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ELLIPTIC PROBLEMS IN $\mathbb{R}^N$ WITH DISCONTINUOUS NONLINEARITIES

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Abstract For a class of elliptic equations in the entire space and with nonlinear terms having a possibly uncountable (but of Lebesgue measure zero) set of discontinuities, the existence of strong solutions is established. Two simple applications are then developed. The approach taken is strictly based on set-valued analysis and fixed-points arguments.

Keywords: elliptic equations in the whole space; discontinuous nonlinearities; elliptic differential inclusions; implicit elliptic equations; strong solutions

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1. Introduction

Nonlinear elliptic equations in the whole space have been widely investigated from several points of view; we cite for instance the recent paper [**11**] and the references given therein. Meaningful results are sometimes obtained by adapting and solving questions previously examined only for equations in bounded domains. Since a branch of the current literature on elliptic boundary-value problems in bounded domains deals with the existence of solutions to equations having discontinuous nonlinearities (see [**8, 13, 18**] for a general reference), it seems of interest to ask what happens whenever elliptic equations of such a kind are considered in the entire space.

The aim of the present paper is to provide a contribution in the above-mentioned direction. Accordingly, here, we study the semilinear elliptic equation

$$\mathcal{L}u = f(x,u), \quad x \in \mathbb{R}^n, \tag{E}$$

where $\mathcal{L}u := -\Delta u + u$, $n \geqslant 3$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is assumed to be only directionally continuous [**4**, p. 460], namely continuous with respect to a given cone of $\mathbb{R}^{n+1}$. We look for solutions of (E) that lie in $W^{2,p}(\mathbb{R}^n)$, $p \in]n, +\infty[$.

Obviously, directionally continuous functions need not be continuous in the usual sense, on the contrary, they may have an uncountable (but of Lebesgue measure zero) set of discontinuities. Nevertheless, in this setting equation (E) becomes easily solvable. Indeed, adapting the approach previously developed in [**4**, **5**] for a class of ordinary differential inclusions, we first consider a suitable upper semicontinuous convex-valued regularization $F(x,u)$ of $f(x,u)$ and, via fixed-points arguments, we get a solution $u \in W^{2,p}(\mathbb{R}^n)$ to the elliptic differential inclusion $\mathcal{L}u \in F(x,u)$, $x \in \mathbb{R}^n$. Next, by using the directional continuity of f, we prove that u also satisfies (E) (see Theorem 3.2).

To the best of our knowledge, very little is known about elliptic equations on the whole space and with discontinuous nonlinear terms. Actually, we can only mention the papers [**2**] and [**8**]. The nonlinearity treated in [**2**] does not depend on $x \in \mathbb{R}^n$ and possesses just one discontinuity point. A quasilinear elliptic equation having a nonlinear term which satisfies appropriate monotonicity conditions is studied in [**8**] through the upper and lower solution method combined with a general fixed-point principle in partially ordered sets. Simple examples show that Theorem 3.2 below and the results of [**2**,**8**] are mutually independent.

We then present two applications. The first of them (Theorem 3.6) deals with the problem

$$u \in W^{2,p}(\mathbb{R}^n), \quad \mathcal{L}u \in G(x,u) \quad \text{in } \mathbb{R}^n. \tag{P$_1$}$$

Here, the right-hand side G takes closed values and is lower semicontinuous. Solutions to (P_1) are easily obtained via Bressan's Directionally Continuous Selection Theorem [**4**, Theorem 1] and our Theorem 3.2. The second application (Theorem 3.7) is an existence result for the implicit problem

$$u \in W^{2,p}(\mathbb{R}^n), \quad \psi(\mathcal{L}u) = \varphi(x,u) \quad \text{in } \mathbb{R}^n, \tag{P$_2$}$$

where φ and ψ are given continuous functions. Through Theorem 2.4 of [**19**] we reduce (P_2) to (P_1), the multi-function G now being a suitable multi-selection from $(x,u) \mapsto \psi^{-1}(\varphi(x,u))$, and next apply Theorem 3.6. The case of bounded domains has previously been investigated in [**15**, **17**] employing a different technique.

For the sake of completeness we finally consider, in Theorem 3.10, the situation when $\mathbb{R}^n$ and $\mathcal{L}$ are, respectively, replaced by a bounded convex domain $\Omega \subseteq \mathbb{R}^n$ and a more general strictly elliptic operator $\mathcal{M}$, providing solutions of (E) that belong to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

As regards the problem $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\mathcal{M}u = f(x,u)$ in Ω, with highly discontinuous f, existence results have recently been established in [**3**,**16**] (see also [**17**,**18**] and the references cited therein) under other assumptions.

