

## LOCALLY FINITE SIMPLE GROUPS WHOSE NONNILPOTENT SUBGROUPS ARE PRONORMAL

M. FERRARA  and M. TROMBETTI  

(Received 17 April 2023; accepted 16 May 2023)

### Abstract

A subgroup  $H$  of a group  $G$  is said to be *pronormal* in  $G$  if each of its conjugates  $H^g$  in  $G$  is already conjugate to it in the subgroup  $\langle H, H^g \rangle$ . The aim of this paper is to classify those (locally) finite simple groups which have only nilpotent or pronormal subgroups.

2020 *Mathematics subject classification*: primary 20D05; secondary 20E32.

*Keywords and phrases*: finite simple group, pronormal subgroup, locally finite simple group.

### 1. Introduction

Let  $G$  be a group. A subgroup  $X$  of  $G$  is said to be *pronormal* in  $G$  if  $X$  is conjugate to  $X^g$  in  $\langle X, X^g \rangle$  for all  $g \in G$ . Pronormal subgroups were introduced by P. Hall in his lectures in Cambridge with the aim of introducing a class of subgroups containing all normal and maximal subgroups of a group; actually, it is clear that for finite groups, all Sylow  $p$ -subgroups are pronormal and for soluble finite groups, all Hall  $\pi$ -subgroups are pronormal as well. It is also very easy to see that any subnormal subgroup which is pronormal must actually be normal. The first relevant results on pronormality were obtained by Rose [24], who proved, among other things, that two pronormal subgroups, one of which normalises the other, have a pronormal product. Later, Peng [22] showed that a finite group having only pronormal subgroups is soluble, but the consideration of the alternating group of degree 5 shows that finite simple groups may have pronormal subgroups. Recently, Revin and Vdovin [23] proved that every Hall  $\pi$ -subgroup of a finite simple group is pronormal; moreover, in [18], they investigated the pronormality of subgroups of odd index in finite simple groups (see also [19]). Based on this work, finite simple groups whose nonpronormal subgroups are abelian have been classified in [3]. Actually, finite soluble groups whose nonabelian subgroups are pronormal began to be studied in [2, 12] with the aim of expanding well-known structural theorems concerning *metahamiltonian groups* (that is, groups whose proper

---

The authors are supported by GNSAGA (INdAM) and are members of the nonprofit association 'AGTA—Advances in Group Theory and Applications' ([www.advgrouptheory.com](http://www.advgrouptheory.com)).

© The Author(s), 2023. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.



subgroups are either abelian or normal) to larger classes of groups (see also [8, 10, 11] for other generalisations of this type) and these results were applied to obtain the results in [3]. Here, we propose a different approach which does not rely on the complex study of the structure of finite soluble groups with only abelian or pronormal subgroups and makes it possible to characterise which finite simple groups have only pronormal or nilpotent subgroups.

**THEOREM 1.1.** *Let  $G$  be a nonabelian finite simple group. Then  $G$  has only pronormal or nilpotent subgroups if and only if it is isomorphic to one of the following groups.*

- $\text{PSL}(2, q)$ , where  $q$  satisfies one of the following properties:
  - (i)  $q = 2^d$  and  $d$  is prime;
  - (ii)  $q = 3^d$  and  $d$  is an odd prime;
  - (iii)  $q$  is prime and if  $q \equiv_{\pm 1} 8$ , then either  $q - 1$  or  $q + 1$  is a power of 2.
- $J_1$ .
- $\text{Sz}(q)$ , where  $q = 2^{2n+1}$  where  $2n + 1$  is a prime number.

**THEOREM 1.2.** *No infinite locally finite simple group has only nilpotent or pronormal subgroups.*

Finally, it should be noted that in Section 3 (by comparison with the main theorem of [3]), our results actually classify (locally) finite simple groups whose subgroups are either nilpotent of class  $\leq c$  or pronormal.

## 2. Preliminaries

In our discussion, we will need the following easy facts on pronormality which are essential to prove that given subgroups are pronormal.

