Canad. Math. Bull. Vol. **58** (1), 2015 pp. 188–195
http://dx.doi.org/10.4153/CMB-2014-037-5

Telescoping Estimates for Smooth Series

Dedicated to my friend Joachim Gräter on the occasion of his 60th birthday

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Abstract. We derive telescoping majorants and minorants for some classes of series and give applications of these results.

1 Introduction

An old and often used theorem to estimate the growth of a series is Cauchy's integral criterion. More powerful and elaborate is the Euler–MacLaurin summation method. In this article, an elementary method of estimation for series that uses telescoping majorants and minorants is presented. These results can be used to ameliorate the theorems derived from Cauchy's method, but naturally they are not as fine as those obtained by the Euler–MacLaurin summation method.

In [2], the authors found elementary bounds for $\sum_{k=n}^{\infty} k^{-s}$, $s > 1$ by use of such methods. The present article is dedicated to the question of which series can be treated in a similar way. In the following we shall show that this is the case for a big class of series $\sum_{k=n}^{\infty} f(k)$. They only have to satisfy certain mild smoothness conditions on the function f. The proofs are based on the comparison of $f(k)$ and

$$\int_{k-c}^{k+1-c} f(x)\,dx, \quad c \in (0,1),$$

that may be regarded as special cases of theorems from the theory of numerical integration. We will demonstrate the usefulness of this method by some applications. Among them there will be a generalization of the Stieltjes constants and an elementary proof of Stirling's formula.

2 The Central Lemma and Telescoping Series

Lemma 1 *Let $F\colon [x_0,\infty) \to \mathbb{R}$ be three times differentiable, and assume*

$$\lim_{x\to\infty} \frac{F''(x)}{F''(x+1)} = 1.$$

Received by the editors January 22, 2014.
Published electronically November 3, 2014.
AMS subject classification: **26D15**, 40A25,97I30.
Keywords: telescoping series, Stietjes constant, Hardy's formula, Stirling's formula.

(i) *If $F''(x) < 0$ and $F'''(x) > 0$ for $x \geq x_0$, then the following assertions are valid.*
  (a) $F'(k) < F(k + \frac{1}{2}) - F(k - \frac{1}{2})$ *for* $k \geq x_0 + \frac{1}{2}$.
  (b) *For any $c \in (0, \frac{1}{2})$ there exists $N(c)$ such that for every $k \geq N(c)$ the inequality*
$$F'(k) > F(k + 1 - c) - F(k - c)$$
  holds.

(ii) *If $F''(x) > 0$ and $F'''(x) < 0$ for $x \geq x_0$, then the following assertions are valid.*
  (a) $F'(k) > F(k + \frac{1}{2}) - F(k - \frac{1}{2})$ *for* $k \geq x_0 + \frac{1}{2}$.
  (b) *For any $c \in (0, \frac{1}{2})$ there exists $N(c)$ such that for every $k \geq N(c)$ the inequality*
$$F'(k) < F(k + 1 - c) - F(k - c)$$
  holds.

(iii) *If $F''(x) > 0$ and $F'''(x) > 0$ for $x \geq x_0$, then the following assertions are valid.*
  (a) $F'(k) < F(k + \frac{1}{2}) - F(k - \frac{1}{2})$ *for* $k \geq x_0 + \frac{1}{2}$.
  (b) *For any $c \in (\frac{1}{2}, 1)$ there exists $N(c)$ such that for every $k \geq N(c)$ the inequality*
$$F'(k) > F(k + 1 - c) - F(k - c)$$
  holds.

(iv) *If $F''(x) < 0$ and $F'''(x) < 0$ for $x \geq x_0$, then the following assertions are valid.*
  (a) $F'(k) > F(k + \frac{1}{2}) - F(k - \frac{1}{2})$ *for* $k \geq x_0 + \frac{1}{2}$.
  (b) *For any $c \in (\frac{1}{2}, 1)$ there exists $N(c)$ such that for every $k \geq N(c)$ the inequality*
$$F'(k) < F(k + 1 - c) - F(k - c)$$
  holds.

