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FREE INVOLUTORIAL COMPLETELY SIMPLE SEMIGROUPS

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1. Introduction and summary. An *involution* $x \to x^*$ of a semigroup S is an antiautomorphism of S of order at most 2, that is $(xy)^* = y^*x^*$ and $x^{**} = x$ for all $x, y \in S$. In such a case, S is called an *involutorial semigroup* if regarded as a universal algebra with the binary operation of multiplication and the unary operation *. If S is also a completely simple semigroup, regarded as an algebra with multiplication and the unary operation $x \to x^{-1}$ of inversion (x^{-1} is the inverse of x in the maximal subgroup of S containing x), then $(S, {}^{-1}, {}^*)$, or simply S, is an *involutorial completely simple semigroup*. All such S form a variety determined by the identities above concerning * and

$$x = xx^{-1}x,\ xx^{-1} = x^{-1}x,\ (x^{-1})^{-1} = x,\ x^0 = (xyx)^0$$

where $x^0 = xx^{-1}$.

Involutorial semigroups satisfying the identity $x = xx^*x$ are called *regular involutorial semigroups*. Their study was essentially started by Nordahl and Scheiblich [**5**] even though they have appeared earlier in several places, notably in regular rings. The second author [**6**] initiated the study of regular involutorial semigroups which are also completely regular semigroups; this includes, in particular, regular involutorial semigroups which are completely simple. Involutions appear in many situations in semigroup theory but are usually not considered as a separate operation on the semigroup.

We arrive here at a Rees matrix representation of the free involutorial completely simple semigroup on a set X by means of the Rees matrix representation of the free completely simple semigroup on a set of double the size of X. As a homomorphic image of this semigroup, we obtain a copy of the free regular involutorial completely simple semigroup on X (where regularity means that the identity $x = xx^*x$ is satisfied).

Section 2 contains a discussion of Rees matrix semigroups, including a new type of normalization of the sandwich matrix and the standard form for an involutorial semigroup. A construction of the free involutorial semigroup belonging to certain classes of semigroups is described in Section 3. This is applied in Section 4 to the class of completely simple semigroups and the corresponding involution is made explicit; the

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standard form of this Rees matrix semigroup is also presented here. A different Rees matrix representation of our semigroup is constructed in Section 5. A family of identities on involutorial completely simple semigroups is characterized in Section 6 in terms of involutorial Rees matrix semigroups in standard form. The free regular involutorial completely simple semigroup is constructed in Section 7 as a quotient of the free involutorial completely simple semigroup by a well-determined congruence. The concluding Section 8 contains a review of the semigroups considered as well as a derivation of the word problem for several free-type semigroups discussed earlier.

2. Rees matrix semigroups. The purpose of this section is to establish a number of statements concerning Rees matrix semigroups (over groups) which will be needed in the main body of the paper. They pertain to a new kind of normalization of the sandwich matrix and a form of involution of a Rees matrix semigroup. We start with a well-known result. For a proof, consult ([**2**], Theorem 3.11).

LEMMA 2.1. *Let* $S = \mathscr{M}(I, G, \Lambda; P)$ *and* $T = \mathscr{M}(J, H, M; Q)$. *Let* $\xi: I \to J$, $u: \mathrm{I} \to \mathrm{H}$, $\omega: G \to H$, $v: \Lambda \to \mathrm{H}$, $\eta: \Lambda \to M$ *be functions where* ω *is a homomorphism,* $u: i \to u_i$, $v: \lambda \to v_\lambda$ *satisfying*

(1) $\quad p_{\lambda i}\omega = v_\lambda q_{\lambda\eta, i\xi} u_i \quad (i \in I, \lambda \in \Lambda).$

Then the mapping

(2) $\quad \chi:(i, g, j) \to (i\xi, u_i(g\omega)v_\lambda, \lambda\eta)$

is a homomorphism of S into T. Conversely, every homomorphism of S into T can be so constructed. Moreover, χ *is a bijection if and only if* ξ, ω *and* η *are bijections.*

As a generalization of the normalization of the sandwich matrix of a Rees matrix semigroup, we offer the following result.

LEMMA 2.2. *Let*

$$S = \mathscr{M}(I, G, \Lambda; P),\ I = \bigcup_{\alpha \in A} I_\alpha,\ \Lambda = \bigcup_{\alpha \in A} \Lambda_\alpha$$

where the I_α *and the* Λ_α *are pairwise disjoint. Fix* $\kappa \in A$ *and for each* $\alpha \in A$, *fix an element in* I_α *and* Λ_α *and denote it by* α. *Define*

$$q_{\lambda i} = p_{\kappa\kappa}^{-1} p_{\kappa\beta} p_{\lambda\beta}^{-1} p_{\lambda i} p_{\alpha i}^{-1} p_{\alpha\alpha} p_{\kappa\alpha}^{-1} p_{\kappa\kappa} \quad (i \in I_\alpha, \lambda \in \Lambda_\beta, \alpha, \beta \in A),$$

and $T = \mathscr{M}(I, G, \Lambda; Q)$ *where* $Q = (q_{\lambda i})$. *Then the mapping*

$$\psi:(i, g, \lambda) \to (i, p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\alpha\alpha}^{-1} p_{\alpha i} g p_{\lambda\beta} p_{\kappa\beta}^{-1} p_{\kappa\kappa}, \lambda)$$

is an isomorphism of S onto T. The matrix Q has the properties:

$$\begin{aligned} q_{\alpha i} = q_{\lambda\alpha} &= 1 \quad \textit{if } i \in I_\alpha, \lambda \in \Lambda_\alpha, \alpha \in A, \\ q_{\kappa\alpha} &= 1 \quad \textit{for all } \alpha \in A. \end{aligned}$$

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Proof. Letting

$$u_i = p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\alpha\alpha}^{-1} p_{\alpha i} \quad \text{and} \quad v_\lambda = p_{\lambda\beta}\, p_{\kappa\beta}^{-1} p_{\kappa\kappa},$$

where $i \in I_\alpha$, $\lambda \in \Lambda_\beta$, to show that ψ is an isomorphism, it suffices to show that

$$p_{\lambda i} = v_\lambda q_{\lambda i} u_i.$$

Indeed,

$$\begin{aligned} v_\lambda q_{\lambda i} u_i &= (p_{\lambda\beta} p_{\kappa\beta}^{-1} p_{\kappa\kappa})(p_{\kappa\kappa}^{-1} p_{\kappa\beta} p_{\lambda\beta}^{-1} p_{\lambda i} p_{\alpha i}^{-1} p_{\alpha\alpha} p_{\kappa\alpha}^{-1} p_{\kappa\kappa}) \\ &\quad \times (p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\alpha\alpha}^{-1} p_{\alpha i}) = p_{\lambda i} \end{aligned}$$

as required. Further, for $i \in I_\alpha$,

$$q_{\alpha i} = p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\alpha\alpha}^{-1} p_{\alpha i} p_{\alpha i}^{-1} p_{\alpha\alpha} p_{\kappa\alpha}^{-1} p_{\kappa\kappa} = 1,$$

and for $\lambda \in \Lambda_\alpha$

$$q_{\lambda\alpha} = p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\lambda\alpha}^{-1} p_{\lambda\alpha} p_{\alpha\alpha}^{-1} p_{\alpha\alpha} p_{\kappa\alpha}^{-1} p_{\kappa\kappa} = 1,$$

and

$$q_{\kappa\alpha} = p_{\kappa\kappa}^{-1} p_{\kappa\kappa} p_{\kappa\kappa}^{-1} p_{\kappa\alpha} p_{\alpha\alpha}^{-1} p_{\alpha\alpha} p_{\kappa\alpha}^{-1} p_{\kappa\kappa} = 1,$$

as required.

As a special case of the above lemma, we record the following well-known result.

COROLLARY 2.3. *Let $S = \mathscr{M}(I, G, \Lambda; P)$. Fix an element both in I and Λ and denote it by z. Define*

$$q_{\lambda i} = p_{\lambda z}^{-1} p_{\lambda i} p_{zi}^{-1} p_{zz} \quad (i \in I, \lambda \in \Lambda),$$

and $T = \mathscr{M}(I, G, \Lambda; Q)$ where $Q = (q_{\lambda i})$. Then the mapping

$$\chi:(i, g, \lambda) \rightarrow (i, p_{zz}^{-1} p_{zi} g p_{\lambda z}, \lambda)$$

is an isomorphism of S onto T. The matrix Q has the property

$$q_{\lambda z} = q_{zi} = 1 \quad \textit{for all } i \in I, \lambda \in \Lambda,$$

that is, Q is normalized at z.

