

# Identities in tensor products of Banach algebras

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Let  $A_1, A_2$  be Banach algebras,  $A_1 \otimes A_2$  their algebraic tensor product over the complex field. If  $\|\cdot\|_\alpha$  is an algebra norm on  $A_1 \otimes A_2$  we write  $A_1 \otimes_\alpha A_2$  for the  $\|\cdot\|_\alpha$ -completion of  $A_1 \otimes A_2$ . In this note we study the existence of identities and approximate identities in  $A_1 \otimes_\alpha A_2$  versus their existence in  $A_1$  and  $A_2$ . Some of the results obtained are already known, but our method of proof appears new, though it is quite elementary.

## 1. Preliminaries

The four results collected here are probably already known in one form or another; the proofs are included for completeness.

**PROPOSITION 1.** *Let  $A$  be a Banach algebra,  $\{e_\lambda\}, \{f_\mu\}$  nets in  $A$  such that  $\|e_\lambda x - x\| \rightarrow 0$ ,  $\|x f_\mu - x\| \rightarrow 0$  uniformly on the unit ball of  $A$ . Then  $A$  has an identity.*

*Proof.* Take  $s \in \{e_\lambda\}$  such that  $\|sx - x\| \leq \frac{1}{2}$  for  $\|x\| \leq 1$ . Then  $s$  is not a left topological divisor of zero, for otherwise there would be  $\{x_n\} \subseteq A$ ,  $\|x_n\| = 1$ , with  $sx_n \rightarrow 0$ , contradicting  $\|sx_n - x_n\| \leq \frac{1}{2}$  for each  $n$ . Similarly there is  $t \in A$  which is not a right topological divisor of zero.

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Now for each positive integer  $n$  take  $e_n \in \{e_\lambda\}$ ,  $f_n \in \{f_\mu\}$  such that  $\|e_n x - x\| \leq \frac{1}{n}$ ,  $\|x f_n - x\| \leq \frac{1}{n}$  for  $\|x\| \leq 1$ . Then certainly  $\|(e_n - e_m)t\| \leq \left(\frac{1}{n} + \frac{1}{m}\right)\|t\|$ ,  $\|s(f_n - f_m)\| \leq \left(\frac{1}{n} + \frac{1}{m}\right)\|s\|$  and so, by the choice of  $s$  and  $t$ ,  $\{e_n\}$ ,  $\{f_n\}$  are Cauchy, and so converge to elements  $e, f \in A$ . But then  $e$  and  $f$  are respectively left and right identities for  $A$ , and so  $e = f$  is an identity.

We will also require the following modification of Proposition 1. As usual  $\nu, \hat{\phantom{x}}$  denote spectral radius and Gelfand transform respectively.

**PROPOSITION 2.** *Let  $A$  be a commutative Banach algebra,  $\{e_\lambda\}$  a net in  $A$  such that  $\nu(e_\lambda x - x) \rightarrow 0$  uniformly for  $\nu(x) \leq 1$ . Then there is an idempotent  $e \in A$  with  $\hat{e} \equiv 1$ .*

*Proof.* By the same argument as in Proposition 1 there is a sequence  $\{e_n\} \subseteq A$  with  $\nu(e_n x - x) \rightarrow 0$  uniformly for  $\nu(x) \leq 1$ , and  $\nu(e_n - e_m) \rightarrow 0$ . It follows that  $\{\hat{e}_n\}$  converges uniformly to the constant function 1 on the carrier space of  $A$ , which is thus compact. Also, for  $n$  sufficiently large  $\hat{e}_n$  is bounded away from zero, and the elementary argument of [5], pp. 171-2 now furnishes the desired idempotent  $e$ .

