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ON THE EXPONENT OF AN OSCULATORY PACKING

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1. Introduction. Suppose that U is an open set in Euclidean N-space which has a finite volume $|U|$. A complete packing of U is a sequence of disjoint N-spheres $C = \{S_n\}$ which are contained in U and whose total volume equals that of U. In an osculatory packing, the spheres are chosen recursively so that for all n larger than a certain value m, S_n has the largest radius of all spheres contained in $U \backslash (S_1^- \cup \ldots \cup S_{n-1}^-)$ (S^- is the closure of S). An osculatory packing is simple if $m = 1$. If r_n denotes the radius of S_n, the exponent of the packing is defined by:

$$e(C, U) = \sup\{t: \sum r_n^t = \infty\}.$$

This quantity is of considerable interest since it measures the effectiveness of the packing of U by C.

In [**7**], Melzak, who introduced $e(C, U)$, gave examples of complete packings of the unit N-sphere B_N for which $e(C, B_N) = N$, which is clearly the largest value possible. For osculatory packings, there are examples of sets U for which $e(C_0, U) = N$, since one may take U to be a disjoint union of spheres with radii $\{r_n\}$ for which $\sum r_n^N < \infty$, but $\sum r_n^t = \infty$ for $t < N$. There is a less trivial example in [**1**] of a set $U \subset E_2$ with $e(C_0, U) = 2$.

In this paper, we give conditions on the boundary of the set U which will ensure that $e(C_0, U) < N$, for all osculatory packings of U. In fact, there are universal constants $\beta_N < N$ so that for most "reasonable" sets U, one has $e(C_0, U) \leqq \beta_N$. More precisely, if one assumes that the volume of the set $U(\delta) = \{x \in U: \operatorname{dist}(x, \partial U) \leqq \delta\}$ is $O(\delta^\gamma)$ as $\delta \to 0+$, for some $\gamma > 0$, then $e(C_0, U) \leqq \max(\beta_N, N - \gamma)$ (Theorem 1). For convex sets, or sets with smooth boundaries of finite surface area, one has $\gamma = 1$, and in this case we have $e(C_0, U) \leqq \beta_N$. Our result is complementary to the result of Larman [**5**], that if $U = I_N$, the unit N-cube, then $e(C, I_N) > N - 1 + 0.03$ for any complete packing C of I_N.

If T_2 is a curvilinear triangle bounded by mutually tangent circular arcs, and if C_0 is the simple osculatory packing of T_2, Melzak [**7**] showed that $1.035 < e(C_0, T_2) < 1.999971$, and these bounds were improved by myself [**2**] to $1.28467 < e(C_0, T_2) < 1.93113$. The set T_2 satisfies the conditions of our theorem, and $\beta_2 = (2 + \sqrt{2})/2 = 1.707\ldots$ so that we obtain an improvement on the upper bound for $e(C_0, T_2)$ as a corollary. A special argument given in § 6 allows us to prove that $e(C_0, T_2) < (9 + \sqrt{41})/10 = 1.5403\ldots$.

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We should mention that we have recently developed an algorithm for computing $e(C_0, T_2)$ to arbitrary accuracy. Details of this will appear in the near future.

(*Added in proof.* Using this algorithm, we have shown that $e(C_0, T_2) < 1.3500$.)

As further corollaries of our theorem, we obtain an estimate of the volume of $R_n = U\backslash(S_1^- \cup \ldots \cup S_n^-)$, and of the Hausdorff dimension of the residual set $U\backslash\bigcup\{S_k: k \geqq 1\}$. The volume of R_n is $O(r_n^{N-\beta_N-\epsilon})$ for any $\epsilon > 0$. This result is complementary to a result of Larman [**6**] who showed that if C is a packing of I_N with $r_1 \geqq r_2 \geqq \ldots$, then $|R_n| \geqq K_N r_n^s$, where K_N and s $(= 0.97)$ are constants.