**LEMMA 2.1** [23, Lemma 5]. *Let  $H$  be a subgroup of a group  $G$ . If  $H$  contains a subgroup  $P$  which is pronormal in  $G$ , then  $H$  is pronormal in  $G$  if and only if  $H$  is conjugate to  $H^g$  in  $\langle H, H^g \rangle$  for all  $g \in N_G(P)$ .*

**PROOF.** The necessity of the condition being obvious, we prove the sufficiency. Let  $g \in G$ . Since  $P$  is pronormal in  $G$ , there exists an element  $x \in \langle P, P^g \rangle \leq \langle H, H^g \rangle$  such that  $P^x = P^g$ . Thus,  $gx^{-1}$  belongs to  $N_G(P)$  and, by hypothesis, there is an element  $y \in \langle H, H^{gx^{-1}} \rangle \leq \langle H, H^g \rangle$  such that  $H^y = H^{gx^{-1}}$ . Therefore,  $H^{yx} = H^g$  and  $yx$  belongs to  $\langle H, H^g \rangle$ . The arbitrariness of  $g$  in  $G$  shows that  $H$  is pronormal in  $G$ .  $\square$

**LEMMA 2.2.** *Let  $p$  be a prime and let  $A$  be a normal abelian  $p$ -subgroup of a group  $G$  having only pronormal or nilpotent subgroups. Let  $B \leq A$  and suppose there is  $1 \neq x \in N_G(B) \setminus C_G(B)$  having order prime to  $p$ . Then,  $N_G(\langle x \rangle) \leq N_G(B)$ .*

**PROOF.** Let  $y \in N_G(\langle x \rangle)$ . Since the subgroup  $\langle x \rangle B$  is nonnilpotent, it is pronormal in  $G$  and hence  $\langle x \rangle B$  is conjugate to  $\langle x \rangle B^y$  in  $X = \langle x, B, B^y \rangle$  through an element  $x_1 \in X$ . However,  $B^y \leq A$  centralises  $B$  showing that  $B$  is normal in  $X$ . Thus,  $B = B^{x_1} = B^y$  and the statement is proved.  $\square$

The following is a well-known result by John Rose, but since we will use it several times, we state it here as a lemma.

**LEMMA 2.3.** *Let  $p$  be a prime and let  $P$  be a  $p$ -subgroup of a finite group  $G$ . Then,  $P$  is pronormal in  $G$  if and only if  $P$  is normal in the normaliser of any Sylow  $p$ -subgroup containing it.*

Since all subgroups of a cyclic Sylow  $p$ -subgroup  $S$  are normal in the normaliser of  $S$ , the above lemma has the following immediate consequence.

**COROLLARY 2.4.** *Let  $G$  be a finite group with a cyclic Sylow  $p$ -subgroup  $S$ . Then, all subgroups of  $S$  are pronormal in  $G$ .*

The following lemma is crucial in showing that many finite simple groups have nonnilpotent, nonpronormal subgroups: in fact, if a group  $G$  has a subquotient containing a nonnilpotent, nonpronormal subgroup, then  $G$  also contains such a subgroup.

**LEMMA 2.5.** *The following statements hold.*

- (i) *The alternating group  $\text{Alt}(6)$  has a nonnilpotent, nonpronormal subgroup.*
- (ii) *A dihedral group  $G = \text{Dih}(2^n d)$  with  $(2, d) = 1$  and  $n \geq 1$  has only nilpotent or pronormal subgroups if and only if either  $d = 1$  or  $n \leq 2$ .*
- (iii) *Let  $n > 3$ . The Weyl group  $W_n$  for type  $D_n$  contains a nonnilpotent, nonpronormal subgroup.*
- (iv) *The Weyl group  $W$  for type  $F_4$  contains a nonnilpotent, nonpronormal subgroup.*
- (v) *If  $n \geq 4$ , the Weyl group  $W$  for type  $B_n / C_n$  contains a nonnilpotent, nonpronormal subgroup.*

**PROOF.** (i) Let  $X = \langle (1, 2, 3), (1, 2)(4, 5) \rangle$ ; in particular,  $X \simeq \text{Sym}(3)$ . Let  $g$  be the permutation  $(1, 4)(2, 5, 3, 6)$ . The subgroup

$$\begin{aligned} \langle X, X^g \rangle &= \langle (1, 2, 3), (1, 2)(4, 5), (4, 5, 6), (4, 5)(1, 3) \rangle \\ &= \langle (1, 2)(4, 5) \rangle \rtimes (\langle (1, 2, 3) \rangle \times \langle (4, 5, 6) \rangle) \end{aligned}$$

contains  $\langle (1, 2, 3) \rangle$  as a normal subgroup. Thus,  $X$  cannot be conjugate to  $X^g$  in  $\langle X, X^g \rangle$ ; in particular,  $X$  is not pronormal in  $G$ . Since it is not even nilpotent, the statement is proved.