2. Basic definitions and preliminary results

Let X and Y be two non-empty sets. The symbols $\Phi : X \to 2^Y$ mean that Φ is a multi-function from X into Y, namely a function which assigns to each point $x \in X$ a non-empty subset $\Phi(x)$ of Y. The graph of Φ is the set $\{(x,y) \in X \times Y : y \in \Phi(x)\}$, while

$\Phi(X) := \cup_{x\in X}\Phi(x)$ represents the range of Φ. For $W \subseteq Y$, define $\Phi^-(W) := \{x \in X : \Phi(x) \cap W \neq \emptyset\}$. If $(X, \mathcal{F})$ is a measurable space, Y is a topological space, and, for any open subset W of Y, one has $\Phi^-(W) \in \mathcal{F}$, we say that Φ is $\mathcal{F}$-measurable. When X and Y are two topological spaces and $\Phi^-(W)$ is closed (respectively, open) for every closed (open) set $W \subseteq Y$, the multi-function Φ is called upper (lower) semicontinuous. In such a case its graph is clearly closed in $X \times Y$ provided that $\Phi(x)$ is closed for all $x \in X$ and Y is regular [**14**, Theorem 7.1.15]. Conversely, if Φ has a relatively compact range and a closed graph, then it is also upper semicontinuous [**14**, Theorem 7.1.16].

The following result is an immediate consequence of Ky Fan's Fixed-Point Theorem (see, for example, [**3**, Theorem 2.1]).

Theorem 2.1. *Let X be a metrizable locally convex topological vector space and let V be a non-empty, weakly compact, convex subset of X. Suppose Φ is a multi-function from V into itself with non-empty convex values and weakly sequentially closed graph. Then there exists $x_0 \in V$ such that $x_0 \in \Phi(x_0)$.*

Let h be a positive integer and let $\mathbb{R}^h$ be the h-dimensional Euclidean space equipped with the norm

$$\|w\| := \sum_{i=1}^{h} |w_i|, \quad w = (w_1, w_2, \ldots, w_h) \in \mathbb{R}^h.$$

If W is a subset of $\mathbb{R}^h$, we write $\operatorname{int}(W)$ for the interior of W, ∂W for the boundary of W, $\bar{W}$ for the closure of W, $\overline{\operatorname{co}}(W)$ for the closed convex hull of W. Moreover, 'measurable' always means Lebesgue measurable and $m(W)$ denotes the measure of W. The symbol $\mathcal{L}(W)$ indicates the Lebesgue σ-algebra of W, while, for any open set $A \subseteq \mathbb{R}$, $\mathcal{B}(A)$ is the Borel σ-algebra of A. When W is non-empty, $w_0 \in \mathbb{R}^h$, and $\delta > 0$, we define $d(w_0, W) := \inf_{w\in W} \|w - w_0\|$ as well as $B(x, \delta) := \{w \in \mathbb{R}^h : \|w - w_0\| < \delta\}$.

The lemma below is easily obtained by using Theorem 7.16 of [**23**]. If $w', w'' \in \mathbb{R}^h$, $w' = (w'_1, w'_2, \ldots, w'_h)$, $w'' = (w''_1, w''_2, \ldots, w''_h)$, we write $w' < w''$ (respectively, $w' \leqslant w''$) whenever $w'_i < w''_i$ ($w'_i \leqslant w''_i$) for each $i = 1, 2, \ldots, h$.

Lemma 2.2. *Let A be a measurable subset of $\mathbb{R}^h$. Then there exists a measurable set $A^* \subseteq A$ having the following properties:*

(p_1) *$m(A^*) = m(A)$;*

(p_2) *for every $x \in A^*$, there is a sequence $\{x_k\} \subseteq A$ converging to x and such that $x_{k+1} < x_k$, $k \in \mathbb{N}$.*

From Vitali's Covering Theorem [**20**, Theorem 3.1, p. 109] and Lusin's Theorem [**23**, Theorem 4.20] we infer the following lemma.

Lemma 2.3. *Let $\mu : \mathbb{R}^h \to \mathbb{R}$ be measurable. Then there exists a sequence $\{A_k\}$ of compact subsets of $\mathbb{R}^h$, no two of which have common points, so that*

$$m\Big(\mathbb{R}^h \setminus \bigcup_{k\in\mathbb{N}} A_k\Big) = 0,$$

and $\mu|_{A_k}$ is continuous for all $k \in \mathbb{N}$.

A non-empty, convex, closed set $\Gamma \subseteq \mathbb{R}^h$ is said to be a cone provided that $\Gamma \cap (-\Gamma) = \{0\}$ and $\lambda w \in \Gamma$ for every $\lambda \geqslant 0$, $w \in \Gamma$. An elementary argument yields the following lemma.

Lemma 2.4. *Let n be a positive integer and let $M > 0$. Define*

$$\Gamma^M := \{(x,z) \in \mathbb{R}^{n+1} : x = (x_1, x_2, \dots, x_n),\ x_i \geqslant 0,\ i = 1, 2, \dots, n,\ |z| \leqslant M\|x\|\}.$$

Then the set Γ^M is a cone of $\mathbb{R}^{n+1}$.

If $f : \mathbb{R}^h \to \mathbb{R}$, $w_0 \in \mathbb{R}^h$, and Γ denotes a cone of $\mathbb{R}^h$, we say that f is Γ-continuous at w_0 (see [**4**, p. 460]) when to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that if $w \in B(w_0, \delta)$ and $w - w_0 \in \Gamma$, then $|f(w) - f(w_0)| < \varepsilon$. The function f is called Γ-continuous when it is Γ-continuous at each point of $\mathbb{R}^h$.