Our method of proof is rather non-geometrical in nature. We first establish a basic inequality involving the sequence $\{r_n\}$. We then develop some integral inequalities which allow us to deduce the behaviour of $\{r_n\}$ from the basic inequality.

2. The basic inequality. To begin with, we must consider a certain function associated with an open set U in N-space. Given such a set U, with boundary ∂U, we write

$$U(\delta) = \{x \in U: \operatorname{dist}(x, \partial U) \leqq \delta\}.$$

We denote the volume of $U(\delta)$ by $V(U, \delta)$. The behaviour of the function $V(U, \delta)$ as $\delta \to 0$ will be of importance in our later deductions. Note that if $U = B_N$ and if ω_N is the volume of B_N, then

$$V(B_N, \delta) = \omega_N(1 - (1 - \delta)^N) = O(\delta) \quad \text{as } \delta \to 0.$$

If U is a bounded convex set, then $V(U, \delta) = O(\delta)$ as $\delta \to 0$. (See [**3**, p. 88]). Finally, it is quite easy to see that if the boundary of U consists of a finite union of compact C^2-surfaces, then $V(U, \delta) = O(\delta)$ as $\delta \to 0$. Thus we can regard this type of behaviour as typical.

LEMMA 1. *Let U be an open set in Euclidean N-space with finite volume. Let $C_0 = \{S_n\}$ be a simple osculatory packing of U, and let r_n be the radius of S_n. Then, for all $n \geqq 0$,*

$$\sum_{k=n+1}^{\infty} r_k^N \leqq \sum_{k=1}^{n} \{(r_k + r_{n+1})^N - r_k^N\} + V(U, r_{n+1})\omega_N^{-1}. \tag{1}$$

Proof. By definition of a simple osculatory packing, if $x \in R_n$, then $\operatorname{dist}(x, \partial R_n) \leqq r_{n+1}$. Hence, if S_k^* is the closed sphere concentric with S_k and with radius $r_k + r_{n+1}$, one has

$$R_n \subset U(r_{n+1}) \cup \bigcup\{S_k^*\backslash S_k: k = 1, 2, \ldots, n\}. \tag{2}$$

An osculatory packing is complete (see [**1**]), and so $|R_n| = \sum(|S_k|: k \geqq n + 1)$. If we now compute the volumes of the sets in (2), we obtain (1).

Our deductions concerning $e(C_0, U)$ will be made on the basis of inequality (1). We first motivate what is to follow by considering what (1) would imply in the case $N = 2$, if $r_n = An^{-\alpha}$ for some constants A and α with $\frac{1}{2} < \alpha < 1$. If $V(U, r_{n+1}) = O(r_{n+1})$ as $n \to \infty$ we have

$$\sum_{k=n+1}^{\infty} r_k^2 \leqq 2r_{n+1} \sum_{k=1}^{n} r_k + nr_{n+1}^2 + O(n^{-\alpha})$$

so that

$$\frac{n^{1-2\alpha}}{2\alpha - 1} \leqq \frac{2n^{1-2\alpha}}{1-\alpha} + n^{1-2\alpha} + o(n^{1-2\alpha})$$

and hence

$$1 - \alpha \leqq (2\alpha - 1)(3 - \alpha)$$

which implies that $\alpha > 2 - \sqrt{2}$. Thus $\sum r_n^t < \infty$ provided $t > (2 + \sqrt{2})/2$. In order to arrive at this conclusion without the assumption that $r_n = An^{-\alpha}$, we shall require an integral inequality (Lemma 3) which is a consequence of an inequality due to Hardy [**4**, § 330].

3. Some integral inequalities.

LEMMA 2 (Hardy [**4**]). *Let h be a non-negative measurable function and let $s > 1$, $r > 1$ be real numbers. Then*

$$\int_0^\infty x^{-r}\left(\int_0^x h(t)\,dt\right)^s dx < \left(\frac{s}{r-1}\right)^s \int_0^\infty x^{-r+s} h(x)^s\,dx$$

unless $h \equiv 0$. The constant is best possible.