(ii) Of course, if  $d = 1$ , then  $G$  is a 2-group and so even nilpotent. However, if  $n \leq 2$ , then all nonnilpotent subgroups of  $G$  are products of a pronormal subgroup of order 2, a normal  $2'$ -subgroup and possibly also of a central subgroup of order 2, so they are pronormal by Lemma 2.3.

Conversely, suppose  $d \neq 1$  and  $n \geq 3$ . Write  $G = \langle a \rangle \rtimes \langle b \rangle$ , where  $a$  has order 2 and  $b$  has order  $2^{n-1}d$ . In this case, it is easy to see that the nonnilpotent subgroup  $\langle a, b^{2^{n-1}} \rangle$  is subnormal but not normal in  $G$ , so it cannot be pronormal.

(iii) It is well known that the group in question is of the form  $\text{Sym}(n) \rtimes \mathbb{Z}_2^{n-1}$ , so the case  $n \geq 6$  follows at once from proof (i). The other cases are straightforward.

(iv) It is evident by an inspection of the Dynkin diagram for type  $F_4$  that  $W$  contains a subgroup which is isomorphic with the Weyl group for type  $D_4$  (see, for instance, [5, page 47]). Thus, the result follows from proof (iii).

(v) It is well known that the Weyl group for type  $B_4$  is isomorphic to the semi-direct product  $\text{Sym}(4) \ltimes (\langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \times \langle a_4 \rangle)$ , where each  $a_i$  has order 2 and  $\text{Sym}(4)$  acts on the  $a_i$  permuting the indexes in a natural way. Now, it is easy to check that the subgroup  $X = \langle (1, 2, 3), a_1 a_2 \rangle$  is isomorphic to  $\text{Alt}(4)$ , so it is not nilpotent. Conjugation by  $g = (1, 2)$  shows that  $X$  is not conjugate to  $X^g$  in  $\langle X, X^g \rangle$ . In fact, the Sylow 2-subgroup  $S$  of  $X$  is normal in  $\langle X, X^g \rangle$ , but  $S^g \neq S$ .

Finally, the result follows from the observation that the Weyl group for type  $B_4$  embeds in that for type  $B_n$  when  $n \geq 4$ , and that the Weyl group for type  $C_n$  coincides with that for type  $B_n$ .  $\square$

### 3. Proof of the main theorems

To prove Theorem 1.1, we proceed along the following lines: first we show that  $J_1$  is the only sporadic group to have only pronormal or nilpotent subgroups (see Theorem 3.1); then we show that the only finite simple groups of untwisted Lie type with only pronormal or nilpotent subgroups are to be found in those of type  $A_n$  (see Theorems 3.2 and 3.3); finally, Theorem 3.4 shows that for finite simple groups of twisted Lie type, we must look at Suzuki groups and then Theorem 3.5 shows that the Suzuki groups we are looking for are precisely the minimal nonsimple ones. Theorem 1.1 clearly follows from a combination of these theorems.

**THEOREM 3.1.**  *$J_1$  is the only sporadic group whose subgroups are either pronormal or nilpotent.*

**PROOF.** It follows from [6] that, apart from  $J_1$ , all sporadic groups contain a subquotient isomorphic to  $\text{Alt}(6)$ ; thus by Lemma 2.5, no sporadic group, apart from possibly  $J_1$ , has only pronormal or nilpotent subgroups. Finally, it follows from [3, Proposition 5] that all subgroups of  $J_1$  are either pronormal or abelian, so in particular they are either pronormal or nilpotent. The proof is complete.  $\square$

We refer to [9] for a detailed list of all isomorphism classes of subgroups of  $\text{PSL}(2, F)$  for a finite field  $F$ .

**THEOREM 3.2.** *Let  $G$  be a projective special linear simple group over a finite field (and let  $c \geq 1$ ). Then,  $G$  has only nilpotent (of class  $\leq c$ ) or pronormal subgroups if and only if it is isomorphic with  $\text{PSL}(2, q)$ , where  $q$  satisfies one of the following properties:*

- (i)  $q = 2^d$  and  $d$  is prime;
- (ii)  $q = 3^d$  and  $d$  is an odd prime;
- (iii)  $q$  is prime and if  $q \equiv_{\pm 1}$ , then either  $q - 1$  or  $q + 1$  is a power  $2^m$  of 2 (and  $m - 3 \leq c$ ).

