow E(b_1, \dots, b_n, c).$$

Proof. Indeed, let $(E, \omega_0) = E(b_1, \dots, b_n)$ be a symplectic ellipsoid and D be the unit disk. Let α be a 1-form on $E \times (D \setminus \{0\})$ such that $d\alpha = -\pi^*\omega_0$ and $\alpha|_{\pi^{-1}(x)} = c d\theta$ for all $x \in E$, where π denotes the projection on the E factor and (r, θ) are polar coordinates on the D factor. Then [Ops07, Lemma 2.1] says that $E(b_1, \dots, b_n, c)$ is symplectomorphic to $(E \times D, \omega)$, where $\omega = \pi^*\omega_0 + d(r^2\alpha)$. It follows that if $\phi: E(a_1, \dots, a_n) \longrightarrow E(b_1, \dots, b_n)$ is a symplectic embedding then $(\text{im}\phi \times D, \omega)$ is symplectomorphic to $E(a_1, \dots, a_n, c)$ and the proposition follows. □

Returning to Theorem 1.1, we start with the following lemma. Recall that $E(a_1, \dots, a_n)$ is the ellipsoid as in Theorem 1.1.

LEMMA 3.6. *There exists a symplectic volume filling embedding $E(a_1, \dots, a_n) \hookrightarrow E(a'_1, \dots, a'_n)$ such that:*

- (i) $a'_n = a_n$;
- (ii) $a'_k/a'_{k-1} < 20$ for all $1 \leq k \leq n - 1$;
- (iii) $a'_n/a'_k > 20^{-(n-2)/2} S^{1/n-1}$ for all $1 \leq k \leq n - 1$.

Proof. We perform a sequence of embeddings fixing throughout the largest factor a_n of the ellipsoid.

Theorem 3.2 implies that if $a_k/a_{k-1} > 20$ then there exists an embedding $E(a_{k-1}, a_k) \rightarrow \sqrt{5a_{k-1}a_k/16}E(1, 16/5)$.

We will apply the embedding of Proposition 3.4 induced by this four-dimensional embedding to any consecutive pair of factors a_{k-1}, a_k for which $k < n$ and whose ratio is greater than 20. After such an operation the product of the first $k - 1$ factors will increase by a factor of at least $\sqrt{100/16}$, but the product of the first $n - 1$ factors is fixed throughout. Thus after a finite number of steps we arrive at an ellipsoid $E(a'_1, \dots, a'_n)$ satisfying condition (ii).

For condition (iii) we estimate

$$\begin{aligned} \left(\frac{a'_n}{a'_k}\right)^{n-1} &\geq \left(\frac{a'_n}{a'_{n-1}}\right)^{n-1} \\ &= \frac{(a'_n)^{n-1}}{20^{(n-1)(n-2)/2}} \cdot \frac{20 \cdot 20^2 \cdots 20^{n-2}}{(a'_{n-1})^{n-1}} \\ &\geq \frac{(a'_n)^{n-1}}{20^{(n-1)(n-2)/2} \cdot a'_{n-1} \cdot a'_{n-2} \cdots a'_1} \end{aligned}$$

using property (ii). Now, as all our embeddings are volume preserving and fix a_n , the product of the first $n - 1$ terms is also preserved under our sequence of embeddings. Thus we have

$$\begin{aligned} \left(\frac{a'_n}{a'_k}\right)^{n-1} &\geq \frac{a_n^{n-1}}{20^{(n-1)(n-2)/2} \cdot a_{n-1} \cdot a_{n-2} \cdots a_1} \\ &> 20^{-(n-1)(n-2)/2} S, \end{aligned}$$

where for the final inequality we used simply that $a_k \leq a_n$ for all $k < n$ and that by hypothesis $a_n > Sa_1$. Our lemma follows. □

Now we drop the primes from the ellipsoid resulting from Lemma 3.6 and write simply $E(a_1, \dots, a_n)$ for our new range.

LEMMA 3.7. $b_k > 6a_k$ for all $1 \leq k \leq n - 1$.