**PROPOSITION 3.** *Let  $A$  be a Banach algebra which does not consist entirely of right (left) topological divisors of zero. If  $A$  has a left (right) approximate identity  $\{d_\rho\}$  then it has a bounded left (right) approximate identity. Indeed, if  $A$  is commutative and  $\{d_\rho\}$  is countable, then  $\{d_\rho\}$  is bounded.*

*Proof.* Let  $F$  be the family of all finite subsets of  $A$ , and define a directed set  $\Lambda = \{(F, \delta) : F \in F, 1 > \delta > 0\}$  where  $(F_1, \delta_1) \leq (F_2, \delta_2)$  if  $F_1 \subseteq F_2$  and  $\delta_2 \leq \delta_1$ . Take  $z \in A$  not a right topological divisor of zero. Then for  $\lambda = (F, \delta) \in \Lambda$ , take  $e_\lambda \in \{d_\rho\}$  such that  $\|e_\lambda y - y\| < \delta$  for  $y \in F \cup \{z\}$ , so that if  $x \in A$ ,  $1 > \varepsilon > 0$ ,  $\|e_\lambda x - x\| < \varepsilon$  provided  $\lambda \geq (\{x\}, \varepsilon)$ . It follows that  $\{e_\lambda\}$  is a left approximate identity in  $A$ . Also,  $\|e_\lambda z\| < 1 + \|z\|$  for all  $\lambda$ , so

that, by the choice of  $z$ ,  $\{e_\lambda\}$  is bounded.

The last statement is proved in [6], p. 279.

REMARK. The converse is false:  $L^1(0, 1)$  is a radical algebra under convolution, so that all elements are topological divisors of zero, but has a (countable) bounded approximate identity.

Now let  $U_1, U_2$  be seminormed spaces under  $p_1, p_2$  respectively. Corresponding to the normed case define seminorms  $p_\gamma, p_\lambda$  on  $U_1 \otimes U_2$  by

$$p_\gamma(x) = \inf \left\{ \sum p_1(u_i)p_2(v_i) : x = \sum u_i \otimes v_i \right\},$$

$$p_\lambda(x) = \sup \left\{ \left| \sum \varphi_1(u_i)\varphi_2(v_i) \right| : x = \sum u_i \otimes v_i, \varphi_j \in U_j^*, \|\varphi_j\|_{p_j} = 1, \right. \\ \left. j = 1, 2 \right\}.$$

As in the normed case  $p_\gamma$  is the greatest seminorm  $p$  on  $U_1 \otimes U_2$  such that  $p(u \otimes v) = p_1(u)p_2(v)$ ,  $u \in U_1, v \in U_2$ ; and so in particular  $p_\lambda \leq p_\gamma$ . Using the terminology of [1] a seminorm  $p$  on  $U_1 \otimes U_2$  will be called admissible if there are positive constants  $m, M$  such that  $mp_\lambda \leq p \leq Mp_\gamma$ .

PROPOSITION 4. Let  $U_j, p_j, j = 1, 2$  be as above, and  $p$  an admissible seminorm on  $U_1 \otimes U_2$ , with  $m$  as above. If

$$x = \sum_{i=1}^n u_i \otimes v_i \in U_1 \otimes U_2 \text{ with } \{v_i\} \text{ (or } \{u_i\}) \text{ linearly independent,}$$

then  $p_1(u_i)p_2(v_i) \leq \frac{1}{m}p(x), i = 1, 2, \dots, n$ .

Proof. Take  $1 \leq i \leq n$ . If  $p_1(u_i)p_2(v_i) = 0$  the result is immediate for this  $i$ , otherwise the Hahn-Banach theorem furnishes  $\varphi_j \in U_j^*, j = 1, 2$  such that  $\|\varphi_j\|_{p_j} = 1, \varphi_2(v_k) = \delta_{ik}p_2(v_k), \varphi_1(u_i) = p_1(u_i)$ . But then for  $w = \sum s_k \otimes t_k \in U_1 \otimes U_2,$   
 $\varphi_1 \otimes \varphi_2(w) = \sum \varphi_1(s_k)\varphi_2(t_k) \leq p_\lambda(w) \leq \frac{1}{m}p(w),$  and so  
 $\varphi_1 \otimes \varphi_2(x) = p_1(u_i)p_2(v_i) \leq \frac{1}{m}p(x).$

2. The general (non-commutative) case

For the remainder of this paper  $A_j$  will denote a Banach algebra with norm  $\|\cdot\|_j$ , spectral radius  $\nu_j$ ,  $j = 1, 2$ ;  $\|\cdot\|_\alpha$  will be an algebra norm on  $A_1 \otimes A_2$  with spectral radius  $\nu_\alpha$ . If  $A$  is commutative its carrier space, with the Gelfand topology, will be denoted  $\Phi_A$ .