Obviously, Γ-continuous functions need not be continuous in the usual sense, as the next straightforward example shows.

Example 2.5. Define, for every $(x,z) \in \mathbb{R}^2$, $f(x,z) = 1$ if $z > x$, $f(x,z) = 2$ otherwise. The function f is discontinuous at all points of the kind (x,x), $x \in \mathbb{R}$, whereas it is Γ^M-continuous with $M \in \,]0,1[$.

In spite of this we have the following lemma.

Lemma 2.6. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be Γ^M-continuous for some $M > 0$. Then the set*

$$D_f := \{(x,z) \in \mathbb{R}^n \times \mathbb{R} : f \text{ is discontinuous at } (x,z)\}$$

has measure zero.

Proof. Arguing by contradiction, suppose $m(D_f) > 0$ and write, for every $z \in \mathbb{R}$, $A(z) := \{x \in \mathbb{R}^n : (x,z) \in D_f\}$. One clearly has

$$A(z) = \bigcup_{k \in \mathbb{N}} A_k(z), \tag{2.1}$$

where

$$A_k(z) := \{x \in A(z) : \limsup_{w', w'' \to (x,z)} |f(w') - f(w'')| \geqslant (1/k)\}.$$

Let us prove that $m(A(z)) = 0$. Pick $k \in \mathbb{N}$. Since f is Γ^M-continuous, to each $x = (x_1, x_2, \dots, x_n)$ there corresponds $\delta_x > 0$ such that if $\xi \in \prod_{i=1}^n]x_i, x_i + \delta_x[$, then

$$\limsup_{w', w'' \to (\xi, z)} |f(w') - f(w'')| < 1/k.$$

Define $C(x,\delta) := \prod_{i=1}^n [x_i, x_i + \delta]$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\delta > 0$, and set $\mathcal{G} := \{C(x,\delta) : x \in \mathbb{R}^n,\ \delta \in \,]0, \delta_x[\}$. The family $\mathcal{G}$ covers $\mathbb{R}^n$ in the sense of Vitali [**20**, p. 109]. Hence, by Vitali's Covering Theorem, there exist $C \subseteq \mathbb{R}^n$ and a sequence $\{C_j\}$ of sets in $\mathcal{G}$, no two of which have common points, satisfying $m(C) = 0$, $\mathbb{R}^n = (\cup_{j\in\mathbb{N}} C_j) \cup C$. Taking into account that

$$A_k(z) \subseteq \Bigl(\bigcup_{j\in\mathbb{N}} \partial C_j\Bigr) \cup C,$$

we get $m(A_k(z)) = 0$ for all $k \in \mathbb{N}$. Consequently, owing to (2.1), $m(A(z)) = 0$. This implies

$$m(D_f) = \int_{\mathbb{R}} m(A(z))\,\mathrm{d}z = 0,$$

against the condition $m(D_f) > 0$. □

3. Existence theorems

Henceforth we suppose $n \geqslant 3$ and denote by p any real number greater than n. Concerning the function spaces we shall use, the notations are standard; so we refer for instance to [**6**, **12**].

Let $\mathcal{L}$ be the linear, second-order, elliptic differential operator defined by

$$\mathcal{L}u := -\Delta u + u.$$

From [**21**, Proposition 4.3] it follows that $\mathcal{L}$ is a one-to-one operator of $W^{2,p}(\mathbb{R}^n)$ onto $L^p(\mathbb{R}^n)$, and there exists a constant c, depending only on n and p, fulfilling

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} \leqslant c\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

for every $u \in W^{2,p}(\mathbb{R}^n)$. Set

$$\gamma = \sup\left\{\frac{\|u\|_{W^{2,p}(\mathbb{R}^n)}}{\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)}} : u \in W^{2,p}(\mathbb{R}^n),\ u \neq 0\right\}. \tag{3.2}$$

Lemma 3.1. *Let $u \in W^{2,p}(\mathbb{R}^n)$. Then*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{\gamma p}{p-n}\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)}. \tag{3.3}$$

Moreover, for every $x', x'' \in \mathbb{R}^n$, one has

$$|u(x') - u(x'')| \leqslant \frac{\gamma p}{p-n}\|\mathcal{L}u\|_{L^p(\mathbb{R}^n)}\|x' - x''\|. \tag{3.4}$$

Proof. We first note that if $v \in W^{1,p}(\mathbb{R}^n)$ then

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{p}{p-n}\|v\|_{W^{1,p}(\mathbb{R}^n)}, \tag{3.5}$$

as the proof of [**6**, Theorem IX.12] shows. Therefore, inequality (3.3) is a simple consequence of (3.2). Next, pick $x', x'' \in \mathbb{R}^n$. By Corollary IX.13 in [**6**] we have

$$|u(x') - u(x'')| \leqslant \Bigl(\sum_{i=1}^{n} \|u_{x_i}\|_{L^\infty(\mathbb{R}^n)}\Bigr)\|x' - x''\|,$$

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while gathering (3.5) and (3.2) yields

$$\sum_{i=1}^{n} \|u_{x_i}\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{\gamma p}{p-n} \|\mathcal{L}u\|_{L^p(\mathbb{R}^n)},$$

which completes the proof. □

To shorten notation, let us write

$$\beta = \frac{\gamma p}{p-n}.$$

We are now in a position to formulate the main result of this paper.