Observe that the Rees matrix representation of the free product of Rees matrix semigroups with normalized sandwich matrices within the class of completely simple semigroups constructed by Jones [**4**] is of the general form described in Lemma 2.2.

If the Rees matrix semigroup $\mathscr{M}(I, G, \Lambda; P)$ admits an antiautomorphism, then one can see easily that I and Λ must have the same cardinality. Thus there is no loss of generality if we assume that $I = \Lambda$, which we will do throughout the paper. Antiautomorphisms of a Rees matrix semigroup are described in the next result; for a proof, consult ([**6**], Theorem 3.2).

LEMMA 2.4. *Let* $S = \mathscr{M}(I, G, I; P)$, ξ *and* η *be permutations of* I, $u:i \rightarrow u_i$, $v:i \rightarrow v_i$ *be functions mapping* I *into* G *and* ω *be an antiautomorphism of* G *satisfying*

$$(1) \quad p_{ji}\omega = u_i p_{i\xi, j\eta} v_j \quad (i, j \in I).$$

Then the mapping

$$(2) \quad \theta:(i, g, j) \rightarrow (j\eta, v_j(g\omega)u_i, i\xi)$$

is an antiautomorphism of S. *Conversely, every antiautomorphism of* S *can be so constructed.*

Based on the above lemma, the next result describes the form of an arbitrary involution on a Rees matrix semigroup with normalized sandwich matrix. For g an element of a group G, let

$$\epsilon_g:x \rightarrow g^{-1}xg \quad (x \in G).$$

LEMMA 2.5. *Let* $S = \mathscr{M}(I, G, I; P)$ *where* P *is normalized at* z, ξ *be a permutation of* I, $i \rightarrow u_i$ *be a function mapping* I *into* G *and* ω *be an antiautomorphism of* G. *Let* $u = u_z$, $c = u^{-1}(u\omega)$ *and assume that*

$$(1) \quad \omega^2 = \epsilon_c, \ p_{ji}\omega = u_i p_{i\xi, j\xi^{-1}} c(u_{j\xi^{-1}}^{-1}\omega) \quad (i, j \in I).$$

Then the mapping

$$(2) \quad \theta:(i, g, j) \rightarrow (j\xi^{-1}, c(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i, i\xi)$$

is an involution of S. *Conversely, every involution of* S *can be so constructed.*

Proof. (Direct part.) We put $\eta = \xi^{-1}$ and $v_i = c(u_{i\xi}^{-1}\omega)$ in Lemma 2.4 and see that θ in the statement of the lemma is indeed an antiautomorphism of S. We compute

$$\begin{aligned}
(i, g, j)\theta\theta &= (j\xi^{-1}, c(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i, i\xi)\theta \\
&= (i, c(u_i^{-1}\omega)[c(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i]\omega u_{j\xi^{-1}}, j) \\
&= (i, c(u_i^{-1}\omega)(u_i\omega)(g\omega^2)(u_{j\xi^{-1}}^{-1}\omega^2)(c\omega)u_{j\xi^{-1}}, j) \\
&= (i, c(c^{-1}gc)(c^{-1}u_{j\xi^{-1}}^{-1}c)(c\omega)u_{j\xi^{-1}}, j) \\
&= (i, gu_{j\xi^{-1}}^{-1}c(c\omega)u_{j\xi^{-1}}, j)
\end{aligned}$$

which together with

$$c(c\omega) = c(u\omega^2)(u^{-1}\omega) = c(c^{-1}uc)(u^{-1}\omega) = uu^{-1}(u\omega)(u^{-1}\omega) = 1$$

proves that θ^2 is the identity mapping.

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(Converse.) Let θ be an involution of S given as in Lemma 2.4. Since θ is of order at most 2, we compute

$$\begin{aligned}(i, g, j)\theta\theta &= (j\eta, v_j(g\omega)u_i, i\xi)\theta \\ &= (i\xi\eta, v_{i\xi}[v_j(g\omega)u_i]\omega u_{j\eta}, j\eta\xi) = (i, g, j)\end{aligned}$$

which implies $\eta = \xi^{-1}$ and

(3) $\quad v_{i\xi}(u_i\omega)(g\omega^2)(v_j\omega)u_{j\xi^{-1}} = g.$

For $g = 1$, we obtain

$$v_{i\xi}(u_i\omega) = u_{j\xi^{-1}}^{-1}(v_j^{-1}\omega),$$

where the left hand side depends only upon i and the right hand side depends only upon j, so we have that

(4) $\quad c = v_{i\xi}(u_i\omega) = u_{j\xi^{-1}}^{-1}(v_j^{-1}\omega)$

is a constant. It follows that $v_{i\xi} = c(u_i^{-1}\omega)$ and writing i for $i\xi$, this gives

(5) $\quad v_i = c(u_{i\xi^{-1}}^{-1}\omega).$

Now formula (1) of Lemma 2.4 gives

$$p_{ji}\omega = u_i p_{i\xi, j\xi^{-1}} c(u_{j\xi^{-1}}^{-1}\omega).$$

For $i = z$ and $j = z\xi$, this gives $1 = uc(u^{-1}\omega)$ and thus $c = u^{-1}(u\omega)$. Now (4) substituted in (3) gives

$$c(g\omega^2)c^{-1} = g \quad \text{whence } \omega^2 = \epsilon_c.$$

Taking into account (5), we see that Lemma 2.4 implies the assertion of the present lemma.

If an involution is given on a Rees matrix semigroup, we can change the Rees matrix representation in such a way that the involution takes on a very simple form. This we do in the next result.

THEOREM 2.6. *Let* $S = \mathscr{M}(I, G, I; P)$ *with* P *normalized at* z. *If* τ *is an involution of* G *satisfying*

(1) $\quad p_{ji}\tau = p_{ij} \quad (i, j \in I),$

then the mapping

(2) $\quad \bar{\tau}:(i, g, j) \to (j, g\tau, i) \quad ((i, g, j) \in S)$

is an involution of S.

Conversely, let θ *be an involution of* S *with parameters as in Lemma* 2.5. *Let* $Q = (q_{ji})$ *where*

(3) $\quad q_{ji} = p_{j\xi,i}p_{z\xi,i}^{-1} \quad (i, j \in I),$

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and let

(4) $\tau = \omega\epsilon_u$.

Then τ *is an involution of* G, Q *is normalized at* z *and* $q_{ji}\tau = q_{ij}$ *for all* $i, j \in I$. *Let*

$$T = \mathscr{M}(I, G, I; Q)$$

and define $\bar{\tau}$ *on* T *by* (2) *so that* $\bar{\tau}$ *is an involution of* T. *Then the mapping*

(5) $\chi:(i, g, j) \rightarrow (i, p_{z\xi,i}g, j\xi^{-1}) \quad ((i, g, j) \in S)$

is an isomorphism of S *onto* T *such that* $\theta\chi = \chi\bar{\tau}$.

Proof. The proof of the direct part consists of a straightforward verification.

For the converse, let θ be an involution of S given by the parameters as in Lemma 2.5. Letting τ be as in (4), we obtain

$$g\tau^2 = g\omega\epsilon_u\omega\epsilon_u = u^{-1}(u^{-1}(g\omega)u)\omega u = u^{-1}(u\omega)(g\omega^2)(u^{-1}\omega)u$$
$$= c(g\omega^2)c^{-1} = g$$

and τ is an antiautomorphism and so an involution. Clearly Q is normalized at z. From (1) of Lemma 2.5, we get

(6) $p_{ji}\omega = u_i p_{i\xi,j\xi^{-1}} u^{-1}(u\omega)(u_{j\xi^{-1}}^{-1}\omega)$.

Writing z for i and j for $j\xi^{-1}$ in (6), we get

$$1 = up_{z\xi,j}u^{-1}(u\omega)(u_j^{-1}\omega)$$

which gives

(7) $p_{z\xi,j} = u^{-1}(u_j\omega)(u^{-1}\omega)u$.