**THEOREM 1.** *Let  $\|\cdot\|_\alpha$  be an admissible algebra norm on  $A_1 \otimes A_2$ . Then  $A_1 \otimes_\alpha A_2$  has an identity  $\iota$  if and only if  $A_1, A_2$  have identities  $e, f$ , and  $\iota = e \otimes f$ .*

**Proof.** Suppose that  $A_1 \otimes_\alpha A_2$  has an identity  $\iota$ , the converse being immediate. Let  $\epsilon > 0$ , and take  $x = x_\epsilon \in A_1 \otimes A_2$  with  $\|x - \iota\|_\alpha < \epsilon$ . Then if  $x = \sum u_i \otimes v_i$  and  $\|s \otimes t\|_\alpha \leq 1$ ,

$$\left\| \sum u_i s \otimes v_i t - s \otimes t \right\|_\alpha < \epsilon, \quad \left\| \sum su_i \otimes tv_i - s \otimes t \right\|_\alpha < \epsilon.$$

Now let  $\sum u'_j s \otimes v'_j, \sum su''_k \otimes v''_k$  be alternative expressions for  $\sum u_i s \otimes v_i t, \sum su_i \otimes tv_i$  respectively, where  $v'_1 = v''_1 = t$  and  $\{v'_j\}, \{v''_k\}$  are linearly independent sets. Then

$$\left\| (u'_1 s - s) \otimes t + \sum_{j \geq 2} u'_j s \otimes v'_j \right\|_\alpha < \epsilon, \quad \left\| (su''_1 - s) \otimes t + \sum_{k \geq 2} su''_k \otimes v''_k \right\|_\alpha < \epsilon,$$

and so by Proposition 4  $\|u'_1 s - s\|_1 \|t\|_2 < \frac{\epsilon}{m}, \|su''_1 - s\|_1 \|t\|_2 < \frac{\epsilon}{m}$ , where  $m \|\cdot\|_\lambda \leq \|\cdot\|_\alpha \leq M \|\cdot\|_\gamma$ . Now  $u'_1, u''_1$  depend on  $t$  only, not upon  $s$ , and so, noting that if  $\|s\|_1 \leq 1$  then  $\|s \otimes t\|_\alpha \leq M \|t\|_2$ , it follows that

$$\|u'_1 s - s\|_1 \leq \frac{M}{m} \epsilon, \quad \|su''_1 - s\|_1 \leq \frac{M}{m} \epsilon, \quad \text{for } \|s\|_1 \leq 1.$$

Taking  $\epsilon = 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots$  we thus obtain sequences  $\{e_n\}, \{e'_n\} \subseteq A_1$  such such that  $\|e_n s - s\|_1 \rightarrow 0, \|se'_n - s\|_1 \rightarrow 0$  uniformly on the unit ball of  $A_1$ , so that by Proposition 1  $A_1$  has an identity  $e$ .

Similarly  $A_2$  has an identity  $f$ , whence  $e \otimes f$  is an identity on

$A_1 \otimes_{\alpha} A_2$  and so must equal  $\iota$ .

By an equally simple argument we have the following.

**THEOREM 2.** *Let  $\|\cdot\|_{\alpha}$  be an admissible algebra norm on  $A_1 \otimes A_2$ .*

*If  $A_1$  and  $A_2$  each possess a bounded left (right) approximate identity then so does  $A_1 \otimes_{\alpha} A_2$ . Conversely, if  $A_1 \otimes_{\alpha} A_2$  has a left (right) approximate identity then so do  $A_1$  and  $A_2$ .*

*Proof.* Let  $m\|\cdot\|_{\lambda} \leq \|\cdot\|_{\alpha} \leq M\|\cdot\|_{\gamma}$ , and take  $\{e_{\lambda}\}$ ,  $\{f_{\mu}\}$  bounded left approximate identities in  $A_1$  and  $A_2$  respectively, with  $\sup\|e_{\lambda}\|_1 \leq C$ ,  $\sup\|f_{\mu}\|_2 \leq C$  for some  $C$ . Then the set

$\{e_{\lambda} \otimes f_{\mu}\} \subseteq A_1 \otimes A_2$  is bounded,  $\sup\|e_{\lambda} \otimes f_{\mu}\|_{\alpha} \leq MC^2$ , and with the product direction is a left approximate identity in  $A_1 \otimes A_2$  (under  $\|\cdot\|_{\alpha}$ ). Since  $A_1 \otimes A_2$  is dense in  $A_1 \otimes_{\alpha} A_2$  it follows easily that  $\{e_{\lambda} \otimes f_{\mu}\}$  is a bounded left approximate identity in  $A_1 \otimes_{\alpha} A_2$ .