Proof. See [**4**, p. 245].

LEMMA 3. *Let f be a non-negative measurable function. Let $a \geqq 0$, $\alpha > 0$, $q \geqq 0$, $p \geqq 0$ be real numbers with $p + q > 1$ and $q\alpha < 1$. Then*

$$(3) \qquad \int_a^\infty x^{(p+q)\alpha-2} f(x)^p \left(\int_a^x f(t)^q\,dt\right) dx \leqq \frac{1}{1-q\alpha} \int_a^\infty x^{(p+q)\alpha-1} f(x)^{p+q}\,dx.$$

The constant is best possible.

Proof. First let $a = 0$, and

$$A = \int_0^\infty x^{(p+q)\alpha-1} f(x)^{p+q}\,dx.$$

Assume also that $p > 0$, $q > 0$. Then, by Hölder's inequality with exponents $r = (p + q)/p$ and $r' = (p + q)/q$, one has

$$(4) \qquad \int_0^\infty x^{(p+q)\alpha-2} f(x)^p \left(\int_0^x f(t)^q\,dt\right) dx$$
$$\leqq \left\{\int_0^\infty x^{-1}(x^{p\alpha} f(x)^p)^r\,dx\right\}^{1/r} \left\{\int_0^\infty x^{-1}\left(x^{q\alpha-1}\int_0^x f(t)^q\,dt\right)^{r'} dx\right\}^{1/r'}.$$

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The first factor in the right member of (4) is $A^{1/r}$. Applying Lemma 2 to the second factor, with $h = f^q$, we find it to be less than

$$[r'/(r' - (p + q)\alpha)]A^{1/r'}.$$

Simplifying the constant yields inequality (3). For $q = 0$, (3) is an equality. For $p = 0$, we can apply Lemma 2 directly with $s = 1$, and $h = f^q$. To obtain (3) for $a > 0$, apply (3) with $a = 0$ to the function which is zero for $0 \leqq x < a$, and equal to $f(x)$ for $x \geqq a$.

To show that the constant is best possible, let $f(x) = x^{-\alpha}$ for $n^{-1} \leqq x \leqq n$, and zero otherwise, and let $n \to \infty$.

4. The main result.

LEMMA 4. *Let $N \geqq 2$ be an integer, and let β_N denote the unique root of the following equation with $N - 1 < \beta_N < N$:*

$$K(x) = \sum_{j=0}^{N} \binom{N}{j} \frac{1}{x - j} = 0.$$

Suppose that f is a non-negative measurable function which satisfies the following inequality for $x \geqq 1$, where c and γ are constants, and $0 < \gamma < N$:

$$\int_x^\infty f(t)^N\,dt \leqq \sum_{j=0}^{N-1} \binom{N}{j} f(x)^{N-j} \int_1^x f(t)^j\,dt + cf^\gamma(x). \tag{5}$$

Suppose that $f(1) < \infty$. Then for $\alpha < \min(\beta_N^{-1}, (N - \gamma)^{-1})$, one has

$$\int_1^\infty t^{N\alpha-1} f(t)^N\,dt < \infty, \tag{6}$$

and, for any real $p > \max(\beta_N, N - \gamma)$, one has

$$\int_1^\infty f(t)^p\,dt < \infty.$$

Proof. It is clear that $K(x) = 0$ has a unique root in each interval $j < x < j + 1$ for $j = 0, 1, \ldots, N - 1$ since $K(x)$ is strictly decreasing except at the points $0, 1, \ldots, N$. For the same reason, if $\beta_N < x < N$, then $K(x) < 0$.

