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ON ORDER PROPERTIES OF ORDER BOUNDED TRANSFORMATIONS

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Introduction. W. A. J. Luxemburg and A. C. Zaanen in [**7**] and W. A. J. Luxemburg in [**5**] have studied the order properties of the order bounded linear functionals of a given Riesz space L. In this paper we consider the vector space $\mathscr{L}_b(L, M)$ of the order bounded linear transformations from a given Riesz space L into a Dedekind complete Riesz space M.

We study the order structure of the Dedekind complete Riesz space $\mathscr{L}_b(L, M)$. Integral and normal integral transformations are considered and the theorems of [**5**] and [**7**] about the different components of an order bounded linear transformation are generalized in this setting. Extensions of order bounded linear transformations are also considered and the theorems of [**7**] are also generalized.

1. Preliminaries. For notation and basic terminology concerning Riesz spaces we refer the reader to [**8**]. Let L and M be two Riesz spaces. We shall denote by $\mathscr{L} = \mathscr{L}(L, M)$ the real linear space of all linear transformations from L into M, and by $\mathscr{L}_b = \mathscr{L}_b(L, M)$ the real subspace of all order bounded linear transformations from L into M, i.e., T is in $\mathscr{L}_b(L, M)$ if $T(A)$ is an order bounded subset of M, whenever A is an order bounded subset of L. A linear transformation T in $\mathscr{L}(L, M)$ is called positive, denoted by $\theta \leqq T$, whenever $\theta \leqq u \in L$, implies $\theta \leqq T(u) \in M$. We write $T_1 \leqq T_2$, T_1, $T_2 \in \mathscr{L}(L, M)$ to indicate that $\theta \leqq T_2 - T_1$. The set of all positive linear transformations of $\mathscr{L}(L, M)$ will be denoted by $\mathscr{L}^+ = \mathscr{L}^+(L, M)$. It is easily seen that $\mathscr{L}^+(L, M) \subseteq \mathscr{L}_b(L, M)$ and that $\mathscr{L}^+$ is a positive cone for $\mathscr{L}_b(L, M)$, and consequently for $\mathscr{L}(L, M)$. Therefore, $(\mathscr{L}_b, \mathscr{L}^+)$ is a (partially) ordered vector space. In the particular case of $M = \mathbf{R}$ we denote the linear space $\mathscr{L}_b(L, \mathbf{R})$ by $L^\sim$, i.e., $\mathscr{L}_b(L, \mathbf{R}) = L^\sim$, and we shall call $L^\sim$ the order dual of L. We remark that in general $\mathscr{L}_b(L, M) \neq \mathscr{L}(L, M)$. (See Example 1.5 below; see also [**6**, Example (iii), p. 440] for an example of a norm bounded linear transformation from l^2 into l^2 which is not order bounded.)

The following Lemma can be found in [**13**, p. 205].

LEMMA 1.1. *Let L and M be two Riesz spaces with M Archimedean. Assume that*

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T is an additive function from L^+ into M^+. Then T is uniquely extendable to a positive linear transformation from L into M.

Note that the extension is given by $T(u) = T(u^+) - T(u^-)$ for all u in L and that Lemma 1.1 may be false if M is not Archimedean. Indeed, let f be an additive function from $\mathbf{R}$ into $\mathbf{R}$ which is not linear, i.e., not of the form $f(x) = cx$, and let L be the lexicographic plane (see [**8**, Example (ii), p. 49]). Consider the mapping $\varphi : \mathbf{R}^+ \rightarrow L^+$ by $\varphi(x) = (x, f(x))$ for all $x \in \mathbf{R}^+$. Note that φ is additive and that if φ would be extendable to a linear mapping from $\mathbf{R}$ into L then f would be linear.

We continue with a fundamental theorem.

THEOREM 1.2 (L. V. Kantorovich [**4**], F. Riesz [**12**]). *Let L and M be two Riesz spaces with M Dedekind complete. Then we have:*

(i) *The ordered space $\mathscr{L}_b(L, M)$ (ordered by the cone $\mathscr{L}^+(L, M)$) is a Dedekind complete Riesz space.*

(ii) *For every $T \in \mathscr{L}_b(L, M)$ and for every $u \in L^+$ we have*

$$\begin{aligned} T^+(u) &= \sup \{T(v) : v \in L \text{ and } \theta \leqq v \leqq u\}, \\ T^-(u) &= \sup \{-T(v) : v \in L \text{ and } \theta \leqq v \leqq u\}, \\ |T|(u) &= \sup \{|Tv| : v \in L \text{ and } |v| \leqq u\}, \end{aligned}$$

where, $T^+ = T \vee \theta$, $T^- = (-T) \vee \theta$ and $|T| = T \vee (-T)$ in $\mathscr{L}_b(L, M)$.