**PROOF.** Suppose  $G$  is isomorphic to  $\text{PSL}(n, F)$  for a finite field  $F$  of order  $q = p^d$  for a prime  $p$ , and has only pronormal or nilpotent subgroups. Since the alternating group on  $n$  elements can always be embedded into  $\text{PSL}(n, F)$  through the consideration of the permutation matrices, it follows from Lemma 2.5 that  $n < 6$ .

Assume now  $n \geq 3$ . If  $p$  is odd, then the subgroup  $X = \langle a, b \rangle$  of  $\text{SL}(2, F)$ , where

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

is easily seen to be dihedral of order  $2p$ , so nonnilpotent. If

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then  $g \in C_G(b)$  and  $H = \langle X, X^g \rangle = \langle b \rangle \rtimes Y$ , where  $Y = \text{UT}(3, p)$ ; in particular,  $Y$  is nonabelian of order  $p^3$  and exponent  $p$ . Since  $a^g \notin \langle a, Y' \rangle \leq H$ , it follows that  $X$  is not conjugate to  $X^g$  in  $H$ . Since  $X$  embeds isomorphically in  $\text{PSL}(2, F)$ , we have a contradiction. Thus,  $p = 2$ . Since it is well known that  $\text{PSL}(3, 4)$  contains a subgroup isomorphic to  $\text{Alt}(6)$ , we are left with  $q = 2$  by Lemma 2.5. Now,  $\text{PSL}(4, 2) \simeq \text{Alt}(8)$  is contained in  $\text{PSL}(5, 2)$  as a subgroup. Again by Lemma 2.5, the only possibility left is  $\text{PSL}(3, 2)$ , but this is isomorphic to  $\text{PSL}(2, 7)$  which has only pronormal or abelian subgroups (see [3, Proposition 4]).

Suppose now  $n = 2$ , so  $q \geq 4$ . Let  $X$  be the image of  $\text{UT}(2, q)$  in  $G$ ; in particular,  $X$  is an elementary abelian  $p$ -group of order  $q$ . It is well known that  $N_G(X)/X$  is a cyclic group of order  $(p^d - 1)/\gcd(2, p^d - 1)$  irreducibly acting on  $X$ . Since  $X = C_G(X)$ , it follows from Lemma 2.2 that all elements of  $N_G(X)/X$  must act irreducibly on  $X$ . This immediately yields that  $N_G(X)/X$  is odd when  $d > 1$ ; in particular,  $q = 9$  is impossible. Moreover, if  $d = ab$  for some  $a, b > 1$ , then

$$p^{ab} - 1 = (p^a - 1)(1 + p^a + \dots + p^{(b-1)a})$$

and so  $p^a - 1 = 1$ , which is a contradiction. Thus,  $d$  is prime when  $d > 1$ .

Suppose  $p > 3$ . In this case,  $N_G(X)/X$  contains a nontrivial subgroup of order  $(p - 1)/2$  and so  $p = q$ . If  $q \equiv_{\pm 1} 8$  and  $q \neq 7$ , then  $q > 13$  and hence  $G$  contains dihedral groups of order  $q - 1$  and  $q + 1$ . By Lemma 2.5, either  $q - 1$  or  $q + 1$  is a power of 2. Now, the result follows easily from the fact that a dihedral group of order  $2^m$  with  $m \geq 4$  has a subgroup of order  $2^{m-2}$  with nilpotency class  $m - 3$  which is not pronormal.

Conversely, suppose  $G$  is isomorphic to one of the groups described in the statement. It follows from [3, Proposition 4] that we may assume  $p > 3$ ,  $p = q$  and either  $q + 1$  or  $q - 1$  being equal to  $2^m$  for some  $m$  with  $3 \leq m (\leq c + 3, c \geq 2)$ . Let  $X$  be a nonnilpotent (of class  $\leq c$ ) and nonpronormal subgroup of  $G$ ; in particular,

$X$  is not a maximal subgroup of  $G$ . Let  $g \in G$  be such that  $X$  is not conjugate to  $X^g$  in  $\langle X, X^g \rangle$ , so  $X < \langle X, X^g \rangle < G$ .