Suppose now that $f \geqq 0$ satisfies (5) and that α satisfies

$$\min(\beta_N^{-1}, (N - \gamma)^{-1}) > \alpha > N^{-1}.$$

Setting $x = 1$ in (5), we see that

$$\int_1^\infty f(x)^N\,dx < \infty.$$

Now, let $a > 1$ be arbitrary but finite and let $g(x) = f(x)$ for $1 \leqq x \leqq a$, $g(x) = 0$ for $x > a$. Then g also satisfies (5) (trivially for $x > a$), and

$$\int_1^\infty t^{N\alpha-1} g(t)^N \, dt < \infty,$$

since

$$\int_1^a g(t)^N \, dt < \infty,$$

and $t^{N\alpha-1}$ is bounded on $[1, a]$.

Multiply (5) through by $x^{N\alpha-2}$, and integrate from 1 to ∞. We observe that, by Fubini's theorem,

$$\int_1^\infty x^{N\alpha-2}\left(\int_x^\infty g(t)^N \, dt\right) dx = (N\alpha - 1)^{-1} \int_1^\infty g(t)^N (t^{N\alpha-1} - 1) \, dt.$$

Also, by Lemma 3, for $j = 0, 1, \ldots, N - 1$,

$$\int_1^\infty x^{N\alpha-2} g(x)^{N-j}\left(\int_1^x g(t)^j \, dt\right) dx \leqq (1 - j\alpha)^{-1} \int_1^\infty x^{N\alpha-1} g(x)^N \, dx.$$

And, using Hölder's inequality, with weight $x^{N\alpha-1}$ and exponents N/γ and $N/(N - \gamma)$, we may treat the error term as follows:

$$\int_1^\infty x^{N\alpha-2} g(x)^\gamma \, dx \leqq \left\{\int_1^\infty x^{N\alpha-1} g(x)^N \, dx\right\}^{\gamma/N} \left\{\int_1^\infty x^{N\alpha-1} x^{-N/(N-\gamma)} \, dx\right\}^{(N-\gamma)/N}.$$

Thus, inequality (5) for g implies, since $\alpha < (N - \gamma)^{-1}$, that

$$\begin{aligned} (7) \quad & \left\{(N\alpha - 1)^{-1} - \sum_{j=0}^{N-1} \binom{N}{j} (1 - j\alpha)^{-1}\right\} \int_1^\infty x^{N\alpha-1} g(x)^N \, dx \\ & \leqq (N\alpha - 1)^{-1} \int_1^\infty g(t)^N \, dt + c(N(N - \gamma)^{-1} - N\alpha)^{-(N-\gamma)/N} \\ & \qquad \times \left\{\int_1^\infty x^{N\alpha-1} g(x)^N \, dx\right\}^{\gamma/N}. \end{aligned}$$

The constant in the left member of (7) is $-\alpha^{-1}K(\alpha^{-1})$ which is greater than zero since $\beta_N < \alpha^{-1} < N$. Dividing (7) through by

$$-\alpha^{-1}K(\alpha^{-1})\left\{\int_1^\infty x^{N\alpha-1} g(x)^N \, dx\right\}^{\gamma/N},$$

using

$$\int_1^\infty g(t)^N \, dt \leqq \int_1^\infty t^{N\alpha-1} g(t)^N \, dt,$$

and letting $a \to \infty$, we have

$$\begin{aligned} (8) \quad & \left\{\int_1^\infty t^{N\alpha-1} f(t)^N \, dt\right\}^{(N-\gamma)/N} \\ & \leqq \alpha(-K(\alpha^{-1}))^{-1}\left[(N\alpha - 1)^{-1}\left\{\int_1^\infty f(t)^N \, dt\right\}^{(N-\gamma)/N} + b\right], \end{aligned}$$

where $b = c(N(N-\gamma)^{-1} - N\alpha)^{-(N-\gamma)/N}$. From (8), we see that (6) holds for $N^{-1} < \alpha < \min(\beta_N^{-1}, (N-\gamma)^{-1})$, and hence (6) holds for