Proof Since the proofs of (i)–(iv) differ only in obvious details, we restrict ourselves to the proof of part (i) here.

To prove (a) we use the fact that there exist $\theta_1 \in (k - \frac{1}{2}, k)$ and $\theta_2 \in (k, k + \frac{1}{2})$ such that
$$F\Big(k + \frac{1}{2}\Big) = F(k) + \frac{1}{2}F'(k) + \frac{1}{8}F''(\theta_2)$$
and
$$F\Big(k - \frac{1}{2}\Big) = F(k) - \frac{1}{2}F'(k) + \frac{1}{8}F''(\theta_1).$$
Hence
$$F\Big(k + \frac{1}{2}\Big) - F\Big(k - \frac{1}{2}\Big) = F'(k) + \frac{1}{8}\big(F''(\theta_2) - F''(\theta_1)\big) > F'(k),$$
since F'' is strictly monotonically increasing.

For the proof of (b) we use the fact that there exist $\theta_1 \in (k - c, k)$ and $\theta_2 \in (k, k + 1 - c)$ such that
$$F(k + 1 - c) - F(k - c) = F'(k) + \frac{1}{2}\big((1 - c)^2 F''(\theta_2) - c^2 F''(\theta_1)\big).$$

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From the monotonicity conditions we derive

$$0 < -F''(k+1-c) < -F''(\theta_2) < -F''(\theta_1) < -F''(k-c),$$

and therefore we have

$$1 < \frac{-F''(\theta_1)}{-F''(\theta_2)} < \frac{-F''(k-c)}{-F''(k+1-c)} \to 1$$

as $k \to \infty$. Hence, there exists $N(c)$ such that for $k \geq N(c)$,

$$\frac{-F''(\theta_1)}{-F''(\theta_2)} < \frac{-F''(k-c)}{-F''(k+1-c)} < \frac{(1-c)^2}{c^2}.$$

This implies

$$(1-c)^2 F''(\theta_2) - c^2 F''(\theta_1) < 0,$$

which proves the assertion. ■

Now, we consider the series $\sum_{k=n}^{\infty} f(k)$ where $F'(x) = f(x)$.

Theorem 1

(i) *If F satisfies the conditions of part* (i) *of Lemma 1, then for any $c \in (0, \frac{1}{2})$ there exists $N(c)$ such that for $n \geq N(c)$,*

$$F\Big(N+\frac{1}{2}\Big) - F\Big(n-\frac{1}{2}\Big) > \sum_{k=n}^{N} f(k) > F(N+1-c) - F(n-c).$$

(ii) *If F satisfies the conditions of part* (ii) *of Lemma 1, then for any $c \in (0, \frac{1}{2})$ there exists $N(c)$ such that for $n \geq N(c)$,*

$$F\Big(N+\frac{1}{2}\Big) - F\Big(n-\frac{1}{2}\Big) < \sum_{k=n}^{N} f(k) < F(N+1-c) - F(n-c).$$

(iii) *If F satisfies the conditions of part* (iii) *of Lemma 1, then for any $c \in (\frac{1}{2}, 1)$ there exists $N(c)$ such that for $n \geq N(c)$,*

$$F\Big(N+\frac{1}{2}\Big) - F\Big(n-\frac{1}{2}\Big) > \sum_{k=n}^{N} f(k) > F(N+1-c) - F(n-c).$$

(iv) *If F satisfies the conditions of part* (iv) *of Lemma 1, then for any $c \in (\frac{1}{2}, 1)$ there exists $N(c)$ such that for $n \geq N(c)$,*

$$F\Big(N+\frac{1}{2}\Big) - F\Big(n-\frac{1}{2}\Big) < \sum_{k=n}^{N} f(k) < F(N+1-c) - F(n-c).$$

3 Stieltjes Constants

The Stieltjes constants γ_ν, $\nu \in \mathbb{N} \cup \{0\}$, are defined by the relation

$$\gamma_\nu = \lim_{N\to\infty}\Big(\sum_{k=1}^{N} \frac{\log^\nu k}{k} - \frac{\log^{\nu+1} N}{\nu+1}\Big), \tag{3.1}$$

where we used the abbreviation $\log^\nu k = (\log k)^\nu$, compare [4] and [5]. These constants are intimately related to the Laurent expansion of the Riemann zeta function at its pole.