Theorem 3.2. *Suppose $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ has the following properties.*

(a_1) *There exist $r > 0$ and $\mu \in L^p(\mathbb{R}^n)$ satisfying $\sup_{|z|<\beta r} |f(x,z)| \leqslant \mu(x)$ for all $x \in \mathbb{R}^n$ as well as $\|\mu\|_{L^p(\mathbb{R}^n)} < r$.*

(a_2) *The function f is Γ^M-continuous, where $M \geqslant \beta r$.*

Then the equation $\mathcal{L}u = f(x,u)$, $x \in \mathbb{R}^n$ has at least one solution $u \in W^{2,p}(\mathbb{R}^n)$ with $|\mathcal{L}u(x)| \leqslant \mu(x)$ almost everywhere in $\mathbb{R}^n$.

Proof. Define, for $(x,z) \in \mathbb{R}^n \times]-\beta r, \beta r[$,

$$F(x,z) = \bigcap_{\delta>0} \overline{\mathrm{co}}(\{f(\xi,\zeta) : \|\xi - x\| < \delta,\ |\zeta - z| < \delta\}).$$

Obviously, $F(x,z)$ is non-empty, because $f(x,z) \in F(x,z)$, convex, and closed. Moreover, owing to hypothesis (a_1), the inclusion

$$F(x,z) \subseteq [-\mu(x), \mu(x)], \quad (x,z) \in \mathbb{R}^n \times]-\beta r, \beta r[, \tag{3.6}$$

holds. Hence, by Lemma 2.3, we obtain a set $A \subseteq \mathbb{R}^n$ together with a sequence $\{A_k\}$ of compact subsets of $\mathbb{R}^n$, no two of which have common points, so that

$$m(A) = 0, \qquad \mathbb{R}^n = \left(\bigcup_{k\in\mathbb{N}} A_k\right) \cup A, \tag{3.7}$$

and the multi-function

$$F_k := F|_{A_k \times]-\beta r, \beta r[}$$

has a bounded range for each $k \in \mathbb{N}$.

Now, pick $(x,z) \in A_k \times]-\beta r, \beta r[$, $y \in \mathbb{R}$, and choose two sequences $\{(x_h, z_h)\} \subseteq A_k \times]-\beta r, \beta r[$, $\{y_h\} \subseteq \mathbb{R}$ fulfilling the conditions:

$$y_h \in F_k(x_h, z_h), \quad h \in \mathbb{N}; \qquad \lim_{h\to\infty} (x_h, z_h) = (x,z); \qquad \lim_{h\to\infty} y_h = y.$$

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Let us prove that $y \in F_k(x,z)$. If, on the contrary, $y \notin F_k(x,z)$, then

$$y \notin \overline{\text{co}}(\{f(\xi,\zeta) : \|\xi - x\| < \delta,\ |\zeta - z| < \delta\}) \tag{3.8}$$

for some $\delta > 0$. Since

$$\{f(\xi,\zeta) : \|\xi - x_h\| < \delta/2,\ |\zeta - z_h| < \delta/2\} \subseteq \{f(\xi,\zeta) : \|\xi - x\| < \delta,\ |\zeta - z| < \delta\}$$

whenever h is sufficiently large and, moreover,

$$y_h \in \overline{\text{co}}(\{f(\xi,\zeta) : \|\xi - x_h\| < \delta/2,\ |\zeta - z_h| < \delta/2\})$$

for all $h \in \mathbb{N}$, we get

$$y \in \overline{\text{co}}(\{f(\xi,\zeta) : \|\xi - x\| < \delta,\ |\zeta - z| < \delta\}),$$

which contradicts (3.8). Therefore, the multi-function F_k has a closed graph and a bounded range. Consequently, it is upper semicontinuous. By [**9**, Corollary III.3] we then infer that F_k is $\mathcal{L}(A_k) \otimes \mathcal{B}(]-\beta r, \beta r[)$-measurable.

The preceding arguments, combined with Example 1.3 of [**10**], produce two $\mathcal{L}(A_k) \otimes \mathcal{B}(]-\beta r, \beta r[)$-measurable functions $\varphi_k, \psi_k : A_k \times]-\beta r, \beta r[\to \mathbb{R}$ having the properties:

$$F_k(x,z) = [\varphi_k(x,z), \psi_k(x,z)], \quad \text{in } A_k \times]-\beta r, \beta r[\,; \tag{3.9}$$

φ_k is lower semicontinuous, while ψ_k is upper semicontinuous.