(iii) *If $\{T_\alpha\} \subseteq \mathscr{L}_b(L, M)$ and $T \in \mathscr{L}_b(L, M)$ then $T_\alpha \uparrow T$ in $\mathscr{L}_b(L, M)$ if, and only if, $T_\alpha(u) \uparrow T(u)$ holds in M for all u in L^+.*

For a proof we refer the reader to [**11**, Proposition 2.3, p. 22,] and to [**2**].

Remark. Theorem 1.2 was proved by F. Riesz in a very special case (see [**12**]). The general Theorem 1.2 as it is stated here was established by L. V. Kantorovich (see [**4**]).

An illustration of the above theorem is given in the next example.

Example 1.3. Let L be the Riesz space of all continuous real valued, piecewise linear functions, defined on $[0, 1]$, with the pointwise ordering and let $M = \mathbf{R}$. Define $\varphi : L \rightarrow \mathbf{R}$ by the formula:

$$\varphi(u) = \int_0^1 u'(x)dx \quad \text{for all } u \in L.$$

It is easily seen that $\varphi(u) = u(1) - u(0)$ for all $u \in L$. Moreover, using Theorem 1.2 (ii), we see that

$$\varphi^+(u) = u(1), \quad \varphi^-(u) = u(0), \quad \text{and} \quad |\varphi|(u) = u(1) + u(0)$$

for all $u \in L^+$.

The following result is a corollary of Theorem 1.2.

COROLLARY 1.4. *Let L and M be as in Theorem 1.2. Then we have:*

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(i) $T_\alpha \xrightarrow{(o)} T$ *in* $\mathscr{L}_b(L, M)$ *implies* $T_\alpha(u) \xrightarrow{(o)} T(u)$ *in* M *for all* $u \in L$.

(ii) *If* $\{T_\alpha\} \subseteq \mathscr{L}_b(L, M)$ *and* $|T_\alpha| \leqq S \in \mathscr{L}_b(L, M)$ *for all* α *and if* $T_\alpha(u) \xrightarrow{(o)} T(u)$ *in* M *for all* $u \in L$, *then* $T \in \mathscr{L}_b(L, M)$.

Proof. (i) This follows immediately from Theorem 1.2 (ii).

(ii) This follows from the elementary fact: If $\{u_\alpha\}$ *is a net of a Riesz space* such that $u_\alpha \leqq v$ for all α and $u_\alpha \xrightarrow{(o)} u$, then $u \leqq v$.

The following example shows that the assumption $|T_\alpha| \leqq S$ for all α in the previous result is essential.

Example 1.5. Let L be the Riesz space of all real sequences which are eventually constant, i.e., $u \in L$ if there exists a real constant $u(\infty)$ such that $u(k) = u(\infty)$ for all $k \geqq k_0$, with the pointwise ordering. Note that the vectors $e_n = (0, \ldots, 0, 1, 0, 0, \ldots)$, $n = 1, 2, \ldots, e = (1, 1, \ldots, 1, \ldots)$ form a Hamel basis for L. Observe also that the element $u = (u(1), \ldots, u(n), u(\infty), u(\infty), \ldots)$ of L can be written in the form $u = (u(1) - u(\infty))e_1 + \ldots + (u(n) - u(\infty))e_n + u(\infty) \cdot e$, with respect to the above Hamel basis. Let φ be the linear functional on L taking on the values $\varphi(e_n) = n$, $n = 1, 2, \ldots$, and $\varphi(e) = 0$ on the Hamel basis. It is easily seen that φ is not order bounded. Now let φ_n be the linear functional on L taking the values $\varphi_n(e_k) = k$, $k = 1, 2, \ldots, n$, $\varphi_n(e_k) = 0$, $k = n + 1, n + 2, \ldots$, $\varphi(e) = 0$ on the Hamel basis. It is not difficult to verify that φ_n is order bounded for all $n = 1, 2, \ldots$ and that $\varphi_n(u) = \varphi(u)$ for all $n \geqq n_0$. This shows that

$$\varphi_n(u) \xrightarrow{(o)} \varphi(u)$$

in $\mathbf{R}$ for all $u \in L$. But $\varphi \notin L^\sim$.