Proof. For any k we have

$$\begin{aligned} 1 &= \frac{a_1 \cdots a_n}{b_1 \cdots b_n} \geq \frac{a_1 \cdots a_{k-1} a_k^{n-k} a_n}{b_1 \cdots b_n} \\ &> \frac{a_k^{n-k}}{b_1 \cdots b_n} \frac{a_k}{20^{k-1}} \cdots \frac{a_k}{20} 20^{-(n-2)/2} S^{1/n-1} a_k \end{aligned}$$

using properties (ii) and (iii) from Lemma 3.6

$$= \left(\frac{a_k}{b_k}\right)^n \frac{20^{-(n-2)/2} S^{1/n-1} (b_k)^n}{20^{k(k-1)/2} b_1 \cdots b_n} \geq \left(\frac{a_k}{b_k}\right)^n 6^n$$

and our lemma follows. □

Now we complete the proof of Theorem 3.1. We will proceed by induction on $k = 0, 1, \dots, n - 1$. After the k th stage we will have found a symplectic volume filling embedding into an ellipsoid of the form $E(b_1, \dots, b_k, a_{k+1}, \dots, a_{n-1}, a_{n,k})$ where the a_i are the factors of our most recent ellipsoid above and $a_{n,k}$ will be a decreasing sequence with $a_{n,0} = a_n$ and $a_{n,k} = (a_{n,k-1}a_k)/b_k$. It is decreasing by Lemma 3.7.

For our inductive step, we claim that there exists an embedding

$$E(a_k, a_{n,k-1}) \longrightarrow E\left(b_k, \frac{a_{n,k-1}a_k}{b_k}\right) = E(b_k, a_{n,k})$$

for all $1 \leq k \leq n - 1$. Thus by repeated applications of Proposition 3.4 we can conclude by induction. After the $(n - 1)$ th step we will automatically have $a_{n,n-1} = b_n$ by the hypothesis $a_1 \cdots a_n = b_1 \cdots b_n$ and the fact that all of our embeddings are volume preserving.

To justify the claim, we will apply Corollary 3.3 with $a = a_k$, $b = a_{n,k-1}$ and $d = b_k$. There are three conditions to check. First

$$\frac{b}{a} = \frac{a_{n,k-1}}{a_k} \geq \frac{a_{n,n-2}}{a_{n-1}} = \frac{a_n a_1 a_2 \cdots a_{n-2}}{b_1 \cdots b_{n-2} a_{n-1}} = \frac{b_{n-1} b_n}{a_{n-1}^2} > 36$$

where the equalities follow from the definitions of the $a_{n,k}$ and the fact that our ellipsoids have equal volume, and the final inequality follows from Lemma 3.7. Secondly, $d = b_k > 6a_k$ again by Lemma 3.7. Finally,

$$\begin{aligned} d^2 &= b_k^2 \leq b_k b_n = \frac{b_1 \cdots b_{k-1} b_k b_n}{b_1 \cdots b_{k-1}} \\ &< \frac{a_1 \cdots a_{k-1} a_k a_n}{b_1 \cdots b_{k-1}} = \cdots = a_k a_{n,k-1} = ab \end{aligned}$$

where the inequality holds since our ellipsoids have equal volume but $b_k > a_k$ for all $k < n$. □

4. Full filling by an ellipsoid

In this section we present Opshtein’s result, which is a refinement of Biran’s polarization theorem, that any rational symplectic manifold can be fully filled by an ellipsoid, that is, we prove the following.

THEOREM 4.1 ([Ops07, Ops09]). *Let (M, ω) be rational. Then there exist b_1, \dots, b_n such that there exists a full filling $E(b_1, \dots, b_n) \longrightarrow (M, \omega)$.*

Given Theorem 4.1 we can prove Theorems 1.5 and 1.8. Theorem 1.3 is also a direct application of Theorems 4.1 and 1.1.

Proof of Theorem 1.5. Given (M, ω) rational, we consider the fully filling ellipsoid $E(b_1, \dots, b_n)$ given by Theorem 4.1. By Theorem 1.1 there exists a constant $S(b_1, \dots, b_n)$ such that there exists a full filling $\gamma E(1, \dots, 1, k) \longrightarrow E(b_1, \dots, b_n)$ for any $k \geq S$. However, the two-dimensional embedding $\bigsqcup_k D(1) \longrightarrow D(k)$ combined with Proposition 3.5 implies that there exists a full filling

$$\bigsqcup_k B(1) \longrightarrow E(1^{\times(n-1)}, k)$$

for any $k \in \mathbb{N}$ (see also [BH11, Lemma 4.1]). Putting the two maps together, we obtain a full filling of $E(b_1, \dots, b_n)$, and hence (M, ω) , by k balls of capacity γ . This means that (M, ω) has packing stability as required. □

Proof of Theorem 1.8. Fix an ellipsoid $D = E(a_1, \dots, a_n)$. We follow an identical line of argument to the proof of Theorem 1.5 above, letting $E(b_1, \dots, b_n)$ be the filling ellipsoid for (M, ω) from Theorem 4.1.