Define

$$V = \{v \in L^p(\mathbb{R}^n) : |v(x)| \leqslant \mu(x) \text{ for almost every } x \in \mathbb{R}^n\}.$$

Evidently, V is a non-empty, convex, weakly compact subset of $L^p(\mathbb{R}^n)$. Furthermore, on account of Lemma 3.1 and assumption (a_1), for any $v \in V$ one has

$$|\mathcal{L}^{-1}(v)(x)| \leqslant \beta \|v\|_{L^p(\mathbb{R}^n)} < \beta r, \quad x \in \mathbb{R}^n.$$

Hence, it makes sense to write

$$\Phi(v) = \{w \in L^p(\mathbb{R}^n) : w(x) \in F(x, \mathcal{L}^{-1}(v)(x)) \text{ almost everywhere in } \mathbb{R}^n\}, \quad v \in V.$$

We claim that $\Phi(v)$ is non-empty. Indeed, the multi-function $x \mapsto F(x, \mathcal{L}^{-1}(v)(x))$, $x \in \mathbb{R}^n$, is measurable because, owing to [**22**, Theorem 1], the same holds for $x \mapsto F_k(x, \mathcal{L}^{-1}(v)(x))$, $x \in A_k$, $k \in \mathbb{N}$. So, by Theorem III.6 of [**9**], there is a measurable function $w : \mathbb{R}^n \to \mathbb{R}$ such that $w(x) \in F(x, \mathcal{L}^{-1}(v)(x))$ for almost every $x \in \mathbb{R}^n$. Inclusion (3.6) leads to $w \in \Phi(v)$, that is $\Phi(v) \neq \emptyset$.

Clearly, $\Phi(v)$ is also convex. Moreover, the multi-function Φ has a weakly sequentially closed graph. To see this, pick $v, w \in V$ and choose two sequences $\{v_h\}$, $\{w_h\}$ in V fulfilling $w_h \in \Phi(v_h)$ for all $h \in \mathbb{N}$ as well as $\lim_{h\to\infty} v_h = v$, $\lim_{h\to\infty} w_h = w$ weakly in $L^p(\mathbb{R}^n)$. Identity (3.9) implies

$$\varphi_k(x, \mathcal{L}^{-1}(v_h)(x)) \leqslant w_h(x) \leqslant \psi_k(x, \mathcal{L}^{-1}(v_h)(x))$$

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almost everywhere in A_k, while the weak convergence of $\{w_h\}$ to w produces

$$\liminf_{h\to\infty} \int_E [\varphi_k(x, \mathcal{L}^{-1}(v_h)(x)) - w(x)]\, dx \leqslant \liminf_{h\to\infty} \int_E [w_h(x) - w(x)]\, dx = 0$$

for any non-empty measurable set $E \subseteq A_k$. Bearing in mind Fatou's Lemma, we get

$$\liminf_{h\to\infty} \varphi_k(x, \mathcal{L}^{-1}(v_h)(x)) - w(x) \leqslant 0, \quad \text{almost everywhere in } A_k. \tag{3.10}$$

Since, by (3.1), $\mathcal{L}^{-1}$ is a continuous linear operator from $L^p(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$, the sequence $\{\mathcal{L}^{-1}(v_h)\}$ converges weakly to $\mathcal{L}^{-1}(v)$ in $W^{2,p}(\mathbb{R}^n)$. Taking a subsequence if necessary, we may suppose that $\lim_{h\to\infty} \mathcal{L}^{-1}(v_h)(x) = \mathcal{L}^{-1}(v)(x)$ at almost all points $x \in \mathbb{R}^n$; vide [**6**, Remark 7, p. 153]. Therefore, due to (3.10) and the lower semicontinuity of the function $z \mapsto \varphi_k(x, z)$, $x \in A_k$,

$$\varphi_k(x, \mathcal{L}^{-1}(v)(x)) - w(x) \leqslant 0, \quad \text{almost everywhere in } A_k.$$

The same arguments, with ψ_k in place of φ_k, yield

$$w(x) - \psi_k(x, \mathcal{L}^{-1}(v)(x)) \leqslant 0.$$

Consequently, in view of (3.9), $w(x) \in F_k(x, \mathcal{L}^{-1}(v)(x))$ for almost all $x \in A_k$ and each $k \in \mathbb{N}$. This implies $w(x) \in F(x, \mathcal{L}^{-1}(v)(x))$ almost everywhere in $\mathbb{R}^n$, namely $w \in \Phi(v)$.

We have thus proved that all the hypotheses of Theorem 2.1 hold. So, there exists a function $v \in V$ complying with $v \in \Phi(v)$. The function $u = \mathcal{L}^{-1}(v)$ lies in $W^{2,p}(\mathbb{R}^n)$ and satisfies $|\mathcal{L}u(x)| \leqslant \mu(x)$ besides

$$\mathcal{L}u(x) \in F(x, u(x)), \quad \text{for almost all } x \in \mathbb{R}^n. \tag{3.11}$$

Fix $k \in \mathbb{N}$ and denote by B_k the set of points $x \in A_k$ with the following properties.

(i) $\mathcal{L}u(x) \in F(x, u(x))$.