If  $X$  is isomorphic to  $\text{Alt}(4)$ , then  $\langle X, X^g \rangle$  is either  $\text{Sym}(4)$  or  $\text{Alt}(5)$ , and both these groups have only one conjugacy class of subgroups isomorphic to  $\text{Alt}(4)$ . Thus,  $X$  is pronormal in these cases and this is a contradiction. A similar argument applies if  $\langle X, X^g \rangle$  is contained in a maximal subgroup  $M$  isomorphic to  $\text{Alt}(5)$  and  $X \simeq \text{Sym}(3), \text{Dih}(10)$ : just note that in these cases,  $X$  is maximal in  $M$ .

Suppose  $X$  is isomorphic to a subgroup of the normaliser of a Sylow  $p$ -subgroup of  $G$ . Then, since it is nonabelian, it must contain the Sylow  $p$ -subgroup of  $G$  and so it is pronormal by Lemma 2.1, which is a contradiction. (If  $X$  is contained in a Sylow 2-subgroup  $S$ , then it must be a normal subgroup of  $S = N_G(S)$  and hence Lemma 2.3 gives a contradiction.)

The only case left is that in which  $X$  is contained in a maximal dihedral subgroup of  $G$  and  $X$  has order  $2r$  for some odd number  $r$  dividing either  $q - 1$  or  $q + 1$ . Since the Sylow  $d$ -subgroups of  $G$  are cyclic for any divisor  $d$  of  $r$ , it follows from Lemma 2.3 that the  $2'$ -component of  $X$  is pronormal in  $G$ . Now, Lemma 2.1 shows that also in this case,  $X$  is pronormal in  $G$ , which is the final contradiction.  $\square$

**THEOREM 3.3.** *Every finite simple group  $G$  of untwisted Lie type  $\neq A_n$  has a nonnilpotent, nonpronormal subgroup.*

**PROOF.** By Lemma 2.5, it is enough to look at finite simple groups of types  $B_n, C_n, E_6, E_7, E_8, G_2$  with  $n \leq 3$ .

Suppose  $G$  is of type  $B_n, n > 1$ . Since  $B_2(2)$  is isomorphic to  $\text{Sym}(6)$ , we may assume by Lemma 2.5 that the characteristic of the ground field is odd. In this case, the subgroup  $X$  of  $B_2$  generated by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is isomorphic to  $\text{SL}(2, 3)$ , so it is not nilpotent. Conjugation by

$$g := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

yields that  $X$  is not pronormal in  $W$ : indeed,  $O = O_2(X)$  is normal in  $\langle X, X^g \rangle$  but  $O^g \neq O$ .

Suppose now  $G$  is a finite simple group of type  $C_n (n \geq 3)$ . If the characteristic of the ground field is even, then  $G$  is isomorphic to the corresponding group of type  $B_n$ . Assume the characteristic of the ground field is odd. It is well known that  $C_n$  contains a

subgroup isomorphic to the general linear group of degree  $n$  over the same field. Thus, the result in this case follows from Theorem 3.2.

For the following discussion concerning finite simple groups of types  $E_6$ ,  $E_7$  and  $E_8$ , we refer to [14, Section 2.12]. The Weyl group for type  $E_6$  has a subgroup which is isomorphic to  $B_2(3)$ , so it has some nonnilpotent, nonpronormal subgroup; note here that the Weyl group for type  $E_6$  embeds in the one for type  $E_7$ . Finally, the Weyl group for type  $E_8$  has a homomorphic image isomorphic to  $D_4(2)$  and so Lemma 2.5 completes the discussion in these cases.

Finally, suppose  $G$  is of type  $G_2$ . If the characteristic is even, then  $G$  contains a subgroup isomorphic to  $\text{Sym}(3) \times \text{Sym}(3)$  (see [7]) which is easily checked to contain a nonnilpotent, nonpronormal subgroup. Suppose therefore the characteristic is odd. In this case,  $G$  contains a subgroup isomorphic to  $SL_3$  over the same field (see for instance [16]) and so Theorem 3.2 yields that  $G$  contains a nonnilpotent, nonpronormal subgroup.  $\square$

**THEOREM 3.4.** *All finite simple groups of twisted Lie type except possibly from type  ${}^2B_2$  have a nonnilpotent, nonpronormal subgroup.*