Now

$$q_{ji}\tau = (p_{j\xi,i}p_{z\xi,i}^{-1})\omega\epsilon_u$$

by (3) and (4)

$$= u^{-1}(p_{z\xi,i}\omega)^{-1}(p_{j\xi,i}\omega)u$$
$$= u^{-1}[u_i p_{i\xi,z}u^{-1}(u\omega)(u^{-1}\omega)]^{-1}[u_i p_{i\xi,j}u^{-1}(u\omega)(u_j^{-1}\omega)]u$$

by (6)

$$= p_{i\xi,j}u^{-1}(u\omega)(u_j^{-1}\omega)u$$
$$= p_{i\xi,j}p_{z\xi,j}^{-1} = q_{ij}$$

by (7) and (3)
which verifies (1) for q_{ji}.

To see that χ is an isomorphism, in Lemma 2.1, we let ξ be the identity mapping, $\eta = \xi^{-1}$, $u_i = p_{z\xi,i}$, ω the identity mapping and $v_j = 1$. In view of (6), we see that (1) in Lemma 2.1 is satisfied and thus χ is a homomorphism, and hence an isomorphism.

It remains to verify that χ respects the involutions. On the one hand,

$$(8)\quad (i, g, j)\chi\bar{\tau} = (i, p_{z\xi,i}g, j\xi^{-1})\bar{\tau} = (j\xi^{-1}, (p_{z\xi,i}g)\tau, i)$$
$$= (j\xi^{-1}, (p_{z\xi,i}g)\omega\epsilon_u, i) = (j\xi^{-1}, u^{-1}(g\omega)(p_{z\xi,i}\omega)u, i)$$

and on the other hand,

$$(9)\quad (i, g, j)\theta\chi = (j\xi^{-1}, u^{-1}(u\omega)(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i, i\xi)\chi$$

by (2) of Lemma 2.5

$$= (j\xi^{-1}, p_{z\xi,j\xi^{-1}}u^{-1}(u\omega)(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i, i)$$
$$= (j\xi^{-1}, u^{-1}(u_{j\xi^{-1}}\omega)(u^{-1}\omega)uu^{-1}(u\omega)(u_{j\xi^{-1}}^{-1}\omega)(g\omega)u_i, i)$$

by (7)

$$= (j\xi^{-1}, u^{-1}(g\omega)u_i, i).$$

Further

$$p_{z\xi,i}\omega = [u^{-1}(u_i\omega)(u^{-1}\omega)u]\omega$$

by (7)

$$= (u\omega)(u^{-1}\omega^2)(u_i\omega^2)(u^{-1}\omega)$$
$$= (u\omega)[\,(u^{-1}\omega)uu^{-1}u^{-1}(u\omega)\,][\,(u^{-1}\omega)uu_iu^{-1}(u\omega)\,](u^{-1}\omega)$$
$$= u_iu^{-1}$$

which implies that the expressions in (8) and (9) are equal. Consequently $\theta\chi = \chi\bar{\tau}$.

This result suggests the following concept.

Definition 2.7. For $S = \mathscr{M}(I, G, I; P)$ and τ an involution of G, the involution

$$\bar{\tau}:(i, g, j) \rightarrow (j, g\tau, i) \quad ((i, g, j) \in S)$$

is said to be in *standard form*. The pair $(S, \bar{\tau})$ will be denoted by $\mathscr{M}(G, I; P, \tau)$ and will be said to be in *standard form*. If $g\tau = g^{-1}$ for all $g \in G$, we write $\mathscr{M}(G, I; P)$.

As a consequence of the above theorem, we have

COROLLARY 2.8. *Every involutorial completely simple semigroup is isomorphic to one in standard form.*

The following lemma will be useful.

LEMMA 2.9. *If a is an element of an involutorial completely simple semigroup, then letting $a^0 = aa^{-1}$,*

$$(a^{-1})^* = (a^*)^{-1}, \quad (a^0)^* = (a^*)^0.$$

Proof. This follows by simple verification in a Rees matrix semigroup in standard form.

3. Free involutorial semigroups. Let C be a semigroup. The *opposite* of C, C^{op}, is the semigroup $C^{\mathrm{op}} = (C, \circ)$ where $a \circ b = ba$ for all $a, b \in C$. Let $\mathscr{C}$ be a class of semigroups such that $C \in \mathscr{C}$ implies $C^{\mathrm{op}} \in \mathscr{C}$. Assume also that $\mathscr{C}$ contains free objects (that is $F\mathscr{C}(X)$ exists for all X). Let $X \neq \emptyset$, let X^* be a set disjoint from X and in a one-to-one correspondence with X, the correspondence being given by $x \to x^*$. Let $I = X \cup X^*$. Since $F\mathscr{C}(I)$ exists, and $F\mathscr{C}(I)^{\mathrm{op}} \in \mathscr{C}$, the map $I \to F\mathscr{C}(I)^{\mathrm{op}}$ obtained by $x \to x^*$ and $x^* \to x$ extends (uniquely) to an isomorphism of $F\mathscr{C}(I)$ onto $F\mathscr{C}(I)^{\mathrm{op}}$. The extended map is thus an antiautomorphism of $F\mathscr{C}(I)$ which is clearly of order 2 and so an involution of $F\mathscr{C}(I)$.

The class $\mathscr{C}^*$ of *involutorial $\mathscr{C}$-semigroups* is defined by

$$\mathscr{C}^* = \{ (C, *) \mid C \in \mathscr{C} \text{ and } * \text{ is an involution of } C \}.$$

The class $\mathscr{C}^*$ is therefore a class of universal algebras with a binary and a unary operation. The binary operation is associative and the unary operation satisfies $x^{**} = x$ and $(xy)^* = y^*x^*$.

THEOREM 3.1. *The involutorial semigroup $(F\mathscr{C}(I), *)$ with $*$ defined above is free on X in $\mathscr{C}^*$. In symbols*

$$F\mathscr{C}^*(X) \cong (F\mathscr{C}(I), *).$$

Proof. Let $(C, *) \in \mathscr{C}^*$. We will show that the diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\varphi"'] & (F\mathscr{C}(I), *) \arrow[d, dashed, "\bar{\varphi}"] \arrow[dl, phantom, "\oplus"] \\ & (C, *) \end{tikzcd}$$

can be completed with a homomorphism $\bar{\varphi}$ which respects both $\cdot$ and $*$. Since X clearly generates $F\mathscr{C}(I)$ (as an involutorial semigroup) this will prove that $(F\mathscr{C}(I), *)$ is free in $\mathscr{C}^*$ on X.

Consider the diagram:

$$\begin{tikzcd} I = X \cup X^* \arrow[r] \arrow[dr, "\varphi \cup \varphi^*"'] & F\mathscr{C}(I) \arrow[d, dashed, "\psi"] \arrow[dl, phantom, "\oplus"] \\ & C \end{tikzcd}$$

where the map $\varphi^*:X^* \to C$ is given by $x^* \to (x\varphi)^*$. Since $F\mathscr{C}(I)$ is free on I in $\mathscr{C}$, there is a homomorphism ψ which completes the diagram. This construction provides us only with a semigroup homomorphism but we will prove that it also respects *. We show that $u^*\psi = (u\psi)^*$ for all $u \in F\mathscr{C}(I)$. It is enough to show that

$$x^*\psi = x^*\varphi^* = (x\varphi)^* = (x\psi)^*,$$

$$x\psi = x\varphi = ((x\varphi)^*)^* = (x^*\varphi^*)^* = (x^*\psi)^*.$$

Therefore ψ respects * and we can take $\bar{\varphi} = \psi$.