(ii) There is a sequence $\{x_h\} \subseteq A_k$ such that $x_{h+1} < x_h$, $\mathcal{L}u(x_h) \in F(x_h, u(x_h))$ for all $h \in \mathbb{N}$, $\lim_{h\to\infty} x_h = x$, $\lim_{h\to\infty} \mathcal{L}u(x_h) = \mathcal{L}u(x)$.

If $B'_k = \{x \in A_k : \text{(i) holds}\}$, then, by (3.11),

$$m(B'_k) = m(A_k). \tag{3.12}$$

Making use of Lusin's Theorem, to each $\varepsilon > 0$ there corresponds a measurable set $B'_{k,\varepsilon} \subseteq B'_k$ such that

$$m(B'_{k,\varepsilon}) > m(B'_k) - \varepsilon, \tag{3.13}$$

and $\mathcal{L}u|_{B'_{k,\varepsilon}}$ is continuous. Lemma 2.2 gives a measurable subset $B^*_{k,\varepsilon}$ of $B'_{k,\varepsilon}$ fulfilling the conditions: $m(B^*_{k,\varepsilon}) = m(B'_{k,\varepsilon})$; $B^*_{k,\varepsilon} \subseteq B_k$. Hence, owing to (3.12) and (3.13), $m(B_k) > m(A_k) - \varepsilon$. As ε was arbitrary, we actually have

$$m(A_k) = m(B_k), \quad k \in \mathbb{N}.$$

On account of the above identity and (3.7), the proof is completed by showing that $\mathcal{L}u(x) = f(x, u(x))$ for every $x \in B_k$, $k \in \mathbb{N}$. Assume, on the contrary, that there exist $k \in \mathbb{N}$, $x \in B_k$ complying with $|\mathcal{L}u(x) - f(x, u(x))| > 0$, and write

$$\varepsilon := |\mathcal{L}u(x) - f(x, u(x))|. \tag{3.14}$$

Hypothesis (a_2) yields $\delta > 0$ such that $|f(\xi, \zeta) - f(x, u(x))| < \varepsilon/2$ whenever $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}$, $\|\xi - x\| < \delta$, $(\xi, \zeta) - (x, z) \in \Gamma^M$. Pick the sequence $\{x_h\}$ given by (ii), and choose $h \in \mathbb{N}$ for which $\|x_h - x\| < \delta$,

$$|\mathcal{L}u(x_h) - \mathcal{L}u(x)| < \tfrac{1}{2}\varepsilon. \tag{3.15}$$

From Lemma 3.1 and assumptions (a_1) and (a_2), we infer $|u(x_h) - u(x)| < M\|x_h - x\|$. Therefore,

$$\mathcal{L}u(x_h) \in F(x_h, u(x_h)) \subseteq \{y \in \mathbb{R} : |y - f(x, u(x))| \leqslant \varepsilon/2\}.$$

Owing to (3.15) this implies $|\mathcal{L}u(x) - f(x, u(x))| < \varepsilon$, which contradicts (3.14). □

Remark 3.3. It is worth noting that, since $p \in]n, +\infty[$, the solution u we find satisfies the condition

$$\lim_{\|x\| \to \infty} u(x) = 0.$$

Moreover, by virtue of [**21**, Proposition 4.3], one has

$$\operatorname{ess\,inf}_{\|x\| \leqslant R} u(x) > 0, \quad \text{for each } R > 0$$

every time that the function f turns out non-negative and not identically zero in $\mathbb{R}^n \times \mathbb{R}$.

Remark 3.4. Hypothesis (a_1) requires that f depends on $x \in \mathbb{R}^n$. Nevertheless, several natural classes of possibly discontinuous nonlinearities fulfil (a_1) and (a_2). Here is an example.

Example 3.5. Pick $f_1 \in L^p(\mathbb{R}^n)$, $\eta \in C^0(\mathbb{R}^n)$, $a, b \in C^0(\mathbb{R})$, and define, for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}$,

$$f_2(x, z) := \begin{cases} a(z), & \text{if } (x, z) \in A_\eta, \\ b(z), & \text{otherwise,} \end{cases}$$

where $A_\eta := \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : z \leqslant \eta(x)\}$. Assume that:

(a_1') there exists $r > 0$ satisfying $\|f_1\|_{L^p(\mathbb{R}^n)} \sup_{|z| < \beta r}(|a(z)| + |b(z)|) < r$;

(a_2') $\lim_{h \to \infty} f_1(x_h) = f_1(x)$ as soon as $x \leqslant x_h$, $h \in \mathbb{N}$, and $\lim_{h \to \infty} x_h = x$;

(a_3') there is $M \geqslant \beta r$ such that to each $w \in \partial A_\eta$ there corresponds $\delta > 0$ for which $w + B(0, \delta) \cap \Gamma^M \subseteq A_\eta$.

Then the function $f(x, z) := f_1(x) f_2(x, z)$, $(x, z) \in \mathbb{R}^n \times \mathbb{R}$, complies with ($a_1$) and ($a_2$) of Theorem 3.2.

Let us next present two simple applications of the above result. The first of them represents an existence theorem for a class of elliptic differential inclusions on the whole space and with lower semicontinuous right-hand sides.