$$\alpha < \min(\beta_N^{-1}, (N-\gamma)^{-1}).$$

Now applying Hölder's inequality, we have

$$(9) \quad \int_1^\infty f(t)^p\,dt \leqq \left\{\int_1^\infty t^{N\alpha-1} f(t)^N\,dt\right\}^{p/N} \left\{\int_1^\infty t^{-(N\alpha-1)p/(N-p)}\,dt\right\}^{(N-p)/N} < \infty,$$

provided $(N\alpha - 1)p > N - p$, or in other words, that $p > \alpha^{-1}$. This is valid for any $\alpha^{-1} > \max(\beta_N, N-\gamma)$ so that

$$\int_1^\infty f(t)^p\,dt < \infty \quad \text{for any } p > \max(\beta_N, N-\gamma).$$

THEOREM 1. *Let U be an open set in Euclidean N-space which has finite volume. Suppose that U satisfies $V(U,\delta) = O(\delta^\gamma)$ as $\delta \to 0$, where $V(U,\delta)$ is the volume of the set $\{x \in U: \operatorname{dist}(x, \partial U) \leqq \delta\}$ and $0 < \gamma \leqq 1$ is a constant. Let β_N be the constant defined in Lemma 4. Let C_0 be an osculatory packing of U. Then the exponent of C_0 satisfies*

$$e(C_0, U) \leqq \max(\beta_N, N-\gamma) < N.$$

Proof. We need only consider simple osculatory packings since in general, for some m, $\{S_{m+1}, S_{m+2}, \ldots\}$ is a simple osculatory packing of

$$R_m = U \backslash (S_1^- \cup \ldots \cup S_m^-),$$

and clearly $V(R_m, \delta) = O(\delta^\gamma)$ as $\delta \to 0$ if $V(U,\delta) = O(\delta^\gamma)$.

Thus, suppose that $r_1 \geqq r_2 \geqq \ldots$ are the radii of the spheres in a simple osculatory packing of U. Then by Lemma 1, for all $n \geqq 0$,

$$(10) \qquad \sum_{k=n+1}^{\infty} r_k^N \leqq \sum_{j=0}^{N-1} \left\{\binom{N}{j} r_{n+1}^{N-j} \sum_{k=1}^{n} r_k^j\right\} + O(r_{n+1}^\gamma).$$

Let f be a non-negative function defined on $[1, \infty[$ as follows: $f(x) = r_n$ if $n \leqq x < n+1$, for $n = 1, 2, \ldots$. Then, using $r_{n+1} \leqq r_n$, we see that there is a constant c such that f satisfies (5) of Lemma 4. Hence, by the conclusion of that lemma,

$$\int_1^\infty f(x)^p\,dx < \infty \quad \text{for all } p > \max(\beta_N, N-\gamma)$$

so that

$$\sum_{k=1}^\infty r_k^p < \infty \quad \text{for } p > \max(\beta_N, N-\gamma)$$

which implies that $e(C_0, U) \leqq \max(\beta_N, N-\gamma)$.

COROLLARY 1. *Let U, C_0, β_N be as in Theorem 1 and let $\gamma_N = \min(\gamma, N-\beta_N)$. Then, the volume of the set $R_n = U\backslash(S_1^- \cup \ldots \cup S_n^-)$ is $O(r_n^{\gamma_N - \epsilon})$ for any $\epsilon > 0$.*

Proof.

$$|R_n| = \omega_N \sum_{k=n+1}^{\infty} r_k^{\,N} \leqq \omega_N r_n^{\,\gamma_N - \epsilon} \sum_{k=n+1}^{\infty} r_k^{\,N-\gamma_N+\epsilon} = O(r_n^{\,\gamma_N-\epsilon}).$$

COROLLARY 2. *Let U, C_0, β_N be as in Theorem* 1. *Let $d(C_0, U)$ be the Hausdorff dimension of $U \backslash \bigcup (S_k\colon k \geqq 1)$. Then*

$$d(C_0, U) \leqq \max(\beta_N, N - \gamma) < N.$$

Proof. This follows from [**1**, Theorem 2], since $d(C_0, U) \leqq e(C_0, U)$.