We shall use Theorem 1 to show that one may use equation (3.1) to define analogous constants γ_s, $s > -1$, and to prove some properties of these constants.

If we let

$$f(x) = \frac{\log^s x}{x}, \quad F(x) = \frac{\log^{s+1} x}{s+1},$$

and differentiate, we easily see that parts (i) of both Lemma 1 and Theorem 1 apply. Hence, for sufficiently big n, one has

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{\log^s k}{k} &+ \frac{\log^{s+1}(N+1-c)}{s+1} - \frac{\log^{s+1}(n-c)}{s+1} - \frac{\log^{s+1} N}{s+1}\\ &< \sum_{k=1}^{N} \frac{\log^s k}{k} - \frac{\log^{s+1} N}{s+1} =: c_N\\ &< \sum_{k=1}^{n-1} \frac{\log^s k}{k} + \frac{\log^{s+1}(N+1/2)}{s+1} - \frac{\log^{s+1}(n-1/2)}{s+1} - \frac{\log^{s+1} N}{s+1}.\end{aligned}$$

If we define

$$\alpha_n = \sum_{k=1}^{n-1} \frac{\log^s k}{k} - \frac{\log^{s+1}(n-c)}{s+1}$$

and

$$\beta_n = \sum_{k=1}^{n-1} \frac{\log^s k}{k} - \frac{\log^{s+1}(n-1/2)}{s+1},$$

we get that for sufficiently big n

$$\alpha_n \le \underline{\lim}_{N\to\infty} c_N \le \overline{\lim}_{N\to\infty} c_N \le \beta_n,$$

since

$$\frac{\log^{s+1}(N+1-c)}{s+1} - \frac{\log^{s+1} N}{s+1} = \frac{\log^{s+1} N}{s+1}\bigg(\Big(1+\frac{\log(1+\frac{1-c}{N})}{\log N}\Big)^{s+1} - 1\bigg) \to 0, \tag{3.2}$$

as $N \to \infty$. Using the inequalities of Lemma 1, part (i), we see that α_n is a strictly increasing sequence for sufficiently big n, and that β_n is a strictly decreasing sequence for sufficiently big n. Similar to (3.2) one can show that

$$\lim_{n\to\infty} (\beta_n - \alpha_n) = 0.$$

This proves the existence of the limit γ_s, $s < -1$.

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In the case $s = 0$, it is well known that the convergence of the sequence β_n to the Euler constant $\gamma = \gamma_0$ is much faster than that of the defining sequence, see [4].

Now, we can use the inequalities

$$c_n - \beta_n < c_n - \gamma_s < c_n - \alpha_n$$

for $c = \frac{1}{2} - \epsilon, \epsilon > 0$, and a sufficiently big n to prove that

$$\lim_{n\to\infty}\Big((c_n - \gamma_s)\frac{n}{\log^s n}\Big) = \frac{1}{2}.$$

In the cases $s \in \mathbb{N} \cup \{0\}$ this relation is well known; compare [5].

In a similar way, one can define γ_{-1} by

$$\gamma_{-1} = \lim_{N\to\infty}\Big(\sum_{k=3}^{N}\frac{1}{k\log k} - \log(\log N)\Big),$$

and prove that

$$\lim_{N\to\infty}\Big(\Big(\sum_{k=3}^{N}\frac{1}{k\log k} - \log(\log N) - \gamma_{-1}\Big)N\log N\Big) = \frac{1}{2}.$$

4 Alternating Series and a Formula Due to Hardy

In this section we will give an elementary proof of a special case of a formula that can be found in Hardy's famous book on divergent series, see [3, p. 333]. Using telescoping series we shall prove that, for $s \in (0, 1)$,

$$\text{(4.1)}\qquad \lim_{m\to\infty}\Big(\sum_{k=1}^{m}\frac{1}{k^s} - \frac{m^{1-s}}{1-s} - \frac{1}{2}m^{-s}\Big) = \frac{1}{1-2^{1-s}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s} = \zeta(s).$$