Theorem 3.6. *Suppose G is a closed-valued lower semicontinuous multi-function from $\mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}$ and there are $r > 0$, $\mu : \mathbb{R}^n \to \mathbb{R}$ lower semicontinuous satisfying $\|\mu\|_{L^p(\mathbb{R}^n)} < r$ as well as*

$$\sup_{|z|\leqslant \beta r} d(0, G(x,z)) < \mu(x), \quad \text{for all } x \in \mathbb{R}^n. \tag{3.16}$$

Then there exists a function $u \in W^{2,p}(\mathbb{R}^n)$ such that $\mathcal{L}u(x) \in G(x, u(x))$ almost everywhere in $\mathbb{R}^n$.

Proof. Define, for $(x,z) \in \mathbb{R}^n \times \mathbb{R}$,

$$\hat{G}(x,z) := \begin{cases} \overline{G(x,z) \cap B(0,\mu(x))}, & \text{if } (x,z) \in \mathbb{R}^n \times [-\beta r, \beta r], \\ \overline{B(0,\mu(x))}, & \text{otherwise}. \end{cases}$$

Obviously, because of (3.16), the set $\hat{G}(x,z)$ is non-empty and compact. Since, by Proposition 5 in [**1**, p. 44], the multi-function $(x,z) \mapsto G(x,z) \cap B(0,\mu(x))$, $(x,z) \in \mathbb{R}^n \times [-\beta r, \beta r]$, is lower semicontinuous, a standard argument (see, for example, [**10**, § 2]) ensures that $\hat{G} : \mathbb{R}^n \times \mathbb{R} \to 2^{\mathbb{R}}$ enjoys the same property.

Now, fix any $M \geqslant \beta r$. Applying Theorem 1 of [**4**] to $\hat{G}$ yields a Γ^M-continuous function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which fulfils the following conditions:

$$g(x,z) \in G(x,z), \quad \text{in } \mathbb{R}^n \times [-\beta r, \beta r]; \qquad \sup_{|z|<\beta r} |g(x,z)| \leqslant \mu(x), \quad \text{for all } x \in \mathbb{R}^n.$$

From Theorem 3.2 we then obtain a solution $u \in W^{2,p}(\mathbb{R}^n)$ of the equation $\mathcal{L}u = g(x,u)$, $x \in \mathbb{R}^n$, such that $|\mathcal{L}u(x)| \leqslant \mu(x)$ almost everywhere in $\mathbb{R}^n$. Owing to Lemma 3.1, the preceding inequality forces $\|u\|_{L^\infty(\mathbb{R}^n)} < \beta r$. This immediately leads to the desired conclusion. □

The second application we wish to point out is concerned with implicit elliptic equations of the type $\psi(\mathcal{L}u) = \varphi(x,u)$ on the whole space.

Theorem 3.7. *Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuous, let Y be a closed real interval, and let $\psi : Y \to \mathbb{R}$ be continuous. Assume that $\varphi(\mathbb{R}^n \times \mathbb{R}) \subseteq \psi(Y)$, for each $\sigma \in \operatorname{int}(\psi(Y))$ the set $\psi^{-1}(\sigma)$ has empty interior, and, moreover, there exist $r > 0$, $\mu : \mathbb{R}^n \to \mathbb{R}$ lower semicontinuous satisfying $\|\mu\|_{L^p(\mathbb{R}^n)} < r$ besides*

$$\psi^{-1}(\varphi(x,z)) \subseteq B(0,\mu(x)), \quad \text{for all } (x,z) \in \mathbb{R}^n \times [-\beta r, \beta r].$$

Then the equation $\psi(\mathcal{L}u) = \varphi(x,u)$, $x \in \mathbb{R}^n$, has at least one solution u belonging to $W^{2,p}(\mathbb{R}^n)$.

Proof. Theorem 2.4 in [**19**] provides a set $Y^* \subseteq Y$ such that $\sigma \mapsto \psi^{-1}(\sigma) \cap Y^*$, $\sigma \in \psi(Y)$, takes non-empty closed values and is lower semicontinuous.

Write, for $(x, z) \in \mathbb{R}^n \times \mathbb{R}$,

$$G(x,z) := \begin{cases} \psi^{-1}(\varphi(x,z)) \cap Y^*, & \text{if } (x,z) \in \mathbb{R}^n \times [\beta r, \beta r], \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

The multi-function $G : \mathbb{R}^n \times \mathbb{R} \to 2^{\mathbb{R}}$ so defined turns out clearly lower semicontinuous. Due to the assumptions, the inclusion

$$G(x,z) \subseteq B(0, \mu(x)), \quad (x,z) \in \mathbb{R}^n \times [-\beta r, \beta r] \tag{3.17}$$

holds. Therefore, by Theorem 3.6, there exists a function $u \in W^{2,p}(\mathbb{R}^n)$ complying with $\mathcal{L}u(x) \in G(x, u(x))$ almost everywhere in $\mathbb{R}^n$. Since Lemma 3.1 and (3.17) yield

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant \beta \|\mu\|_{L^p(\mathbb{R}^n)} < \beta r,$$

we actually have $\mathcal{L}u(x) \in \psi^{-1}(\varphi(x, u(x))$ for almost all $x \in \mathbb{R}^n$, which completes the proof. □

Remark 3.8. In the case of bounded domains, results somewhat similar to Theorems 3.6 and 3.7 have previously been established, inside a different abstract framework, by the second author [**15**].