**PROOF.** Suppose first that  $G$  is of type  ${}^2A_n$ . We can rule out this case precisely as in [3]. Here, we describe an alternative approach. If  $n = 3$  and  $p = q = 5$ , then a look at [1, Table 8.6] shows that  $G$  involves an alternating group of degree 6 and the conclusion is reached by Lemma 2.5. Assume  $n = 3$  and  $q \neq 5$ , and let  $B$  be a Borel subgroup of  $G$  and  $X$  a Levi complement of  $B$ , so  $B = XS$ , where  $S$  is a Sylow  $p$ -subgroup of  $G$ . In particular,  $X$  is cyclic and  $|X| = (q^2 - 1)/d$ , where  $d = (q + 1, 3)$  (see [1, Table 8.5]). Put  $\bar{B} = B/Z(S)$ . Now it is not difficult to see (using also [25, Table 2]) that there exist a subgroup  $\bar{H}$  of  $\bar{S} = S/Z(S)$  and elements  $\bar{x}$  and  $\bar{y}$  of  $\bar{X} = XZ(S)/Z(S)$ , such that  $\bar{x}$  normalises  $\bar{H}$ , while  $\langle \bar{x}, \bar{H} \rangle$  and  $\langle \bar{x}, \bar{H} \rangle^{\bar{y}}$  are not conjugate in the subgroup they generate. Therefore, we may assume  $n > 3$ . Now, the discussion follows the notation in [4]. Let  $H$  be the simple adjoint algebraic group over  $\overline{\mathbb{F}}_q$  with associated Dynkin diagram of type  $A_n$ ,  $\lambda = \sigma_q$  and  $\mu = {}^2\sigma_q$ . Then  $H_\lambda = \text{PGL}_n(q)$ ,  $H_\mu = \text{PGU}_{n+1}(q)$ ,  $O^{p'}(H_\lambda) = L_{n+1}(q)$ ,  $O^{p'}(H_\mu) = U_{n+1}(q) = G$ ,

$$X = O^{p'}(H_\mu \cap H_\lambda) = \begin{cases} \text{PSp}_{n+1}(q) & \text{if } n \text{ is odd} \\ \Omega_{n+1}(q) & \text{if } n \text{ is even and } q \text{ is odd} \\ \text{Sp}_n(q) & \text{if } n \text{ and } q \text{ are even.} \end{cases}$$

Since  $n > 3$ ,  $X$  is either simple of type  $B$  or  $C$  (and we apply Theorem 3.3) or it is  $\text{Sp}_4(2) \simeq \text{Sym}(6)$  (and we apply Lemma 2.5).

Assume now  $G \simeq {}^2G_2(q)$  for  $q = 3^{2n+1} \geq 27$ . It follows from [16] that  $G$  has a subgroup  $X$  of type  $\mathbb{Z}_3 \ltimes (V \times D)$ , where  $D$  is dihedral of order  $(1/2)(q + 1)$  and the elements of order 3 normalise but do not centralise the four-group  $V \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ . Since a Sylow 2-subgroup of  $G$  is elementary abelian of order 8, it follows that  $D$  is nonnilpotent; but it can be easily seen that  $X$  contains a subgroup isomorphic to  $D$  which is not even pronormal in  $X$ .



If  $G$  is of type  ${}^2F_4$ , then it contains a subgroup isomorphic to a finite simple group of type  $B_2$  (see [21]) and Theorem 3.3 shows that  $G$  contains a nonnilpotent, nonpronormal subgroup.

If  $G \simeq {}^3D_4(q^3)$ , then it contains a subgroup  $X$  isomorphic to  $G_2(q)$  (see [17]) and so again Theorem 3.3 shows that  $G$  contains a nonnilpotent, nonpronormal subgroup in the case where  $q > 2$ . If  $q = 2$ , then  $G' \simeq {}^2A_2(3^2)$  and we are done in any case.

The case  ${}^2E_6$  can be easily handled noticing that it contains as a subgroup a finite simple group of type  $F_4$  (see [20, Table 1]).

Assume  $G$  is of type  ${}^2D_n$  for  $n \geq 4$ : the discussion here follows again the notation in [4]. Let  $H$  be the simple adjoint algebraic group over  $\overline{\mathbb{F}}_q$  with associated Dynkin diagram of type  $D_n$ ,  $\lambda = \sigma_q$  and  $\mu = {}^2\sigma_q$ . Then,  $O^{p'}(H_\lambda) = P\Omega_{2n}^+(q)$ ,  $O^{p'}(H_\mu) = P\Omega_{2n}^-(q) = G$  and

$$X = O^{p'}(H_\mu \cap H_\lambda) = \begin{cases} \Omega_{2n-1}(q) & q \text{ odd,} \\ \text{Sp}_{2n-2}(q) & q \text{ even.} \end{cases}$$