LEMMA 4.2. *There exists a full filling*

$$\bigsqcup_k E(a_1, \dots, a_{n-1}, a_n) \longrightarrow E(a_1, \dots, a_{n-1}, ka_n)$$

for any $k \in \mathbb{N}$.

This follows immediately from Proposition 3.5 given the two-dimensional embedding $\bigsqcup_k D(a_n) \longrightarrow D(ka_n)$ (see [Sch05, Lemma 5.3.1(i)] for a different approach).

But by Theorem 1.1 there exists a constant $S(b_1, \dots, b_n)$ such that there exists a full filling $\gamma E(a_1, \dots, a_{n-1}, ka_n) \longrightarrow E(b_1, \dots, b_n)$ for any $k \geq Sa_1/a_n$. This map and the embedding from our lemma together give packing stability for (M, ω) with respect to D . \square

Now we proceed to prove Theorem 4.1 by induction on the dimension of the manifold M . Observe that Moser’s theorem implies that a surface can be fully filled by any number of balls, so we assume that the theorem holds for all manifolds of dimension $2(n - 1)$ and let (M, ω) be rational of dimension $2n$.

Using the rationality assumption, a theorem of Donaldson, [Don96], says that (M, ω) can be polarized in the sense that there exists a connected symplectic codimension 2 submanifold N which is Poincaré dual to $m[\omega]$ for a suitable multiple m . Let $\tau = \omega|_N$.

Let $SDB(N, \tau, m)$ be a standard symplectic disk bundle over N . This is a symplectic manifold whose underlying smooth manifold is the disk bundle over N with Euler class $[m\tau] \in H^2(N, \mathbb{Z})$ and which has a symplectic form restricting to τ on N and with fibers of area $1/m$. As a symplectic manifold, $SDB(N, \tau, m)$ is well defined up to isotopy. For more details see [Bir01, § 2], or [Ops09, § 1]. Given the above, we can now state the Biran decomposition.

THEOREM 4.3 ([Ops09, Theorem 1], [Bir01]). *There exists a full filling $SDB(N, \tau, m) \longrightarrow (M, \omega)$.*

By our induction hypothesis, there exists b_1, \dots, b_{n-1} and a full filling $E(b_1, \dots, b_{n-1}) \longrightarrow (N, \tau)$. Now we apply the construction of Opshtein.

LEMMA 4.4 [Ops07, Lemma 2.1]. *A symplectic embedding $E(b_1, \dots, b_{n-1}) \longrightarrow (N, \tau)$ can be extended to a symplectic embedding $E(b_1, \dots, b_{n-1}, 1/m) \longrightarrow SDB(N, \tau, m)$.*

If $E(b_1, \dots, b_{n-1}) \longrightarrow (N, \tau)$ is a full filling then the embedding $E(b_1, \dots, b_{n-1}, 1/m) \longrightarrow SDB(N, \tau, m)$ of Lemma 4.4 is also a full filling. Thus, combining Theorem 4.3 and Lemma 4.4 we have a full filling $E(b_1, \dots, b_{n-1}, 1/m) \longrightarrow (M, \omega)$ and we conclude the proof of Theorem 4.1 by induction.

4.1 Full fillings without ECH

In fact, as was also explained to us by Opshtein, a similar argument can generalize Theorem 4.1 as follows.

THEOREM 4.5 (Opshtein). *Let (M, ω) be integral, that is, $[\omega] \in H^2(M, \mathbb{Z})$, and let $V = \text{vol}(M)$. For all sufficiently large l we can fully fill M by the ellipsoid $(1/l)E(1, \dots, 1, n!l^n V)$.*

Given this, following the same lines as above, to establish packing stability we need only show that for all large k there exists a $\gamma > 0$ and a full filling $\gamma E(1, \dots, 1, k) \rightarrow E(1, \dots, 1, n!l^n V)$ for an l which is also sufficiently large for Theorem 4.5 to hold. But this follows directly from the methods of [BH11], in particular avoiding ECH. Indeed, assuming $n \geq 3$, formulas (19) and (20) from [BH11] together imply that $\gamma E(1, \dots, 1, k)$ fully fills any $E(1, \dots, 1, z)$ provided $(2/3)^{n/2(n-1)} k^{(n-2)/2(n-1)} < z < k^{(n-2)/2(n-1)}$ and one can check that there exist numbers $z = n!l^n V$ in this range with $l \rightarrow \infty$ as $k \rightarrow \infty$.