The next theorem generalizes the statement 1.5.8 of [**3**, p. 20]. The proof is similar and so we omit it.

THEOREM 1.6. *Let* L *and* M *be two Riesz spaces with* M *Dedekind complete and let* $T \in \mathscr{L}_b(L, M)$. *Then for every* $u \in L^+$ *we have:*

$$\sup T([\theta, u]) + \inf T([\theta, u]) = T(u).$$

A mapping p from a Riesz space L into another Riesz space M is called sublinear if $p(u + v) \leqq p(u) + p(v)$ and $p(\lambda u) = \lambda p(u)$ for all u, v of L and all $\lambda \geqq 0$.

The next theorem is a generalization of the Hahn-Banach theorem. (See [**11**, Proposition 2.1, p. 78].)

THEOREM 1.7 (Hahn-Banach). *Let* L *and* M *be two Riesz spaces with* M *Dedekind complete and let* $p : L \to M$ *be a sublinear mapping. Assume that* T *is a linear mapping defined on a linear subspace* K *of* L *with range in* M *such that* $T(u) \leqq p(u)$ *for all* u *in* K. *Then* T *can be extended to a linear mapping* T_1 *of* L *into* M *such that* $T_1(u) \leqq p(u)$ *for all* u *in* L.

2. The space $\mathscr{L}_b(L, M)$. In the next theorem we derive some formulas which are the "dual" formulas to those of Theorem 1.2 (ii).

THEOREM 2.1. *Let L and M be two Riesz spaces with M Dedekind complete. For every $u \in L$ and for every $\theta \leqq T \in \mathscr{L}_b(L, M)$ we have:*

(i) $T(u^+) = \sup \{S(u) : S \in \mathscr{L}_b(L, M); \theta \leqq S \leqq T\}$,

(ii) $T(u^-) = \sup \{-S(u) : S \in \mathscr{L}_b(L, M); \theta \leqq S \leqq T\}$,

(iii) $T(|u|) = \sup \{|S(u)| : S \in \mathscr{L}_b(L, M); |S| \leqq T\}$.

Proof. We prove the third formula first. Assume u in L and define the function T_u on L^+ into M by the formula $T_u(w) = \sup \{T(w \wedge n|u|) : n = 1, 2, \ldots\}$ for all $w \in L^+$. It follows easily that the function defined by $p(w) = T_u(|w|)$ for all w in L is a sublinear mapping such that $p(w) \leqq T(|w|)$ for all w in L. Let, now $K = \{\lambda u : \lambda \in \mathbf{R}\}$ and let S be the linear mapping from K into M defined by $S(\lambda u) = \lambda p(u)$ for all λ in $\mathbf{R}$. According to Theorem 1.7 there is an extension S_1 of S to all of L such that $S_1(g) \leqq p(g)$ for all g in L. It follows easily from the last relation that $S_1 \in \mathscr{L}_b(L, M)$ and $|S_1| \leqq T$. Hence $T(|u|) \leqq \sup \{|S(u)| : S \in \mathscr{L}_b(L, M); |S| \leqq T\}$. Since the other inequality it is obvious the proof is finished.

For the first formula we apply the same arguments using as sublinear mapping $p_1(w) = \sup \{T(w^+ \wedge nu^+) : n = 1, 2, \ldots\}$ for all $w \in L$. The second formula follows from the first by noting that $f^- = (-f)^+$.

Remarks. (i) It can be seen easily from the above proof that the above suprema are actually maxima.

(ii) Theorem 2.1 is a generalization of a Theorem of Luxemburg and Zaanen (see [**7**, Note VI, Theorem 19.6, p. 662]).

We continue with a Theorem which is a kind of converse of Theorem 1.2 (i).