5. Remarks. (1) For the first few values of N, we have (with truncated values)

$$\begin{aligned}
\beta_2 &= 1.7071\ldots = (2 + \sqrt{2})/2,\\
\beta_3 &= 2.8228\ldots = (3 + \sqrt{7})/2,\\
\beta_4 &= 3.8923\ldots = 2 + (2 + \sqrt{(5/2)})^{\frac{1}{2}},\\
\beta_5 &= 4.9350\ldots = (5 + (15 + \sqrt{76})^{\frac{1}{2}})/2,\\
\beta_6 &= 5.9612\ldots,\\
\beta_7 &= 6.9772\ldots,\\
\beta_8 &= 7.9867\ldots,\\
\beta_9 &= 8.9924\ldots.
\end{aligned}$$

It is apparent that $N - \beta_N \to 0$ fairly rapidly. In fact, it is easy to see that

$$\frac{1}{N - \beta_N} = \sum_{j=0}^{N-1} \binom{N}{j} \frac{1}{\beta_N - j} > \frac{1}{\beta_N} \sum_{j=0}^{N-1} \binom{N}{j} = \frac{2^N - 1}{\beta_N}$$

which implies that

$$0 < N - \beta_N < N \cdot 2^{-N}.$$

(2) An example of a set U satisfying the conditions of Theorem 1 with a constant $\gamma < 1$ is

$$U = \{(x, y)\colon 1 < x < \infty, 0 < y < x^{-s}\} \quad \text{with } s > 1.$$

Here $V(U, \delta) = O(\delta^{(s-1)/s})$, so that

$$e(C_0, U) \leqq \max((2 + \sqrt{2})/2, (s + 1)/s).$$

It can be shown that, in fact, $e(C_0, U) = (s + 1)/s$ provided $(s + 1)/s \geqq (2 + \sqrt{2})/2$ (i.e. $s \leqq \sqrt{2}$).

(3) Reading an earlier draft of this paper, P. R. Beesack suggested to me that the proof of Lemma 1 could be used to give a proof of the completeness of an osculatory packing different from that given in [**1**]. Note that, by equation (2),

$$(*) \qquad |R_{n+1}| \leqq \omega_N \sum_{k=1}^{n} \{(r_k + r_{n+1})^N - r_k^{\,N}\} + V(U, r_{n+1}).$$

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Applying the mean value theorem, and then Hölder's inequality, to the first term of the right member of (*), one has:

$$\sum_{k=1}^{n} \{(r_k + r_{n+1})^N - r_k^N\} \leqq 2^{N-1} N r_{n+1} \sum_{k=1}^{n} r_k^{N-1}$$
$$\leqq (2^{N-1}N) r_{n+1} n^{1/N} \left\{ \sum_{k=1}^{n} r_k^N \right\}^{(N-1)/N}.$$

The sum $\sum_{k=1}^{n} r_k^N$ is dominated by the volume of U, hence the series converges, and this, together with the fact that $\{r_n\}$ is decreasing implies that $n r_{n+1}^N \to 0$. Thus, the first factor on the right of (*) converges to zero. To show that the second factor converges to zero we simply note that

$$V(U, r_{n+1}) = |U(r_{n+1})| = \sum_{k=n}^{\infty} |U(r_{k+1}) \backslash U(r_{k+2})|,$$

so that $V(U, r_{n+1})$ is the tail of a convergent series.

6. The exponent for a curvilinear triangle. Theorem 1 is essentially the best result we can expect to deduce from Lemma 1, as the example $r_n = An^{-\alpha}$ shows. It is possible, however, to improve Theorem 1 in case $N = 2$ and $U = T_2$ is a curvilinear triangle bounded by mutually tangent circular arcs, as we now show.