We start with the equation

$$\text{(4.2)}\qquad \sum_{k=1}^{m}\frac{1}{k^s} = \frac{1}{2^{1-s}-1}\Big(\sum_{k=m+1}^{2m}\frac{1}{k^s} - \sum_{k=1}^{2m}\frac{(-1)^{k-1}}{k^s}\Big).$$

The proof of this formula is an exercise in elementary calculations with series, see [1, p. 127]. It should be remarked that (4.2) can be used to prove that for $s > 1$, the identity

$$\frac{1}{1-2^{1-s}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s} = \zeta(s)$$

holds.

If in Lemma 1 and Theorem 1 we take

$$f(x) = \frac{1}{x^s}, \quad F(x) = \frac{x^{1-s}}{1-s}, \quad s \in (0, 1),$$

we see by differentiation that again part (i) applies. Now, we take $N = 2m$ and $n = m + 1$, m sufficiently big, in (4.2), and let $m \to \infty$. This results in the identities

$$\lim_{m\to\infty}\Big(\sum_{k=1}^{m}\frac{1}{k^s} - \frac{m^{1-s}}{1-s}\Big) = \frac{1}{2^{1-s}-1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}$$

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and

$$\lim_{m\to\infty}\Big(\sum_{k=1}^{m}\frac{1}{k^s}-\frac{m^{1-s}}{1-s}+\frac{1}{2^{1-s}-1}\sum_{k=1}^{2m}\frac{(-1)^{k-1}}{k^s}\Big)m^s=\frac{(2^{1-s}-2)(s+1)}{4(2^{1-s}-1)}. \tag{4.3}$$

In order to simplify this formula we apply our method of telescoping series to alternating series. The basis of this application is the following lemma.

Lemma 2 *Let $s>0$.*

(i) *For any $k\in\mathbb{N}$ the inequality*

$$k^{-s}-(k+1)^{-s}<\frac{1}{2}\Bigg(\Big(k-\frac{1}{2}\Big)^{-s}-\Big(k+\frac{3}{2}\Big)^{-s}\Bigg) \tag{4.4}$$

is valid.

(ii) *For any $c\in(0,\frac{1}{2})$ there exists $N(c)$ such that for all $k\geq N(c)$ the inequalities*

$$k^{-s}-(k+1)^{-s}>\frac{1}{2}\big((k-c)^{-s}-(k+2-c)^{-s}\big)$$

are valid.

Sketch of the proof We only mention the crucial steps in the proof of part (i) and omit the proof of part (ii), since the ideas are essentially the same.

We multiply the inequality (4.4) by k^s and replace $\frac{1}{k}$ by x. To prove

$$1-(1+x)^{-s}<\frac{1}{2}\Bigg(\Big(1-\frac{x}{2}\Big)^{-s}-\Big(1+\frac{3x}{2}\Big)^{-s}\Bigg)\quad x\in(0,1],$$

we remark that both sides vanish at $x=0$, and we differentiate both sides. This procedure reveals that it is sufficient to prove

$$1<\frac{1}{4}\Big(\frac{1+x}{1-\frac{x}{2}}\Big)^{1+s}+\frac{3}{4}\Big(\frac{1+x}{1+\frac{3x}{2}}\Big)^{1+s},\quad x\in(0,1].$$

Since both sides are identical for $x=0$, we differentiate again. The resulting inequality

$$0<\frac{3(1+x)^s(1+s)}{8}\Bigg(\Big(1-\frac{x}{2}\Big)^{-s-2}-\Big(1+\frac{3x}{2}\Big)^{-s-2}\Bigg),\quad x\in(0,1]$$

is obvious and proves the assertion. ■

Summation of the inequalities in Lemma 2 for $k=2m+1,2m+3,\dots,2N-1$, and letting firstly $N\to\infty$, and then $m\to\infty$, prove the following.