4. Free involutorial completely simple semigroups. A matrix representation of the free completely simple semigroup on $I = X \cup X^*$, $F\mathscr{C}\mathscr{S}(I)$, was given by Clifford [**1**] and Rasin [7] as follows. Let $z \in I$, let G be the free group on

$$I \cup \{p_{ij} | i, j \in I, i \neq z \neq j\}$$

where this is a disjoint union, and let $p = (p_{ij})$ where $p_{zi} = p_{jz} = 1 \in G$. Then

$$S = \mathscr{M}(I, G, I; P) \cong F\mathscr{C}\mathscr{S}(I).$$

The isomorphism is given by extending $i \to (i, i, i)$ for $i \in I$. The involution of $F\mathscr{C}\mathscr{S}(I)$ given in Section 3 can be transported by the isomorphism to an involution of S. On the generators of S this involution is given by

$$(i, i, i) \to (i^*, i^*, i^*).$$

THEOREM 4.1. *Let ω be the antihomomorphism of G defined by extending the following map on the free generators of G:*

$$i \to p_{z^*i^*}i^*p_{i^*z^*}p_{z^*z^*}^{-1},$$

$$p_{ij} \to p_{z^*z^*}p_{j^*z^*}^{-1}p_{j^*i^*}p_{z^*i^*}^{-1}.$$

Then ω is an antiautomorphism of G and the involution of S described above is given by

$$(i, g, j)^* = (j^*, p_{z^*j^*}^{-1}(g\omega)p_{z^*z^*}p_{i^*z^*}^{-1}, i^*).$$

Proof. We will apply Lemma 2.5 to show that $(i, g, j) \to (i, g, j)^*$ is an involution of S. In the notation of Lemma 2.5 let

$$u_i = p_{z^*z^*}p_{i^*z^*}^{-1}$$

and let

$$i\xi = i^* \quad \text{for all } i \in I;$$

then $u = u_z = 1$ and therefore $c = 1$. Then

$$\begin{aligned}p_{ij}\omega^2 &= (p_{z^*z^*}p_{j^*z^*}^{-1}p_{j^*i^*}p_{z^*i^*}^{-1})\omega\\ &= (p_{z^*i^*}\omega)^{-1}(p_{j^*i^*}\omega)(p_{j^*z^*}\omega)^{-1}(p_{z^*z^*}\omega)\\ &= (p_{z^*z^*}p_{iz^*}^{-1}p_{iz}p_{z^*z}^{-1})^{-1}(p_{z^*z^*}p_{iz^*}^{-1}p_{ij}p_{z^*j}^{-1})\\ &= (p_{z^*z^*}p_{zz^*}^{-1}p_{zj}p_{z^*j}^{-1})^{-1}(p_{z^*z^*}p_{zz^*}^{-1}p_{zz}p_{z^*z}^{-1})\\ &= p_{ij}.\end{aligned}$$

A similar calculation shows that $i\omega^2 = i$. Therefore $\omega^2 = \epsilon_1$, the identity map on G. In particular, ω is a permutation of G. We can also show that $u_{j^*}\omega = p_{z^*j^*}$. Therefore

$$p_{ij}\omega = u_i p_{i^*j^*}(u_{j^*}^{-1}\omega).$$

These are the conditions necessary to apply Lemma 2.5.

It is also straightforward to check that

$$(i, i, i)^* = (i^*, i^*, i^*).$$

Since the involution of the theorem agrees with the involution inherited from $(F\mathscr{C}\mathscr{S}(I), {}^*)$ on $\{(i, i, i)\,|i \in I\}$, and $F\mathscr{C}\mathscr{S}(I)$ is free, the involutions coincide.

THEOREM 4.2. (Standardization). *Let S be defined as above, let*

$$r_{ij} = p_{i^*j}p_{z^*j}^{-1} \quad (i, j \in I)$$

and let $V = \mathscr{M}(G, I; R, \omega)$ where $R = (r_{ij})$. Then

$$\chi:(i, g, j) \rightarrow (i, p_{z^*i}g, j^*)$$

*is a *-isomorphism of S onto V.*

In addition, the set $I \cup \{r_{ij}|i \neq z \neq j\}$ freely generates G, the involution of V has the form

$$(i, g, j) \rightarrow (j, g\omega, i)$$

and ω is the involution of G whose action on generators is given by

$$i \rightarrow r_{z^*i}^{-1}i^*r_{iz^*},$$

$$r_{ij} \rightarrow r_{ji}.$$

Proof. The isomorphism χ is obtained by specializing Theorem 2.6 and using the parameters in the proof of Theorem 4.1. It is also straightfoward to show that

$$p_{ij} = r_{i^*j}r_{z^*j}^{-1}.$$

In particular, the set $I \cup \{r_{ij}|i \neq z \neq j\}$ generates G.

To show that the set $I \cup \{r_{ij} | i \neq z \neq j\}$ freely generates G let

$$r_k = r_{ij}^{\epsilon} \quad \text{for some } i, j \in I \backslash \{z\} \text{ and } \epsilon \in \{-1, 1\}.$$

We show that if $r_1 \ldots r_n = 1$, then

$$r_k r_{k+1} = 1 \quad \text{for some } 1 \leqq k < n.$$

Assume that n is the smallest integer such that

$$r_1 \ldots r_n = 1 \text{ and } r_k r_{k+1} \neq 1 \text{ for all } 1 \leqq k < n.$$

Of course $n \geqq 2$ since $r_1 \neq 1$. Assume

$$r_1 = p_{j^*i} p_{z^*i}^{-1}.$$

(The proof in the other case is similar.) Since $r_1 \ldots r_n = 1, p_{z^*i}$ must occur in $r_2 \ldots r_n$ which means by inspection that

$$r_k = p_{z^*i} p_{u^*i}^{-1} \text{ for some } k \geqq 2 \text{ and some } u.$$

If r_k contains the p_{z^*i} which occurs in the reduction of $r_1 \ldots r_n$ to 1 using the freeness of the p_{ij}'s, then $r_2 \ldots r_{k-1} = 1$. This is impossible since $r_1 \ldots r_n$ is the shortest product with the stated properties. Therefore $k = 2$. If $n = 2$, then $u^* = j^*$ and $r_1 = r_2^{-1}$, again a contradiction. If $n > 2$ then

$$r_3 = p_{u^*i} p_{z^*i^*}^{-1}$$

(by the same kind of argument which showed that $r_2 = p_{z^*i} p_{u^*i}^{-1}$). But then $r_2 = r_3^{-1}$. This contradiction completes the proof by showing that $r_1 \ldots r_n = 1$ and $r_k r_{k+1} \neq 1$ for all $1 \leqq k < n$ is impossible. Using Theorem 4.1, we obtain

$$\begin{aligned} i\omega &= p_{z^*i^*} i^* p_{i^*z^*} p_{z^*z^*}^{-1} = r_{zi^*} r_{z^*i^*}^{-1} i^* r_{iz^*} r_{z^*z^*}^{-1} (r_{zz^*} r_{z^*z^*}^{-1})^{-1} \\ &= r_{z^*i^*}^{-1} i^* r_{iz^*}, \\ r_{ij}\omega &= (p_{i^*j} p_{z^*j}^{-1})\omega = (p_{z^*j}\omega)^{-1} (p_{i^*j}\omega) \\ &= (p_{z^*z^*} p_{j^*z^*}^{-1} p_{j^*z} p_{z^*z}^{-1})^{-1} (p_{z^*z^*} p_{j^*z^*}^{-1} p_{j^*i} p_{z^*i}^{-1}) \\ &= p_{j^*i} p_{z^*i}^{-1} = r_{ji} \end{aligned}$$

and thus ω has the form as in the statement of the theorem.

5. Another Rees matrix representation of $F\mathscr{CS}^*(X)$. Lemma 2.2 asserts that every Rees matrix semigroup has a representation in which the matrix is "normalized" in a special way. The particular "normalization", which may be termed "local" in view of its form, appears in the Rees matrix representation of the free product of Rees matrix semigroups in the category of completely simple semigroups constructed by Jones [**4**]. Recalling that $I = X \cup X^*$ and $z \in X$ is fixed, we say that the Rees

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matrix semigroup $\mathscr{M}(I, G, I; Q)$ has a matrix *locally normalized at* z, z^* if

$$q_{xz} = q_{zx} = q_{x^*z^*} = q_{z^*x^*} = q_{zz^*} = 1 \quad \text{for all } x \in X.$$

We provide here such a representation for $F\mathscr{C}\mathscr{S}^*(X)$. The corresponding involution is similar to that given in Theorem 4.1. The copy of $F\mathscr{C}\mathscr{S}^*(X)$ so obtained seems a natural one as it exhibits the interchange of x and x^* under the involution better than the standard representation and retains the simplicity of its group mapping. Our approach consists of first producing such a semigroup and then proving that it is isomorphic to S of Theorem 4.1.

LEMMA 5.1. *Let $Q = (q_{ij})$, $i, j \in I = X \cup X^*$ be locally normalized at z, z^*. Let H be the free group on the set*

$$I \cup (\{q_{ij} | i, j \in I\} \backslash \{q_{zx}, q_{xz}, q_{x^*z^*}, q_{z^*x^*}, q_{zz^*} | x \in X\}).$$

Set

(1) $d_{ij} = q_{iz}^{-1} q_{ij} q_{zj}^{-1}$.

Then H is the free group on $I \cup \{d_{ij} | i \neq z \neq j\}$.