Outline of the proof of Theorem 4.5. Let L be a complex line bundle over M with first Chern class $[\omega]$. Donaldson’s theorem has been generalized by Donaldson [Don99] and Auroux [Aur01, Theorem 5.1] to show that for all l sufficiently large we can find approximately holomorphic sections s_1, \dots, s_{n-1} of $L^{\otimes l}$ which intersect the zero section transversally in submanifolds H_1, \dots, H_{n-1} such that all intersections $N_i = H_1 \cap \dots \cap H_i$ for $1 \leq i \leq n - 1$ are transverse and N_i gives a polarization of N_{i-1} .

Now, as N_{n-1} is Poincaré dual to $l^{n-1}[\omega]^{n-1}$ it is a symplectic surface of area $n!l^{n-1}V$ and hence is fully filled by a ball $B(n!l^{n-1}V)$. Next, each N_i gives a polarization of N_{i-1} Poincaré dual to $l[\omega]_{N_{i-1}}$ and so arguing by induction as above we get a full filling of M by an ellipsoid $E(1/l, \dots, 1/l, n!l^{n-1}V)$ as required.

5. Examples

5.1 The complex projective space $\mathbb{C}P^n$

Complex projective space $\mathbb{C}P^n$ is fully filled by one ball. Recall also that $E(1, \dots, 1, k)$ can be fully filled by k balls, see [BH11, Lemma 4.1]. Therefore we can establish our stability bound for $\mathbb{C}P^n$ by producing an embedding $E(1, \dots, 1, k) \hookrightarrow B(k^{1/n})$ for all $k \geq (8(1/36))^{n/2}$.

We consider the following sequence of $n - 1$ potential embeddings:

$$\begin{aligned} & E(1, \dots, 1, k) \rightarrow E(k^{1/n}, 1, \dots, 1, k^{(n-1)/n}) \\ \rightarrow & E(k^{1/n}, k^{1/n}, 1, \dots, 1, k^{(n-2)/n}) \rightarrow \dots \rightarrow E(k^{1/n}, \dots, k^{1/n}, 1, k^{2/n}) \\ & \rightarrow E(k^{1/n}, \dots, k^{1/n}) = B(k^{1/n}). \end{aligned}$$

The first $n - 2$ embeddings will exist by Theorem 3.2 and Proposition 3.4 provided we have

$$k^{(n-i)/n} \geq \frac{(5k^{(n-i-2)/n} + 16)^2}{16k^{(n-i-2)/n}} = \frac{25}{16}k^{(n-i-2)/n} + 10 + 16k^{-(n-i-2)/n} \tag{7}$$

for all $0 \leq i \leq n - 3$.

For the final embedding, as the target is a ball, we get an improved bound by appealing directly to [MS12, Theorem 1.1.2(iv)]. This says that $E(1, k^{2/n}) \rightarrow B(k^{1/n})$ whenever $k^{2/n} \geq 8(1/36)$. Therefore by Proposition 3.4 the final embedding will also exist provided $k^{2/n} \geq 8(1/36)$.

For the first $n - 2$ embeddings we require

$$\frac{25}{16}k^{-2/n} + 10k^{-(n-i)/n} + 16k^{-2(n-i-1)/n} \leq 1 \tag{8}$$

for all $0 \leq i \leq n - 3$. The left-hand side is an increasing function of i , so it suffices that

$$\frac{25}{16}k^{-2/n} + 10k^{-3/n} + 16k^{-4/n} \leq 1.$$

A calculation shows that this also holds when $k^{2/n} \geq 8(1/36)$ and so Theorem 1.6(i) is proved. \square

There is in fact a better bound valid in dimension 6.

PROPOSITION 5.1. $N_{\text{stab}}(\mathbb{C}P^3) \leq 21$.

Proof. Part (i) of Theorem 1.6 already implies that $N_{\text{stab}}(\mathbb{C}P^3) \leq 23$. Thus we just need to check that the same embeddings exist when $k = 21$ and $k = 22$. Direct computation shows that the inequality (8) continues to hold for these k , so we are reduced to showing $E(1, k^{2/3}) \rightarrow B(k^{1/3})$ when $k = 21, 22$. We appeal again to McDuff–Schlenk’s results from [MS12], Theorem 5.2.3 and its corresponding Table 5.2. This says that $f_1(\alpha) = \sqrt{\alpha}$ on the interval $[7\frac{1}{9}, 8]$ with the exception of eight intervals that are explicitly provided in their statement. The numbers $k^{2/3}$ for $k = 21, 22$ do not lie in these intervals, thus this embedding also exists and hence the whole sequence of embeddings provided in the proof of Theorem 1.6. \square

5.2 The hypersurface H_d^n

We first make the following observation.