For the sake of completeness we finally consider the situation when $\mathbb{R}^n$ and $\mathcal{L}$ are, respectively, replaced with a bounded convex domain $\Omega \subseteq \mathbb{R}^n$ having a boundary of class $C^{1,1}$ and the more general strictly elliptic operator

$$\mathcal{M}u = -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u,$$

where $a_{ij} \in C^0(\bar{\Omega})$ and $a_{ij} = a_{ji}$, $i,j = 1,2,\ldots,n$; $b_i \in L^\infty(\Omega)$, $i = 1,2,\ldots,n$; $c \in L^\infty(\Omega)$ and $c(x) \geqslant 0$ almost everywhere in Ω.

Theorem 9.15 of [**12**] ensures that $\mathcal{M} : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ is bijective, while Lemma 9.17 in [**12**] gives a constant c fulfilling

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c \|\mathcal{M}u\|_{L^p(\Omega)}$$

for every $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Set

$$\gamma_0 = \sup\left\{ \frac{\|u\|_{W^{2,p}(\Omega)}}{\|\mathcal{M}u\|_{L^p(\Omega)}} : u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),\ u \neq 0 \right\},$$

and write $d(\Omega)$ for the diameter of Ω. Reasoning as in the proof of Lemma 3.1, with the inequality (vide [**7**, Theorem 1])

$$\|v\|_{L^\infty(\Omega)} \leqslant \frac{1}{[m(\Omega)]^{1/p}} \max\left\{1, \frac{d(\Omega)}{n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p}\right\} \|v\|_{W^{1,p}(\Omega)}, \quad v \in W^{1,p}(\Omega),$$

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in place of (3.5), we achieve the lemma below, where

$$\beta_0 = \frac{\gamma_0}{[m(\Omega)]^{1/p}} \max\left\{1, \frac{d(\Omega)}{n^{1/p}} \left(\frac{p-1}{p-n}\right)^{1-1/p}\right\}.$$

Lemma 3.9. *If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, then*

$$\|u\|_{L^\infty(\Omega)} \leqslant \beta_0 \|\mathcal{M}u\|_{L^p(\Omega)}.$$

Moreover, for every $x', x'' \in \Omega$, one has

$$|u(x') - u(x'')| \leqslant \beta_0 \|\mathcal{M}u\|_{L^p(\Omega)} \|x' - x''\|.$$

Now, arguments somewhat similar to those employed in establishing Theorem 3.2 produce the following result.

Theorem 3.10. *Suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has the following properties.*

(b_1) *There are $r > 0$ and $\mu \in L^p(\Omega)$ satisfying $\sup_{|z|<\beta r} |f(x,z)| \leqslant \mu(x)$ for all $x \in \Omega$ as well as $\|\mu\|_{L^p(\Omega)} < r$.*

(b_2) *The function f is Γ^M-continuous, with $M \geqslant \beta_0 r$.*

Then there exists a function $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{M}u(x) = f(x, u(x))$ and $|\mathcal{M}u(x)| \leqslant \mu(x)$ almost everywhere in Ω.

Remark 3.11. Because of Lemma 2.6, hypothesis (b_2) clearly forces $m(D_f) = 0$, where

$$D_f := \{(x,z) \in \Omega \times \mathbb{R} : f \text{ is discontinuous at } (x,z)\}.$$

Remark 3.12. Concerning the problem $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\mathcal{M}u = f(x,u)$ in Ω, with highly discontinuous f, existence results have recently been obtained in [**3, 16**] (see also [**17, 18**] and references cited therein) by adopting, instead of (b_2), the following assumptions:

(c_1) $c(x) = 0$ for almost all $x \in \Omega$;

(c_2) there is $\Omega_0 \subseteq \Omega$ such that $m(\Omega_0) = 0$ and the set

$$D_f^* := \bigcup_{x \in \Omega \setminus \Omega_0} \{z \in \mathbb{R} : f(x,\cdot) \text{ is discontinuous at } z\}$$

has measure zero;

(c_3) for almost every $x \in \Omega$ and very $z \in D_f^*$, the condition

$$\lim_{\delta \to 0^+} \inf_{|\zeta - z| \leqslant \delta} f(x,\zeta) \leqslant 0 \leqslant \lim_{\delta \to 0^+} \sup_{|\zeta - z| \leqslant \delta} f(x,\zeta)$$

implies $f(x,z) = 0$.

We note that, in this setting, neither can the simpler hypothesis $m(D_f) = 0$ take the place of (c_2) nor may (c_3) be omitted, as Examples 4.3 of [**17**] and 3.3 of [**16**], respectively, show. Nevertheless, Theorem 3.6 provides a class of functions f, which, in spite of their possibly large set of discontinuities, allow us to dispense with (c_1)–(c_3) when solving the above-mentioned problem.

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