Proof. Note that $d_{zi} = d_{jz} = 1$. We first show that the set

$$I \cup \{d_{ij} | i \neq z \neq j\}$$

generates H by showing the q_{ij} can be written as a product of d_{ij}'s for all i, j. Equation (1) can be rewritten to give

(2) $q_{ij} = q_{iz} d_{ij} q_{zj}$.

It is therefore enough to express q_{iz} and q_{zj} in terms of the d_{ij}'s. By direct substitution in (1) we can show that

$$q_{x^*z} = d_{x^*z^*}^{-1}.$$

It is also easy to show from (1) that

$$q_{zx^*} = d_{z^*x^*}^{-1} d_{z^*z^*}.$$

Since $q_{xz} = q_{zx} = 1$ we have all possibilities for q_{iz} and q_{zj} needed in (2).

For each $1 \leqq k \leqq m$, let

$$d_k = d_{ij}^{\epsilon} \quad \text{for some } i, j \in I, \epsilon \in \{1, -1\}.$$

In order to show that H is freely generated by $I \cup \{d_{ij} | i \neq z \neq j\}$, it is sufficient to show that if $d_1 \ldots d_n = 1$, then

$$d_k = d_{k+1}^{-1} \text{ for some } k.$$

We can assume that m is the least integer such that $d_1 \ldots d_m = 1$, $d_k \neq 1$ for $1 \leqq k \leqq m$ and $d_k \neq d_{k+1}^{-1}$ for all $1 \leqq k < m$. Note that $m \geqq 2$ since

$d_1 \neq 1$. The following is a list of the possibilities for d_{ij} and d_{ij}^{-1}. For $x, y \in X$,

	d_{ij}	d_{ij}^{-1}
$i = x, j = y$	q_{xy}	q_{xy}^{-1}
$i = x, j = y^*$	$q_{xy^*}q_{zy^*}^{-1}$	$q_{zy^*}q_{xy^*}^{-1}$
$i = x^*, j = y$	$q_{x^*z}^{-1}q_{x^*y}$	$q_{x^*y}^{-1}q_{x^*z}$
$i = x^*, j = y^*$	$q_{x^*z}^{-1}q_{x^*y^*}q_{zy^*}^{-1}$	$q_{zy^*}q_{x^*y^*}^{-1}q_{x^*z}$

We first show that q_{xy} does not occur in $d_1 \dots d_m$. If q_{xy} did occur, then

$$d_1 \dots d_m = Aq_{xy}Bq_{xy}^{-1}C = 1,$$

where A, B, C are possibly empty products of d_k's. The displayed pair q_{xy}, q_{xy}^{-1} is a pair which is cancelled in a reduction of $d_1 \dots d_m$ to 1 using the freeness of the q_{ij}'s. Therefore B is the empty word and

$$d_1 \dots d_m = Aq_{xy}q_{xy}^{-1}C$$

which contradicts the assumption that $d_k \neq d_{k+1}^{-1}$ for all $1 \leqq k < m$.

Let $h(d_k)$ be the first q_{ij} or q_{ij}^{-1} to occur in d_k and let $t(d_k)$ be the last q_{ij} or q_{ij}^{-1} to occur. We show that

$$t(d_k)h(d_{k+1}) = 1 \quad \text{for } 1 \leqq k < m.$$

The proof is by induction on k. Notice that, from the form of these elements, $(t(d_k))^{-1}$ can only occur as $h(d_l)$ and that $(t(d_k))^{-1}$ does not occur in d_k. By the freeness of the q_{ij}'s, it follows that

$$d_1 \dots d_m = At(d_1)B(t(d_1))^{-1}C = 1.$$

The displayed pair $t(d_1)$ and $(t(d_1))^{-1}$ cancel in the reduction of the q_{ij}'s to 1. These remarks show that B is a product of d_k's and therefore is the empty word. Consequently $t(d_1) = h(d_2)^{-1}$.

Assume now that

$$t(d_k)h(d_{k+1}) = 1 \quad \text{for } 1 \leqq k < n < m - 1.$$

To reduce $d_1 \dots d_n$ to 1 as a product of q_{ij}'s we can cancel the pairs $t(d_k)$, $h(d_{k+1})$ for $1 \leqq k < n$. In the resulting product $t(d_n)$ must cancel $h(d_j)$ for some $j \geqq n + 1$. An argument similar to that used for $k = 1$ shows that

$$j = n + 1 \quad \text{and} \quad t(d_n)h(d_{n+1}) = 1.$$

We next show that if d_k is a product of two q_{ij}'s, then d_{k+1} is a product of three q_{ij}'s. Assume d_{k+1} is a product of two q_{ij}'s. First let

$$d_k = q_{xy^*}q_{zy^*}^{-1}.$$

If $k > 1$, then

$$d_{k-1} = q_{zy^*}q_{xy^*}^{-1}$$

since $t(d_{k-1})h(d_k) = 1$. But then $d_{k-1} = d_k^{-1}$, a contradiction. If $k = 1$, then

$$d_2 = q_{zy^*}q_{uy^*}^{-1}.$$

If $m = 2$, then $u = x$ (in order that the q_{ij}'s cancel) and therefore $d_2 = d_1^{-1}$. If $m > 2$ and $u \neq x$, then

$$d_3 = q_{uy^*}q_{zy^*}^{-1} \quad \text{and} \quad d_2 = d_3^{-1}.$$

In either case we have a contradiction. In case

$$d_k = q_{x^*z}^{-1}q_{x^*y} \quad \text{or} \quad d_k = q_{zy^*}q_{xy^*}^{-1}$$

it follows that $d_{k+1} = d_k^{-1}$, a contradiction. Finally if

$$d_k = q_{x^*y}^{-1}q_{x^*z}$$

we proceed as in the first case.

The final step is to show that no d_k is of length three. This will complete the proof since $m \geqq 2$ and d_k and d_{k+1} are not both of length two for any k.

If some d_k is of length three, then either d_k or d_k^{-1} is $q_{x^*z}^{-1}q_{x^*y^*}q_{zy^*}^{-1}$ for some $x, y \in X$. Both $q_{x^*y^*}$ and $q_{x^*y^*}^{-1}$ must occur in $d_1 \dots d_m$, since the q_{ij}'s all cancel in the product. If

$$d_k = q_{x^*z}^{-1}q_{x^*y^*}q_{zy^*}^{-1},$$

for example, then $q_{x^*y^*}^{-1}$ occurs in some d_i and therefore

$$d_i = q_{zy^*}q_{x^*y^*}^{-1}q_{x^*z} = d_k^{-1}.$$

Assume that d_k and d_i in fact contain $q_{x^*y^*}$ and $q_{x^*y^*}^{-1}$, respectively, which cancel in the reduction of the q_{ij}'s to 1. Also let $i > k$. (The proof in the other case is similar.) Now $i \neq k + 1$ since $d_{k+1} \neq d_k^{-1}$. If $i = k + 2$, then

$$h(d_{k+1}) = q_{zy^*} \quad \text{and} \quad t(d_{k+1}) = q_{zy^*}^{-1}$$

which is impossible. Therefore $i \geqq k + 3$. Since no two factors of length two are adjacent, there is at least one d_j of length three between d_k and d_i. In order for the middle element of d_j to cancel in the product of q_{ij}'s, there must be at least two d_j's of length 3 between d_k and d_i. But then by repeating the argument there would have to be infinitely many which is impossible. Consequently no d_k of length three occurs which completes the proof as indicated above.

LEMMA 5.2. *Let H be the free group defined in Lemma* 5.1, *and let τ be the antihomomorphism of H which extends the mapping*

$$x \to q_{zx^*}^{-1} x^* q_{x^*z}^{-1},$$

$$x^* \to q_{z^*x} x q_{xz^*},$$

$$q_{ij} \to q_{j^*i^*}.$$

Then τ is an involution of H and the mapping

$$\hat{\tau}:(i, g, j) \to (j^*, g\tau, i^*)$$

is an involution of $T = \mathscr{M}(I, H, I; Q)$.

Proof. Direct calculation shows that τ is of order two on the generators and therefore on H. Hence τ is a permutation of H and thus an involution. In Lemma 2.5, let $i\xi = i^*$ and $u_i = 1$ for all $i \in I$. Then $c = 1$ and $\omega = \tau$ whence the last assertion follows.