THEOREM 2.2. *Let L and M be two Riesz spaces with $L^\sim \neq \{\theta\}$. Let $\mathscr{L}_b = \mathscr{L}_b(L, M)$ denote the real vector space of all order bounded linear mappings from L into M. Then we have:*

(i) *If the ordered vector space $(\mathscr{L}_b, \mathscr{L}^+)$ is a Dedekind complete Riesz space then M is Dedekind complete.*

(ii) *If the ordered vector space $(\mathscr{L}_b, \mathscr{L}^+)$ is a super Dedekind complete Riesz space then M is super Dedekind complete.*

Proof. (i) Assume $\theta \leqq u_\alpha \uparrow \leqq u_0$ in M. We have to show that $u_\alpha \uparrow u$ in M for some u in M. Let φ be a non-zero positive linear functional of L, and let $f_0 \in L^+$ be such that $\varphi(f_0) = 1$. For each α we define a linear mapping T_α in $\mathscr{L}_b(L, M)$ as follows:

$$T_\alpha(f) = \varphi(f)u_\alpha \quad \text{for all } f \text{ in } L.$$

It is easily seen that $\theta \leqq T_\alpha \uparrow \leqq T$ in $\mathscr{L}_b(L, M)$, where $T \in \mathscr{L}_b(L, M)$, $T(f) = \varphi(f)u_0$ for all f in L. Since $\mathscr{L}_b(L, M)$ is a Dedekind complete Riesz space $\theta \leqq T_\alpha \uparrow S \leqq T$ for some $S \in \mathscr{L}_b(L, M)$. In particular we have $T_\alpha(f_0) = u_\alpha \uparrow \leqq S(f_0)$. We show next that $S(f_0)$ is the least upper bound of the net $\{u_\alpha\}$ in M. Suppose that $u_\alpha \leqq w$ for all α. Then we have $T_\alpha \leqq T_w \in \mathscr{L}_b(L, M)$ for all α, where $T_w(f) = \varphi(f)w$ for all f in L. Hence $S \leqq T_w$ and so $S(f_0) \leqq T_w(f_0) = w$. This shows that $u_\alpha \uparrow S(f_0)$, i.e., M is Dedekind complete.

(ii) The proof of (ii) is similar.

The next example shows that Theorem 2.2 may be false if $L^\sim = \{\theta\}$.

Example 2.3†. Let $L = L_p([0, 1], 0 < p < 1$, and let $M = C_{[0,1]}$. It is known that $L^\sim = \{\theta\}$ (see [**1**] and [**11**, p. 86]). It is not difficult to verify that $\mathscr{L}_b(L, M) = \{\theta\}$, which is a super Dedekind complete Riesz space. Note that M is not even σ-Dedekind complete.

Given two Riesz spaces L and M with $L^\sim \neq \{\theta\}$ and with M Dedekind complete we pick $\theta < \varphi \in L^\sim, \theta < f_0 \in L, \varphi(f_0) = 1$ and we define the mapping $T : M \rightarrow \mathscr{L}_b(L, M)$ by $u \rightarrow T_u$, $T_u(v) = \varphi(v)u$ for all $v \in L$. Some properties of this mapping T are included in the next theorem.

THEOREM 2.4. *Let L and M be two Riesz spaces with $L^\sim \neq \{\theta\}$ and with M Dedekind complete and let $T : M \rightarrow \mathscr{L}_b(L, M)$ be defined as above. Then we have:*

(i) *T is a one-to-one Riesz homomorphism from M into $\mathscr{L}_b(L, M)$.*

(ii) *T preserves arbitrary suprema and arbitrary infima, i.e., T is a normal Riesz homomorphism.*

Proof. (i) It is obvious that T is a positive linear mapping from M into $\mathscr{L}_b(L, M)$. To see that T is one-to-one let $T_u = \theta$ for some $u \in M$. Then we have $\varphi(v)u = \theta$ for all $v \in L$ and so $u = \varphi(f_0)u = \theta$. To see that T is a Riesz homomorphism let u, w be in M such that $u \wedge w = \theta$. Then for every $\theta \leqq v \in L$ we have $\theta \leqq (T_u \wedge T_w)(v) \leqq T_u(v) \wedge T_w(v) = \varphi(v)u \wedge w = \theta$, i.e., $T_u \wedge T_w = \theta$ and this completes the proof.