Conversely, let  $\{d_{\rho}\}$  be a left approximate identity in  $A_1 \otimes_{\alpha} A_2$ . Let  $F$  be a finite subset of  $A_1$ ,  $K = \max\{\|s\|_1 : s \in F\} + 1$ ,  $\delta > 0$ , and take  $t \in A_2$ ,  $\|t\|_2 = 1$ . Choose  $x \in \{d_{\rho}\}$  such that

$$\|x(s \otimes t) - s \otimes t\|_{\alpha} < \frac{\delta m}{2M}, \quad s \in F, \text{ and then take } \sum u_i \otimes v_i \in A_1 \otimes A_2$$

with  $\left\|x - \sum u_i \otimes v_i\right\|_{\alpha} < \frac{\delta m}{2KM^2}$ . But then  $\left\|\sum u_i s \otimes v_i t - s \otimes t\right\|_{\alpha} < \frac{\delta m}{M}$

for all  $s \in F$ . Proceeding as in Theorem 1 it follows that there is  $u \in A_1$  with  $\|us - s\|_1 < \delta$  for  $s \in F$ . Now proceed as in Proposition 3, but without the element  $z$ , to obtain a net  $\{e_{\lambda}\}$ , consisting of such  $u$ , which is a left approximate identity in  $A_1$ .

Similarly  $A_2$  has a left approximate identity.

**REMARK.** The first half of this result appears known, it is used implicitly in [4], Theorem 2.2. The present author has been unable to determine whether addition of the hypothesis of boundedness of  $\{d_{\rho}\}$  in the converse half would ensure boundedness of the resulting nets  $\{e_{\lambda}\}$ ,  $\{f_{\mu}\}$  in  $A_1, A_2$  respectively. However, if  $A_1$  and  $A_2$  are commutative

and  $\{d_\rho\}$  is countable and unbounded then not both  $\{e_\lambda\}$ ,  $\{f_\mu\}$  are bounded, for otherwise  $\{e_\lambda \otimes f_\mu\}$  is a bounded approximate identity in  $A_1 \otimes_\alpha A_2$ , contradicting [6], p. 279. In the general case, if  $A_1$  and  $A_2$  do not consist entirely of right (left) topological divisors of zero then Proposition 3 shows that they have bounded left (right) approximate identities, and hence so does  $A_1 \otimes_\alpha A_2$ . Thus if  $A_1$  and  $A_2$  are commutative and  $\{d_\rho\}$  is countable, then  $\{d_\rho\}$  is bounded.

### 3. The commutative case

The first result concerning identities in  $A_1 \otimes_\alpha A_2$  was that of Gelbaum [2], Theorem 4, who considered the case  $A_1, A_2$  commutative semisimple, and  $\|\cdot\|_\alpha = \|\cdot\|_\gamma$ . This case is included in Theorem 1 above. Recently Lardy and Lindberg [3] have defined an algebra norm  $\|\cdot\|_\alpha$  on  $A_1 \otimes A_2$  to be a spectral tensor norm in the case  $v_\alpha(u \otimes v) = v_1(u)v_2(v)$ ,  $u \in A_1, v \in A_2$ . They showed that the natural map of  $\Phi_{A_1 \otimes_\alpha A_2}$  into  $\Phi_{A_1} \times \Phi_{A_2}$  is surjective if and only if  $\|\cdot\|_\alpha$  is spectral, and in this case  $A_1 \otimes_\alpha A_2$  has an identity if and only if  $A_1$  and  $A_2$  have identities. In this section we obtain an elementary proof of this result.