Since  $n \geq 4$ ,  $X$  is a finite simple group of type  $B_{n-1}$  or of type  $C_{n-1}$ , Theorem 3.3 completes the proof in this case. □

**THEOREM 3.5.** *Let  $q = 2^{2n+1}$  for some positive integer  $n$ . Then  $\text{Sz}(q)$  has only nilpotent (of class  $\leq 2$ ) or pronormal subgroups if and only if  $2n + 1$  is a prime.*

**PROOF.** It is well known that a subgroup of  $G$  is either isomorphic to  $\text{Sz}(s)$  with  $q$  a power of  $s$ , or conjugate to a subgroup of one of the following groups.

- (1) A solvable Frobenius group  $F$  of cardinality  $q^2(q - 1)$ ; note that  $F = D \rtimes S$ , where  $D$  is cyclic of order  $q - 1$  and  $S$  is a Sylow 2-subgroup of  $G$ .
- (2)  $\text{Dih}(2(q - 1))$ ; this is actually the normaliser of the diagonal subgroup of order  $q - 1$ .
- (3) The normaliser  $N$  of a cyclic group  $A$  of cardinality  $q \pm r + 1$  with  $r^2 = 2q$ ; in this case,  $N$  has order  $4(q \pm r + 1)$ .

We also remark that the order of  $G$  is  $q^2(q^2 + 1)(q - 1)$  and that the numbers  $q - 1$ ,  $q + r + 1$  and  $q - r + 1$  are odd and pairwise relatively prime. Moreover, a Sylow  $p$ -subgroup  $S$  of  $G$  is such that  $S' = \Omega_1(S) \leq Z(S)$ ,  $S/S'$  is elementary abelian of order  $q$ ; a cyclic subgroup  $D \leq N_G(S)$  of order  $q - 1$  acts on  $S' \setminus \{1\}$  transitively. It is therefore clear that if  $d \in D$  normalises some proper nontrivial subgroup  $L$  of  $S'$ , then  $\langle d \rangle L$  is nonpronormal and not even nilpotent; since the former condition is equivalent to the requirement that  $2n + 1$  is a prime, the necessity of the condition is proved.

Suppose now that  $2n + 1$  is a prime and let  $H$  be a nonnilpotent, nonpronormal subgroup of  $G$ . It follows from Lemma 2.1 and Corollary 2.4 that  $H$  cannot be contained in a subgroup of type (2) or (3). The only possibility left is that  $H$  is contained in a subgroup  $F = D \rtimes S$  of type (1). In such circumstances,  $H \geq S'$ . Since  $H$  is not a maximal subgroup of  $G$ , we may also assume that  $H \cap S = S'$ . Let  $g \in G$  be such that  $H$  is not conjugate to  $H^g$  in  $J = \langle H, H^g \rangle$ . Since  $G > J > H$ , it is easy to see



that  $J \leq S$ ; moreover, since  $S' = \Omega_1(S)$ , it follows that  $g \in N_G(S')$  and  $S'$  is normal in  $J$ . Finally, since all subgroups of  $S$  of order dividing  $q - 1$  are conjugate in  $S$ , it follows that  $H$  is conjugate to  $H^g$  in  $J$ , which is a contradiction.  $\square$

**PROOF OF THEOREM 1.2.** Let  $G$  be an infinite locally finite simple group with only nilpotent or pronormal subgroups. Using [15, Theorem 4.4], we may assume that  $G$  is countably infinite. Let  $\mathcal{K}$  be a Kegel cover of  $G$ ; using Theorem 1.1, we may certainly arrange this in such a way that the Kegel factors are either all of type  $A_n$  or all of type  ${}^2B_2$ .

Since every finite subgroup of  $G$  is isomorphic to a subgroup of some Kegel factor, [13, Theorem 1] and Malcev's representation theorem yield that  $G$  is linear over some field. Now, it follows from [15, Proposition 4.6] that  $G$  is the union of an ascending chain of finite simple groups and this is impossible by Theorem 1.1.  $\square$