Fact. H_d^n embeds as a polarizing hypersurface in H_d^{n+1} Poincaré dual to the symplectic form.

Indeed, $H_d^n = H_d^{n+1} \cap \mathbb{C}P^{n+1}$ where $H_d^{n+1} \subset \mathbb{C}P^{n+2}$ and we think of $\mathbb{C}P^{n+1}$ as a hyperplane intersecting H_d^{n+1} transversally.

Now, H_d^1 is a curve of degree d and so is fully filled by $B^2(d)$, normalizing the Fubini–Study form on the $\mathbb{C}P^n$ so that lines have area 1. Hence, combining the above fact with Lemma 4.4 and arguing by induction we see that $E(1, \dots, 1, d)$ fully fills H_d^n .

Therefore, arguing as in case (i) by applying Lemma 4.1 of [BH11], to demonstrate Theorem 1.6(ii) it suffices to show that

$$E(1, \dots, 1, k) \rightarrow \frac{k^{1/n}}{d^{1/n}} E(1, \dots, 1, d)$$

for all integers $k \geq ((25/16)d + 10d^{-(n-2)/n} + 16d^{-2(n-1)/n})^{n/2}$.

We now consider the sequence of $(n - 1)$ potential embeddings

$$\begin{aligned} E(1, \dots, 1, k) &\rightarrow E\left(\frac{k^{1/n}}{d^{1/n}}, 1, \dots, 1, k^{(n-1)/n}d^{1/n}\right) \rightarrow \\ E\left(\frac{k^{1/n}}{d^{1/n}}, \frac{k^{1/n}}{d^{1/n}}, 1, \dots, 1, k^{(n-2)/n}d^{2/n}\right) &\rightarrow \dots \rightarrow E\left(\frac{k^{1/n}}{d^{1/n}}, \dots, \frac{k^{1/n}}{d^{1/n}}, k^{1/n}d^{(n-1)/n}\right) \\ &= \frac{k^{1/n}}{d^{1/n}} E(1, \dots, 1, d). \end{aligned}$$

By Proposition 3.4 the $(i + 1)$ th embedding exists provided there exists an embedding

$$E(1, k^{(n-i)/n}d^{i/n}) \rightarrow E\left(\frac{k^{1/n}}{d^{1/n}}, k^{(n-i-1)/n}d^{(i+1)/n}\right)$$

which by Theorem 3.2 exists provided

$$k^{(n-i)/n}d^{i/n} \geq \frac{25}{16}k^{(n-i-2)/n}d^{(i+2)/n} + 10 + \frac{16}{k^{(n-i-2)/n}d^{(i+2)/n}}$$

or equivalently

$$\frac{25}{16}k^{-2/n}d^{2/n} + 10k^{-(n-i)/n}d^{-i/n} + 16k^{-2(n-i-1)/n}d^{-2(i+1)/n} \leq 1.$$

For $k \geq d$ and n fixed, the left-hand side of this last inequality is an increasing function of i . Therefore it suffices to check it when $i = n - 2$. In this case we get

$$\frac{25}{16}k^{-2/n}d^{2/n} + 10k^{-2/n}d^{-(n-2)/n} + 16k^{-2/n}d^{-2(n-1)/n} \leq 1$$

which is equivalent to

$$k^{2/n} \geq \frac{25}{16}d^{2/n} + 10d^{-(n-2)/n} + 16d^{-2(n-1)/n}$$

and so holds in our range as required.

Remark 5.2. Still following our methods here as well as those from [BH11], the estimates for the upper bound on the stability numbers presented here may be improved if we can improve, by a much more complicated number theoretical investigation of the ECH vectors, the four-dimensional embedding result from Theorem 1.4. However, in the case when the target manifold is $\mathbb{C}P^3$ the inequality $N_{\text{stab}} \leq 21$ cannot be improved in this way as it relies at the last step on a sharp estimate from McDuff–Schlenk giving the precise range for volume filling ellipsoids in the unit ball. It remains a very interesting question to find methods to obstruct symplectic volume fillings of ellipsoids or disjoint balls into the unit six dimensional ball. As will be shown in [BH], it appears that obstructions arising from symplectic field theory that have been successful in [HK09] when targets were cylinders fail here.

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