THEOREM 5.3. *Let the notation be as at the beginning of Section* 4 *and in Lemma* 5.2. *Let $\sigma:H \to G$ be the homomorphism obtained by extending $i \to i$, $d_{ij} \to p_{ij}$. Then σ is an isomorphism. Define $\psi:T \to S$ by*

$$(i, g, j) \to (i, (q_{zi} g q_{jz})\sigma, j).$$

*Then ψ is a *-isomorphism of T onto S.*

Proof. To prove that ψ is a semigroup homomorphism we must show, by (1) of Lemma 2.1, that

$$q_{ij}\sigma = (q_{iz}\sigma)p_{ij}(q_{zj}\sigma).$$

This follows by applying σ to statement (2) in the proof of Lemma 5.1. It is clear that ψ is one-to-one and onto.

To show that ψ respects involutions it is sufficient (since $\{ (x, x, x) | x \in X\}$ generates S as an involutorial semigroup) to show that

$$(x, x, x)^*\psi^{-1} = (x, x, x)\psi^{-1}\hat{\tau}.$$

Now

$$\begin{aligned}(x, x, x)\psi^{-1}\hat{\tau} &= (x, q_{zx}^{-1} x q_{xz}^{-1}, x)\hat{\tau} \\ &= (x^*, (q_{xz}^{-1}\tau)(x\tau)(q_{zx}^{-1}\tau), x^*) \\ &= (x^*, q_{z^*x^*}^{-1} q_{zx^*}^{-1} x^* q_{x^*z}^{-1} q_{x^*z^*}^{-1}, x^*) \\ &= (x^*, q_{zx^*}^{-1} x^* q_{x^*z}^{-1}, x^*)\end{aligned}$$

since Q is locally normalized at z, z^*

$$= (x^*, x^*, x^*)\psi^{-1} = (x, x, x)^*\psi^{-1},$$

as required.

In the light of ([**4**], Theorem 2.1), we deduce the following interesting consequence of the above result.

THEOREM 5.4. *The underlying semigroup of the free involutorial completely simple semigroup on X is the free product of two copies of free completely simple semigroups on X.*

6. Examples of identities. We consider here certain identities on involutorial completely simple semigroups implied by the identity $a = aa^*a$. We find the conditions on an involutorial Rees matrix semigroup S in standard form which are necessary and sufficient for S to satisfy each of these identities.

THEOREM 6.1. *Let $S = \mathscr{M}(G, I; P, \tau)$. Then for $n = -1, 0, 1, 2, \dots$, S satisfies the identity $a^{-n} = a(a^*)^{n+2}a$ if and only if*

(1) $g^{n+2}(g\tau)^{n+2} = p_{ij}p_{ji} = p_{ii} = 1 \quad (g \in G, i, j \in I).$

Proof. We first compute, for $a = (i, g, j)$,

(2) $a^{-1} = (i, p_{ji}^{-1}g^{-1}p_{ji}^{-1}, j),$

(3) $a^0 = (i, p_{ji}^{-1}, j),$

and for $n > 0$, taking into account (2),

(4) $$a^{-n} = ((i, g, j)^{-1})^n = (i, p_{ji}^{-1}g^{-1}p_{ji}^{-1}, j)^n$$
$$= (i, (p_{ji}^{-1}g^{-1})^n p_{ji}^{-1}, j),$$

and on the other hand, for $n = -1, 0, 1, \dots$,

(5) $$a(a^*)^{n+2}a = (i, g, j)(j, g\tau, i)^{n+2}(i, g, j)$$
$$= (i, g, j)(j, [(g\tau)p_{ij}]^{n+1}(g\tau), i)(i, g, j)$$
$$= (i, gp_{jj}[(g\tau)p_{ij}]^{n+1}(g\tau)p_{ii}g, j).$$

Direct part. Case $n = -1$. By (4) and (5), we get

$$g = gp_{jj}(g\tau)p_{ii}g$$

so that

$$g\tau = p_{jj}^{-1}g^{-1}p_{ii}^{-1}.$$

For $g = 1$ and $j = z$, we obtain $p_{ii} = 1$ which then implies that $g\tau = g^{-1}$. In particular,

$$p_{ij} = p_{ji}\tau = p_{ji}^{-1}.$$

Case $n = 0$. By (3) and (5), we obtain

$$p_{ji}^{-1} = gp_{jj}(g\tau)p_{ij}(g\tau)p_{ii}g\ .$$

For $g = 1$ and $j = z$, we get $p_{ii} = 1$. Now for $g = 1$, it follows that $p_{ji}^{-1} = p_{ij}$. For $j = z$, we get

$$g(g\tau)^2 g = 1$$

and thus $g^2(g\tau)^2 = 1$.

Case $n > 1$. From (4) and (5), we have

$$(p_{ji}^{-1}g^{-1})^n p_{ji}^{-1} = gp_{jj}[(g\tau)p_{ij}]^{n+1}(g\tau)p_{ii}g.$$

For $g = 1$ and $j = z$, this gives $p_{ii} = 1$ and thus

$$(p_{ji}^{-1}g^{-1})^{n+2} = p_{ji}^{-1}[(g\tau)p_{ij}]^{n+1}(g\tau) = p_{ji}^{-1}[(g\tau)p_{ij}]^{n+2}p_{ij}^{-1}.$$

For $j = z$, we get

$$g^{n+2}(g\tau)^{n+2} = 1.$$

For $g = p_{ji}^{-1}$, we get

$$1 = p_{ji}^{-1}[(p_{ji}^{-1}\tau)p_{ij}]^{n+2}p_{ij}^{-1} = p_{ji}^{-1}p_{ij}^{-1}$$

so that $p_{ij}p_{ji} = 1$.

Converse. Case $n = -1$. Since $g\tau = g^{-1}$, then

$$\begin{aligned} aa^*a &= (i, g, j)(j, g\tau, i)(i, g, j) \\ &= (i, gp_{jj}g^{-1}p_{ii}g, j) = (i, g, j) = a. \end{aligned}$$

Case $n = 0$. We compute

$$\begin{aligned} aa^*a^*a &= (i, g, j)(j, g\tau, i)(j, g\tau, i)(i, g, j) \\ &= (i, gp_{jj}(g\tau)p_{ij}(g\tau)p_{ii}\, g, j) = (i, g(gp_{ji}g)\tau g, j). \end{aligned}$$

By hypothesis,

$$\begin{aligned} (p_{ji}\tau)(gp_{ji}g)\tau &= [(gp_{ji})\tau]^2 = [(gp_{ji})^{-1}]^2 = p_{ji}^{-1}(gp_{ji}g)^{-1} \\ &= p_{ij}(gp_{ji}g)^{-1} = (p_{ji}\tau)(gp_{ji}g)^{-1} \end{aligned}$$

which implies that

$$(gp_{ji}g)\tau = (gp_{ji}g)^{-1}.$$

Hence

$$aa^*a^*a = (i, g(gp_{ji}g)^{-1}g, j) = (i, p_{ji}^{-1}, j) = a^0.$$

Case $n > 0$. We calculate

$$\begin{aligned} g[(g\tau)p_{ij}]^{n+1}(g\tau)g &= g[(g\tau)p_{ij}]^{n+2}p_{ij}^{-1}g \\ &= g[(p_{ji}g)\tau]^{n+2}p_{ij}^{-1}g \\ &= g(p_{ji}g)^{-(n+2)}p_{ij}^{-1}g \end{aligned}$$

$$
\begin{aligned}
&= g(p_{ji}g)^{-1}(p_{ji}g)^{-n}(p_{ji}g)^{-1}p_{ij}^{-1}g \\
&= p_{ji}^{-1}(g^{-1}p_{ji}^{-1})^n g^{-1}p_{ji}^{-1}p_{ij}^{-1}g \\
&= (p_{ji}^{-1}g^{-1})^n p_{ji}^{-1}
\end{aligned}
$$

which in view of (4) and (5) implies that $a^{-n} = a(a^*)^{n+2}a$.

The special case $n = -1$ in the above theorem is of particular interest.