LEMMA. *Let  $A_1, A_2$  be commutative Banach algebras,  $\|\cdot\|_\alpha$  an algebra norm on  $A_1 \otimes A_2$ . Then  $\|\cdot\|_\alpha$  is a spectral tensor norm if and only if  $v_\alpha$  is an admissible seminorm on  $A_1 \otimes A_2$ , taking the seminorms  $v_1, v_2$  on  $A_1, A_2$ . Indeed,  $\|\cdot\|_\alpha$  is spectral if and only if  $v_\alpha = v_\lambda$ .*

Proof. Suppose  $\|\cdot\|_\alpha$  is spectral. Then by [3], Theorem 1, every multiplicative linear functional on  $A_1 \otimes A_2$  is  $\|\cdot\|_\alpha$ -continuous, and so if  $x \in A_1 \otimes A_2$ ,

$$\begin{aligned} v_\alpha(x) &= \sup\{|\varphi(x)| : \varphi \in (A_1 \otimes A_2)^*, \varphi \text{ multiplicative}\} \\ &= \sup\{|\varphi \otimes \psi(x)| : (\varphi, \psi) \in \Phi_{A_1} \times \Phi_{A_2}\} \\ &\leq v_\lambda(x). \end{aligned}$$

Now let  $x = \sum u_i \otimes v_i$ , and take  $\epsilon > 0$ . Then there are  $\phi_j \in A_j^*$ ,  $\|\phi_j\|_{v_j} = 1$ ,  $j = 1, 2$ , with  $v_\lambda(x) \leq \left| \sum \phi_1(u_i)\phi_2(v_i) \right| + \epsilon$ . Since  $|\phi_1(u)| \leq v_1(u)$ ,  $u \in A_1$ , there is  $\phi_1' \in \Phi_{A_1}$  with  $\left| \phi_1' \left( \sum u_i \phi_2(v_i) \right) \right| \geq \left| \phi_1 \left( \sum u_i \phi_2(v_i) \right) \right|$  so that  $v_\lambda(x) \leq \left| \sum \phi_1'(u_i)\phi_2(v_i) \right| + \epsilon$ . Similarly, there is  $\phi_2' \in \Phi_{A_2}$  with  $v_\lambda(x) \leq \left| \sum \phi_1'(u_i)\phi_2'(v_i) \right| + \epsilon \leq v_\alpha(x) + \epsilon$ . Since  $\epsilon > 0$  is arbitrary it follows that  $v_\lambda(x) \leq v_\alpha(x)$ . Thus  $v_\alpha = v_\lambda$  and so  $v_\alpha$  is certainly admissible.

Suppose conversely that  $v_\alpha$  is admissible, with  $mv_\lambda \leq v_\alpha$ , and that  $\|\cdot\|_\alpha$  is not spectral. Since  $v_\alpha(u \otimes v) \leq v_1(u)v_2(v)$  for all  $u \in A_1$ ,  $v \in A_2$ , there must be  $u, v$  with  $v_\alpha(u \otimes v) \leq kv_1(u)v_2(v)$  for some  $k$ ,  $0 < k < 1$ . But then

$v_\alpha(u^n \otimes v^n) \leq k^n v_1(u^n)v_2(v^n) = k^n v_\lambda(u^n \otimes v^n) \leq \frac{k^n}{m} v_\alpha(u^n \otimes v^n)$ , which is impossible for  $n$  sufficiently large. Thus  $\|\cdot\|_\alpha$  is spectral, and so by the above  $v_\alpha = v_\lambda$ .

**THEOREM 3** (Lardy and Lindberg). *Let  $A_1, A_2$  be commutative Banach algebras,  $\|\cdot\|_\alpha$  an algebra norm on  $A_1 \otimes A_2$ . If  $\|\cdot\|_\alpha$  is a spectral tensor norm then  $A_1 \otimes_\alpha A_2$  has an identity if and only if  $A_1$  and  $A_2$  have identities.*

*Proof.* Suppose  $A_1 \otimes_\alpha A_2$  has an identity, the converse being immediate. Arguing as in Theorem 1, but using  $v_1, v_2, v_\alpha$  in place of  $\|\cdot\|_1, \|\cdot\|_2$  and  $\|\cdot\|_\alpha$ , there is  $\{e_n\} \subseteq A_1$  with  $v_1(e_n s - s) \rightarrow 0$  uniformly for  $v_1(s) \leq 1$ . But then by Proposition 2 there is an idempotent  $e \in A_1$  with  $\hat{e} \equiv 1$ . Similarly there is an idempotent  $f \in A_2$  with  $\hat{f} \equiv 1$ . The result now follows as in [3], Theorem 4.

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