(ii) Assume that $u_\alpha \downarrow \theta$ in M and that $T_{u_\alpha} \geqq S \geqq \theta$ for all α and some S in $\mathscr{L}_b(L, M)$. Then we have $\varphi(v) \cdot u_\alpha \geqq S(v) \geqq \theta$ for all $v \in L^+$ and this implies $S(v) = \theta$ for all $v \in L^+$, i.e., $S = \theta$. Hence $T_{u_\alpha} \downarrow \theta$ in $\mathscr{L}_b(L, M)$ and this completes the proof of (ii).

Some more properties of $\mathscr{L}_b(L, M)$ are included in the next theorem.

THEOREM 2.5. *If L and M are two Riesz spaces with $L^\sim \neq \{\theta\}$ and M Dedekind complete then the following hold:*

(i) *If $\mathscr{L}_b(L, M)$ has a strong unit then M also has a strong unit.*

(ii) *If $\mathscr{L}_b(L, M)$ is universally complete then M is also universally complete.*

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†This example was exhibited by Professor W. A. J. Luxemburg during a discussion in a seminar at the California Institute of Technology.

Proof. (i) Let $\theta < \varphi \in L^{\sim}$ be as above, and let $\theta \leqq T_0 \in \mathscr{L}_b(L, M)$ be a strong unit for $\mathscr{L}_b(L, M)$. Given $u \in M$, determine $n \in N$ such that $T_u \leqq nT_0$. Hence $u \leqq nT_0(f_0)$. This shows that $T_0(f_0)$ is a strong unit of M.

(ii) Let $\{u_\alpha\}$ be a mutually disjoint system of M^+. Then $\{T_{u_\alpha}\}$ is a mutually disjoint system of $\mathscr{L}_b^+(L, M)$ (the proof is similar to that of Theorem 2.4 (i)). Hence, since $\mathscr{L}_b(L, M)$ is a universally complete Riesz space $S = \sup \{T_{u_\alpha}\}$ exists in $\mathscr{L}_b(L, M)$. It is easily seen now that $S(f_0) = \sup \{u_\alpha\}$ in M. This shows that M is universally complete and the proof is finished.

3. Extension of order bounded linear transformations. Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Assume that T is an order bounded linear transformation from A into M. The order bounded transformation S from L into M is called an extension of T, if $S(u) = T(u)$ for all u in A, i.e., $S = T$ on A. In this case we shall call T an extendable transformation. It is easy to verify that if $\theta \leqq T \in \mathscr{L}_b(A, M)$ and if T is extendable, then T has a positive extension on L. Indeed, let S be an extension of T. Then if $u \in A^+$ we have

$$S^+(u) = \sup \{S(v) : v \in L; \theta \leqq v \leqq u\} = \sup \{T(v) : v \in A; \theta \leqq v \leqq u\} = T^+(u) = T(u),$$

i.e., S^+ is a positive extension of T.

More generally, if S is an extension of T then S^+ is an extension of T^+ and S^- is an extension of T^-. In other words, T is extendable if and only if T^+ and T^- are both extendable.

It is not true that every operator of $\mathscr{L}_b(A, M)$ is extendable to $\mathscr{L}_b(L, M)$. As an example take $L = L_p([0, 1])$, $0 < p < 1$, $M = \mathbf{R}$ and A the ideal of all bounded (a.e.) Lebesgue measurable functions on $[0, 1]$. The linear mapping $\varphi : A \to \mathbf{R}$ defined by

$$\varphi(u) = \int_0^1 u(x)dx \quad \text{for all } u \in L$$

is a positive one, but φ cannot be extended to L as an order bounded linear mapping, since $L^{\sim} = \{\theta\}$ (see [**11**, p. 86]).

More details about extensions are included in the next theorem.

THEOREM 3.1. *Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Then we have:*

(i) *The set of all extendable transformations of $\mathscr{L}_b(A, M)$ forms an ideal of $\mathscr{L}_b(A, M)$, which we shall denote by $\mathscr{L}_b^e(A, M)$.*

(ii) *For every $\theta \leqq T \in \mathscr{L}_b^e(A, M)$ there exists a smallest positive extension T_m, in the sense that for every positive extension S of T on L we have $T_m \leqq S$ in $\mathscr{L}_b(L, M)$. Moreover*

$$T_m(u) = \sup \{T(v) : v \in A; \theta \leqq v \leqq u\}$$

for all u in L^+. In particular we have $T_m(u) = \theta$ for all $u \in A^d$.