THEOREM 2. *Let T_2 be a curvilinear triangle bounded by mutually tangent circular arcs and let C_0 be a simple osculatory packing of T_2. Then*

$$e(C_0, T_2) \leqq (9 + \sqrt{41})/10 = 1.5403\ldots.$$

Proof. We begin by giving an improvement of Lemma 1, and will use the notation of the proof of that lemma, so that $S_1, S_2, \ldots$ are the disks in the packing and $R_n = T_2 \backslash (S_1^- \cup \ldots \cup S_n^-)$. Let S_{-2}, S_{-1}, S_0 be the disks of radii r_{-2}, r_{-1}, r_0, respectively, which bound T_2. We may assume that these disks are externally tangent since any curvilinear triangle may be inverted into one for which this is the case, without altering the exponent. We observe, by induction, that R_n is the union of $2n + 1$ curvilinear triangles $K_1, \ldots, K_{2n+1}$, each with in-radius at most r_{n+1}. For a given K_i, let L_i be the (rectilinear) triangle whose vertices are the centres of the sides of K_i. By induction, the L_i have mutually disjoint interiors, and $L_i \cap R_n = K_i$ $(i = 1, 2, \ldots, 2n + 1)$. Let S_k^* $(k \geqq -2)$ be the disk with the same centre as S_k and radius $r_k + r_{n+1}$. We shall show that

$$2\,\mathrm{area}(R_n) < \sum_{k=-2}^{n} \mathrm{area}(S_k^* \backslash S_k), \tag{11}$$

by showing that for each $i = 1, 2, \ldots, 2n + 1$,

$$2\,\mathrm{area}(R_n \cap L_i) < \sum_{k=-2}^{n} \mathrm{area}((S_k^* \backslash S_k) \cap L_i). \tag{12}$$

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For a fixed i, let K_i have in-radius w and sides of radii x, y, z centred at A, B, C, respectively. Let α, β, and γ be the angles at A, B, C, respectively, in the triangle L_i. If we let the sum in the right member of (12) be denoted by $v(i)$, then, since the vertices of L_i are among the centres of the annuli $S_k^*\backslash S_k$, and since $r_{n+1} \geqq w$, we have

$$\begin{aligned} v(i) &\geqq \tfrac{1}{2}\alpha((x+w)^2 - x^2) + \tfrac{1}{2}\beta((y+w)^2 - y^2) + \tfrac{1}{2}\gamma((z+w)^2 - z^2) \\ &= (\alpha x + \beta y + \gamma z)w + \tfrac{1}{2}\pi w^2. \end{aligned}$$

On the other hand, $R_n \cap L_i = K_i$. To show that $2\,\text{area}(K_i) < v(i)$, we break L_i up into six triangles by joining D, the in-centre of K_i, to A, B, C and to the vertices of K_i. Consider one of the triangles so formed, say $\triangle\, ADE$. Let δ be the angle DAE. Then, using $\sin\delta < \delta$, we have

$$\text{area}(K_i \cap \triangle\, ADE) = \tfrac{1}{2}(\sin\delta)x(x+w) - \tfrac{1}{2}\delta x^2 < \tfrac{1}{2}\delta xw.$$

Summing over the six triangles, we have

$$\text{area}(K_i) < \tfrac{1}{2}(\alpha x + \beta y + \gamma z)w < \tfrac{1}{2}v(i).$$

Summing over i proves (11). From (11) we deduce that

$$2\sum_{k=n+1}^{\infty} r_k^{\,2} < \sum_{k=1}^{n} \left((r_k + r_{n+1})^2 - r_k^{\,2}\right) + O(r_{n+1}). \tag{13}$$

Equation (13) is an improvement of equation (1).

Now, we can repeat the proof of Theorem 1, using (13) instead of (1), and we see that $\sum r_n^{\,t} < \infty$ if $t \geqq (9 + \sqrt{41})/10$ which proves Theorem 2.

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