## References

- [1] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups* (Cambridge University Press, Cambridge, 2013).
- [2] M. Brescia, M. Ferrara and M. Trombetti, 'Groups whose subgroups are either abelian or pronormal', *Kyoto J. Math.*, to appear.
- [3] M. Brescia and M. Trombetti, 'Locally finite simple groups whose non-abelian subgroups are pronormal', *Comm. Algebra* **51**(8) (2023), 3346–3353.
- [4] N. Burgoyne, R. Griess and R. Lyons, 'Maximal subgroups and automorphisms of Chevalley groups', *Pacific J. Math.* **71** (1977), 365–403.
- [5] R. W. Carter, *Simple Groups of Lie Type* (Wiley, New York, 1972).
- [6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups* (Clarendon, Oxford, 1985).
- [7] B. N. Cooperstein, 'Maximal subgroups of  $G_2(2^n)$ ', *J. Algebra* **70** (1981), 23–36.
- [8] M. De Falco, F. de Giovanni, C. Musella and R. Schmidt, 'Groups in which every non-abelian subgroup is permutable', *Rend. Circ. Mat. Palermo (2)* **52** (2003), 70–76.
- [9] L. E. Dickson, *Linear Groups: With an Exposition of the Galois Field Theory* (Teubner, Leipzig, 1901).
- [10] M. Ferrara and M. Trombetti, 'Groups whose non-permutable subgroups are metaquasihiltonian', *J. Group Theory* **23** (2019), 513–529.
- [11] M. Ferrara and M. Trombetti, 'A local study of group classes', *Note Mat.* **40**(2) (2020), 1–20.
- [12] M. Ferrara and M. Trombetti, 'Groups with many pronormal subgroups', *Bull. Aust. Math. Soc.* **105** (2022), 75–86.
- [13] J. I. Hall and B. Hartley, 'A group theoretical characterization of simple, locally finite finitary linear groups', *Arch. Math.* **60** (1993), 108–114.
- [14] J. E. Humphreys, *Reflection Groups and Coxeter Groups* (Cambridge University Press, Cambridge, 1992).
- [15] O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups* (Elsevier, New York, 1973).
- [16] P. B. Kleidman, 'The maximal subgroups of the Chevalley groups  $G_2(q)$  with  $q$  odd, the Ree groups  ${}^2G_2(q)$ , and their automorphism groups', *J. Algebra* **117** (1988), 30–71.
- [17] P. B. Kleidman, 'The maximal subgroups of the Steinberg triality groups  ${}^3D_4(q)$  and their automorphism groups', *J. Algebra* **115** (1988), 182–199.
- [18] A. S. Kondrat'ev, N. V. Maslova and D. O. Revin, 'On the pronormality of subgroups of odd index in finite simple groups', *Sib. Math. J.* **56** (2015), 1101–1107.
- [19] A. S. Kondrat'ev, N. V. Maslova and D. O. Revin, 'On the pronormality of subgroups of odd index in finite simple symplectic groups', *Sib. Math. J.* **58** (2017), 467–475.

- [20] M. W. Liebeck and J. Saxl, 'On the orders of maximal subgroups of the finite exceptional groups of Lie Type', *Proc. Lond. Math. Soc. (3)* **55** (1987), 299–330.
- [21] G. Malle, 'The maximal subgroups of  ${}^2F_4(q^2)$ ', *J. Algebra* **139** (1991), 52–69.
- [22] T. A. Peng, 'Finite groups with pro-normal subgroups', *Proc. Amer. Math. Soc.* **20** (1969), 232–234.
- [23] D. O. Revin and E. P. Vdovin, 'Pronormality of Hall subgroups in finite simple groups', *Sib. Math. J.* **53** (2012), 419–430.
- [24] J. S. Rose, 'Finite soluble groups with pronormal system normalizers', *Proc. Lond. Math. Soc. (3)* **17** (1967), 447–469.
- [25] W. A. Simpson and J. S. Frame, 'The character tables for  $SL(3, q)$ ,  $SU(3, q^2)$ ,  $PSL(3, q)$ ,  $PSU(3, q^2)$ ', *Canad. J. Math.* **3** (1973), 486–494.

M. FERRARA, Dipartimento di Matematica e Fisica,  
Università degli Studi della Campania 'Luigi Vanvitelli',  
viale Lincoln 5, Caserta, Italy  
e-mail: [maria.ferrara1@unicampania.it](mailto:maria.ferrara1@unicampania.it)

M. TROMBETTI, Dipartimento di Matematica e Applicazioni 'Renato Caccioppoli',  
Università di Napoli Federico II, Complesso Universitario Monte S. Angelo,  
Via Cintia, Napoli, Italy  
e-mail: [marco.trombetti@unina.it](mailto:marco.trombetti@unina.it)