Definition 6.2. An involution * of a semigroup S is a *regular involution* if S satisfies the identity $a = aa^*a$; in such a case, S is a *regular involutorial semigroup.*

COROLLARY 6.3. *The involution* $\bar{\tau}$ *of* $\mathscr{M}(G, I; P, \tau)$ *is regular if and only if*

$$g(g\tau) = p_{ij}p_{ji} = p_{ii} = 1 \quad \textit{for all } g \in G,\ i, j \in I.$$

PROPOSITION 6.4. *The following conditions on* $S = \mathscr{M}(G, I; P, \tau)$ *are equivalent.*

(i) $p_{ij}p_{ji} = p_{ii} = 1$ *for all* $i, j \in I$.
(ii) S *satisfies the identity* $a^0a^{*0} = (aa^*)^0$.
(iii) S *satisfies the identity* $a^0 = (aa^*)^0(a^*a)^0$.
(iv) S *satisfies the identity* $a^0 = a^0a^{0*}a^0$.
(v) *The restriction of * to the subsemigroup of* S *generated by the idempotents is a regular involution.*

Proof. The proof of the equivalence of item (i) and each of (ii), (iii) and (iv) consists of a simple verification and is omitted. Also item (v) trivially implies item (iv). It remains to prove that (i) implies (v).

Assume that (i) holds. Let $i_k, j_k \in I$ for $k = 1, 2, \ldots, n$ and let

$$t = p_{j_1i_1}^{-1}p_{j_1i_2}p_{j_2i_2}^{-1} \ldots p_{j_{n-1}i_n}p_{j_ni_n}^{-1}.$$

Then

$$(i_1, p_{j_1i_1}^{-1}, j_1)(i_2, p_{j_2i_2}^{-1}, j_2) \ldots (i_n, p_{j_ni_n}^{-1}, j_n) = (i_1, t, j_n)$$

and using $p_{ij}\tau = p_{ij}^{-1}$ and $p_{ii} = 1$, we get

$$
\begin{aligned}
&(i_1, t, j_n)(i_1, t, j_n)^*(i_1, t, j_n) = (i_1, t, j_n)(j_n, t\tau, i_1)(i_1, t, j_n) \\
&= (i_1, tp_{j_nj_n}(t\tau)p_{i_1i_1}t, j_n) \\
&= (i_1, t(p_{j_ni_n}^{-1}\tau)(p_{j_{n-1}i_n}\tau) \ldots (p_{j_2i_2}^{-1}\tau)(p_{j_1i_2}\tau)(p_{j_1i_2}^{-1}\tau)t, j_n) \\
&= (i_1, tp_{j_ni_n}p_{j_{n-1}i_n}^{-1} \ldots p_{j_2i_2}p_{j_1i_2}^{-1}p_{j_1i_1}t, j_n) \\
&= (i_1, t, j_n),
\end{aligned}
$$

as required.

7. Free regular involutorial completely simple semigroups. In order to describe the semigroups in the title of this section, we will characterize the least congruence ρ on $V = \mathscr{M}(G, I; R, \omega)$, introduced in Theorem 4.2, for which V/ρ is a regular involutorial completely simple semigroup. A simplified version of this characterization will also yield the least congruence λ on V for which V/λ satisfies the identities in Proposition 6.4.

LEMMA 7.1. *Let $V = \mathscr{M}(G, I; R, \omega)$ and N be the normal subgroup of G generated by the set*

(1) $\{r_{ii}, r_{ij}r_{ji}, r_{z^*i^*}^{-1}i^*r_{iz^*}i \mid i, j \in I\}.$

Then the relation ρ defined on V by

(2) $(i, g, j)\rho(k, h, l) \Leftrightarrow i = k, gh^{-1} \in N, j = l$

*is the least *-congruence on V for which the involution on V/ρ is regular.*

Proof. We let ρ be the least *-congruence on V with the property that the involution on V/ρ is regular; we will show that ρ may be described as in the statement of the lemma. By hypothesis, ρ is the least *-congruence on V for which $a\rho aa^*a$. Hence, in view of Theorem 4.2, ρ must be idempotent separating. Every idempotent separating congruence on the Rees matrix semigroup V is determined by a normal subgroup N of G according to formula (2). It remains to prove that N is generated, qua normal subgroup, by the set in (1).

We first observe that

$$(i, g, j)\rho(i, g, j)(i, g, j)^*(i, g, j) = (i, gr_{jj}(g\omega)r_{ii}g, j)$$

and by (2) this is equivalent to

$$gr_{jj}(g\omega)r_{ii} \in N.$$

This is, in particular, true for $g = 1$ and $j = z$, which then gives $r_{ii} \in N$. But then

$$gr_{jj}(g\omega) \in N,$$

and since N is a normal subgroup, we deduce that $(g\omega)g \in N$. For $g = i$ and $g = r_{ij}$ respectively, by Theorem 4.2, we get

$$r_{z^*i}^{-1}i^*r_{iz^*}i, \ r_{ij}r_{ji} \in N.$$

Conversely, if

$$r_{ii}, \ r_{ij}r_{ji}, \ r_{z^*i^*}^{-1}i^*r_{iz^*}i \in N$$

for all $i, j \in I$, then since $I \cup \{r_{ij} \mid i \neq z \neq j\}$ generates G, it follows by Theorem 4.2 that

$$gr_{ji}(g\omega)r_{ii} \in N.$$

Furthermore, from Theorem 4.2 we get

$$r_{ii}\omega = r_{ii},$$

$$(r_{ij}r_{ji})\omega = (r_{ji}\omega)(r_{ij}\omega) = r_{ij}r_{ji},$$

$$\begin{aligned}(r_{z^*i^*}^{-1}i^*r_{iz^*}i)\omega &= (i\omega)(r_{iz^*}\omega)(i^*\omega)(r_{z^*i^*}\omega)^{-1}\\ &= (r_{z^*i^*}^{-1}i^*r_{iz^*})r_{z^*i}(r_{z^*i}^{-1}ir_{i^*z^*})r_{i^*z^*}^{-1}\\ &= r_{z^*i^*}^{-1}i^*r_{iz^*}i\end{aligned}$$

which gives $N\omega = N$. This implies that ρ respects the *-operation.

We aim next at a suitable Rees matrix representation of V/ρ in Lemma 7.1.

LEMMA 7.2. *Let I be linearly ordered by $\leq$. Let N be the normal subgroup of G defined in Lemma 7.1. Then G/N is freely generated by*

$$\{xN|x \in X\} \cup \{r_{ij}N|i > j, i \neq z \neq j\}.$$

Proof. It is clear that G/N is generated by this set since the generators of G can be expressed modulo N in terms of these elements. Consider the following diagram:

$$\begin{tikzcd}
X \cup \{r_{ij}|i > j, i \neq z \neq j\} \arrow[r, "\theta"] \arrow[dr, "\subseteq"'] & G/N \arrow[d, dashed, "\psi"'] & G \arrow[l] \arrow[dl, "\bar{\varphi}"] & I \cup \{r_{ij}|i \neq z \neq j\} \arrow[l] \arrow[dll, "\varphi"] \\
 & F & &
\end{tikzcd}$$

where F is the free group on $X \cup \{r_{ij}|i > j, i \neq z \neq j\}$,

$$\varphi: I \cup \{r_{ij}|i \neq z \neq j\} \to F$$

is defined by

$$x \to x,$$

$$x^* \to (r_{xz^*}xr_{z^*x^*}^{-1})^{-1} = r_{z^*x^*}x^{-1}r_{xz^*}^{-1}$$

$$r_{ij} \to r_{ij} \text{ if } i > j, i \neq z \neq j,$$

$$r_{ii} \to 1,$$

$$r_{ij} \to r_{ji}^{-1} \text{ if } i > j, i \neq z \neq j,$$

and

$$\theta: X \cup \{r_{ij} | i > j, i \neq z \neq j\} \rightarrow G/N$$

is defined by $t \rightarrow tN$ for all t.

The map $\overline{\varphi}$ is the unique extension of φ to G. To prove that G/N is free it is sufficient to prove that θ is one-to-one and to define a homomorphism $\psi: G/N \rightarrow F$ rendering the left part of the above diagram commutative.

The elements of N are of the form

$$n = \prod_{i=1}^{k} a_i^{-1} w_i a_i$$

where w_i or w_i^{-1} is a generator of N. To show that θ is one-to-one, it is enough to show that if u and v are distinct elements in the domain of θ, then $uv^{-1} \notin N$, that is uv^{-1} cannot be expressed in the above way. There are several cases. If $x, y \in X$, $x \neq y$, and if $xy^{-1} = n$, then in n there is an occurrence of x not matched with some x^{-1}. At the same time all occurrences of x^* must be matched with some occurrence of $(x^*)^{-1}$. Since all occurrences of x in a_i can be matched with x^{-1} in a_i^{-1} and similarly for x^*, and since in w_i, x^* and x (or their inverses) both occur once or neither occurs then $xy^{-1} \neq n \in N$. The other cases, that is $xr_{ij}^{-1} \notin N$ and $r_{ij}r_{kl}^{-1} \notin N$ require similar arguments.