Corollary 1 *Let $s>0$.*

(i) *For any $m\in\mathbb{N}\cup\{0\}$ the inequality*

$$\sum_{k=2m+1}^{\infty}\frac{(-1)^{k-1}}{k^s}<\frac{1}{2}\Big(2m+\frac{1}{2}\Big)^{-s}$$

is valid.

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(ii)

$$\lim_{m\to\infty}\left(\Big(\sum_{k=2m+1}^{\infty}\frac{(-1)^{k-1}}{k^s}\Big)m^s\right)=2^{-s-1}.$$

An immediate consequence of formula (4.3) and Corollary 1 is the desired identity (4.1).

5 An Elementary Proof of the Stirling Formula

By similar considerations as in the preceding section one can get information on

$$\log(m!)=\sum_{k=2}^{m}\log k.$$

It is easy to see that the identity

$$\sum_{k=2}^{m}\log k=-2m\log 2+\sum_{k=m+1}^{2m}\log k+\sum_{k=2}^{2m}(-1)^k\log k \tag{5.1}$$

holds. If we let

$$f(x)=\log x,\quad F(x)=x(\log x-1),$$

we see that case (ii) of Lemma 1 and Theorem 1 applies. Hence, setting $n=m+1$ and $N=2m$, the above considerations imply that for any $c\in(0,1/2)$ and m sufficiently big the inequalities

$$\begin{aligned}&\Big(2m+\frac{1}{2}\Big)\Big(\log\Big(2m+\frac{1}{2}\Big)-1\Big)-\Big(m+\frac{1}{2}\Big)\Big(\log\Big(m+\frac{1}{2}\Big)-1\Big)\\&\quad<\sum_{k=m+1}^{2m}\log k\\&\quad<(2m+1-c)\big(\log(2m+1-c)-1\big)-(m+1-c)\big(\log(m+1-c)-1\big)\end{aligned}$$

are valid. If in the upper bound we let $c=\frac{1}{2}-\epsilon$, $\epsilon>0$, and consider the upper and lower bound for $m\to\infty$, we get

$$\lim_{m\to\infty}\left(\sum_{k=m+1}^{2m}\log k-\Big(2m\log 2+m(\log m-1)+\frac{1}{2}\log 2\Big)\right)=0. \tag{5.2}$$

The third term on the right hand side of (5.1) can be written as

$$\frac{1}{2}\sum_{k=1}^{m}\log\Big(\frac{4k^2}{4k^2-1}\Big)+\frac{1}{2}\log(2m+1).$$

The application of the product theorem of Wallis herein immediately implies

$$\lim_{m\to\infty}\left(\sum_{k=2}^{2m}(-1)^k\log k-\Big(\frac{1}{2}\log\frac{\pi}{2}+\frac{1}{2}\log m+\frac{1}{2}\log 2\Big)\right)=0. \tag{5.3}$$

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Inserting (5.2) and (5.3) into (5.1) delivers an elementary proof of the Stirling formula

$$\lim_{m\to\infty}\left(\log(m!)-\left(\left(m+\frac{1}{2}\right)\log m-m+\log\sqrt{2\pi}\right)\right)=0.$$

6 Further Applications

If we consider $\sum_{k=1}^{\infty} k^s$ for $s>0$ and let

$$f(x)=x^s,\quad F(x)=\frac{x^{s+1}}{s+1},$$

we see that for $s\in(0,1)$, parts (ii) of Lemma 1 and Theorem 1 apply, whereas for $s>1$ part (iii) of Lemma 1 and Theorem 1 play the decisive rôle. In both cases we can use the resulting inequalities to get

$$\lim_{N\to\infty}\left(\frac{1}{N^s}\sum_{k=1}^{N}k^s-\frac{N}{s+1}\right)=\frac{1}{2}.$$

For $s=1$ the idea of telescoping series and the identity

$$k^1=\frac{(k+\frac{1}{2})^2-(k-\frac{1}{2})^2}{2}$$

delivers a very short proof of the famous equation

$$\sum_{k=n}^{N}k=\frac{(N+\frac{1}{2})^2-(n-\frac{1}{2})^2}{2}=\frac{N(N+1)}{2}-\frac{n(n-1)}{2}.$$

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