It is a routine calculation to show that $n\overline{\varphi} = 1$ for the generators of N and therefore that $(N)\overline{\varphi} = 1$. It follows that the required ψ can be defined by

$$(aN)\psi = a\overline{\varphi}$$

since the resulting mapping is well-defined.

THEOREM 7.3. *Let K be the free group on the set*

$$X \cup \{w_{ij} | i > j, i \neq z \neq j\}.$$

Let $w_{ij} = w_{ji}^{-1}$ if $i < j$, $i \neq z \neq j$ and let $w_{iz} = w_{zi} = 1$ for all $i, j \in I$. Then $Z = \mathcal{M}(K, I; W)$ is a free regular involutorial completely simple semigroup on X.

Proof. This follows immediately from the above two lemmas and their proofs.

The arguments above for the treatment of a regular involution simplify in an obvious way giving the following results which we state without proof.

LEMMA 7.4. *Let $V = \mathcal{M}(G, I; R, \omega)$ and M be the normal subgroup G generated by the set*

(1) $\{r_{ii}, r_{ij}r_{ji} | i, j \in I\}$.

Then the relation λ *defined on* V *by*

(2) $(i, g, j)\lambda(k, h, l) \Leftrightarrow i = k, gh^{-1} \in M, j = l$

is the least $*$*-congruence on* V *for which* V/λ *satisfies the identity*

$$a^0 a^{*0} = (aa^*)^0.$$

LEMMA 7.5. *Let* I *be linearly ordered by* $\leqq$. *Let* M *be the normal subgroup of* G *defined in Lemma* 7.4. *Then* G/M *is freely generated by*

$$\{iN | i \in I\} \cup \{r_{ij}N | i > j, i \neq z \neq j\}.$$

THEOREM 7.6. *Let* L *be the free group on the set*

$$I \cup \{l_{ij} | i > j, i \neq z \neq j\}.$$

Let $l_{ij} = l_{ji}^{-1}$ *if* $i < j, i \neq z \neq j$ *and let* $l_{iz} = l_{zi} = l_{ii} = 1$ *for all* $i, j \in I$. *Then* $\mathcal{M}(L, I; W)$ *is a free involutorial completely simple semigroup satisfying the identity*

$$a^0 a^{*0} = (aa^*)^0.$$

8. Word problems. We discuss here the word problem for the free involutorial completely simple semigroup, relative to the free involutorial unary semigroup and for the free regular involutorial completely simple semigroup relative to the free involutorial semigroup on a fixed nonempty set.

In order to keep track of various semigroups and mappings, we illustrate the situation by the diagram below. As usual, we write $I = X \cup X^*$.

I^+ is the free semigroup on I thought of as the set of all words over the alphabet I with concatenation as product.

$(I^+, *)$ is the free involutorial semigroup on X as constructed by Wagner [**8**]; the involution is defined by $x \leftrightarrow x^*$ on I and then extended to all of I^+ by the requirement that it be an antihomomorphism.

$U(I)$ is the free unary semigroup on I as constructed by Clifford [**1**]; it may be defined as the smallest subset of the free semigroup on $I \cup \{(,)^{-1}\}$, where (and $)^{-1}$ are two extra symbols satisfying the properties:

(i) $I \subseteq U(I)$,

(ii) $w \in U(I) \Rightarrow (w)^{-1} \in U(I)$,

(iii) $u, v \in U(I) \Rightarrow uv \in U(I)$.

$(U, *)$, where $U = U(I)$, is the free involutorial unary semigroup on X; the involution is defined by $x \leftrightarrow x^*$ on I and then extended to all of U by the requirement that it be an antihomomorphism (this is an obvious analogue of Theorem 3.1).

$\mathcal{M}(I, G, I; P)$ is the free completely simple semigroup on I as specified by Clifford [**1**] and Rasin [**7**].

$(S, *)$, where $S = \mathcal{M}(I, G, I; P)$, is the free involutorial completely simple semigroup on X; the matrix P is normalized.

$(T, *)$, where $T = \mathcal{M}(I, G, I; Q)$, is a *-isomorphic copy of $(S, *)$ with the matrix Q locally normalized at (z, z^*).

$V = \mathcal{M}(G, I; R, \omega)$ is a *-isomorphic copy of $(S, *)$ in standard form.

$\mathcal{M}(G/M, I; R/M, \omega')$ is the (relatively) free involutorial completely simple semigroup on X in the variety determined by the identity

$$a^0 a^{*0} = (aa^*)^0.$$

$\mathcal{M}(G/N, I; R/N)$ is the free regular involutorial completely simple semigroup on X.

The signatures of the algebras depicted in the diagram are by columns: $(\cdot)$, $(\cdot, *)$, $(\cdot, {}^{-1}, *)$, $(\cdot, {}^{-1})$.

$$\begin{tikzcd}
I^+ \arrow[r] & (I^+, *) \arrow[r] \arrow[dddd] & (U, *) & U = U(I) \arrow[l] \arrow[dd] \\
& & (T = \mathcal{M}(I, G, I; Q), *) \arrow[d] & \\
& & (S, *) \arrow[d] & S = \mathcal{M}(I, G, I; P) \arrow[l] \\
& & V = \mathcal{M}(G, I; R, \omega) \arrow[d] \arrow[dl, dashed] & \\
& \mathcal{M}(G/N, I; R/N) & \mathcal{M}(G/M, I; R/M, \omega') \arrow[l] &
\end{tikzcd}$$

Diagram 8.1

In [**3**] we solved the word problem for $F\mathcal{CS}(I)$. We defined two maps $h, t: U(I) \to I$ and the map $m: U(I) \to G$ where G is the free group on

$$I \cup \{p_{ij} | i \neq z \neq j\}$$

as follows. Let $p_{iz} = p_{zi} = 1$. Let $w \in U(I)$; then

$h(w)$ is the first element of I to occur in w,

$t(w)$ is the last element of I to occur in w,

$m(i) = i$ for all $i \in I$,

$m(uv) = m(u)p_{t(u)h(v)}m(v)$,

$m((u)^{-1}) = p^{-1}_{t(u)h(v)}[m(u)]^{-1}p^{-1}_{t(u)h(v)}$.

We showed that if $w \sim w'$ is defined by $h(w) = h(w')$, $m(w) = m(w')$ and $t(w) = t(w')$, then

$$U(I)/\sim \ \cong F\mathscr{C}\mathscr{S}(I),$$

solving the word problem for $F\mathscr{C}\mathscr{S}(I)$ in terms of $U(I)$.

The involution defined on $U(I)$ by extending $i \to i^*$ gives rise to an involution on $U(I)/\sim$ and

$$(U(I)/\sim, {}^*) \cong F\mathscr{C}\mathscr{S}^*(I).$$

Thus the word problem for $F\mathscr{C}\mathscr{S}^*(I)$ relative to $U(I)$ is the same as the word problem for $F\mathscr{C}\mathscr{S}(I)$ relative to $U(I)$.

The word problem for the free objects in the variety determined by the identity $a^0a^{*0} = (aa^*)^0$ and in the variety of regular involutorial completely simple semigroups can be described in a similar way. For the former we simply modify m to

$$m'(w) = m(w)M \quad (w \in U(X)).$$

For the variety of regular involutorial completely simple semigroups we modify m to

$$m''(w) = m(w)N \quad (w \in U(X)).$$

In the case of the free regular involutorial completely simple semigroup, $F\mathscr{R}\mathscr{C}\mathscr{S}^*(X)$, we can also give a solution of the word problem in terms of I^+. In fact $I^+ \subseteq U(I)$ and we can restrict the maps h, t and m'' defined above to I^+. If $\approx$ is the congruence on I^+ so induced we will show that

$$(I^+/\approx, {}^*) \cong F\mathscr{R}\mathscr{C}\mathscr{S}^*(X).$$

The congruence $\approx$ can be obtained from the maps given in Diagram 8.1 through U. That the resulting map is onto $\mathscr{M}(G/N, I; R/N)$ follows from the fact that in a regular involutorial completely simple semigroup

$$a^{-1} = aa^*a^*a^*a.$$

Using Corollary 6.3, this formula can be immediately verified in a Rees matrix semigroup with regular involution in standard form.

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