Proof. (i) It is evident that $\mathscr{L}_b^e(A, M)$ is a vector subspace of $\mathscr{L}_b(A, M)$. Now let $\theta \leqq S \leqq T$ in $\mathscr{L}_b(A, M)$ with $T \in \mathscr{L}_b^e(A, M)$. Without loss of generality we can assume that T is defined on all of L. Then we have $S(u) \leqq |S(u)| \leqq S(|u|) \leqq T(|u|)$ for all u in A, and that the function $p : L \to M$, $p(u) = T(|u|)$ is a sublinear mapping. It follows from Theorem 1.7 that S is extendable to all of L as a linear transformation S_1 satisfying $S_1(f) \leqq T(|f|)$ for all f in L. It is easily seen that $S_1 \in \mathscr{L}_b(L, M)$ and so $S \in \mathscr{L}_b^e(A, M)$. The conclusion that $\mathscr{L}_b^e(A, M)$ is an ideal of $\mathscr{L}_b(A, M)$, now follows from the earlier observation that $T \in \mathscr{L}_b^e(A, M)$ if and only if T^+ and T^- are both in $\mathscr{L}_b^e(A, M)$, and so, in particular $T \in \mathscr{L}_b^e(A, M)$ implies $|T| = T^+ + T^-$ in $\mathscr{L}_b^e(A, M)$.

(ii) Since T is extendable, it is easy to see that $\sup \{T(v) : v \in A; \theta \leqq v \leqq u\}$ exists in M for all u in L^+. So, let $T_m(u) = \sup \{T(v) : v \in A; \theta \leqq v \leqq u\}$, $u \in L^+$. It is easily verified that T_m is an additive mapping from L^+ into M^+. Consequently, by Lemma 1.1 T_m is extendable uniquely to a positive linear transformation on L, which we shall denote also by T_m. Obviously T_m is a positive extension of T. Now let S be a positive extension of T, $u \in L^+$ and $v \in A$ such that $\theta \leqq v \leqq u$. Then $T(v) = S(v) \leqq S(u)$ and so $T_m(u) \leqq S(u)$, i.e., $T_m \leqq S$ in $\mathscr{L}_b(L, M)$ and this completes the proof.

Given two Riesz spaces L and M with M Dedekind complete the Riesz annihilator A° of a subset A of L is defined by

$$A^\circ = \{T \in \mathscr{L}_b(L, M) : T = \theta \text{ on } A\}.$$

It is obvious that A° is a linear subspace of $\mathscr{L}_b(L, M)$. The inverse Riesz annihilator ${}^\circ B$ of a subset B of $\mathscr{L}_b(L, M)$ is defined by

$${}^\circ B = \{u \in L : T(u) = \theta \text{ for all } T \in B\}.$$

Evidently ${}^\circ B$ is a linear subspace of L.

THEOREM 3.2. *Assume that L and M are two Riesz spaces with M Dedekind complete. Then we have*:

(i) *If A is an ideal of L, then A° is a band of $\mathscr{L}_b(L, M)$.*

(ii) *If B is an ideal of $\mathscr{L}_b(L, M)$, th n ${}^\circ B$ is an ideal of L.*

Proof. Part (i) is a straightforward application of Theorem 1.2 and part (ii) follows immediately from Theorem 2.1.

It is not difficult to see that the mapping $T \to T_m$ from $(\mathscr{L}_b^e(A, M))^+$ into $(\mathscr{L}_b(L, M))^+$ is an additive one. Indeed, given $\theta \leqq T, S \in \mathscr{L}_b^e(A, M)$ we obviously have $(T + S)_m \leqq T_m + S_m$. On the other hand if U is a positive extension of $T + S$ then $U - T_m$ is a positive extension of S (note that $\theta \leqq T \leqq S$ in $\mathscr{L}_b(A, M)$ implies according to Theorem 3.1 (ii) $\theta \leqq T_m \leqq S_m$ in $\mathscr{L}_b(L, M)$) and so $U - T_m \geqq S_m$, i.e., $S_m + T_m \leqq U$ for all positive extensions U of $S + T$. Hence $S_m + T_m \leqq (S + T)_m$ and this shows that $(T + S)_m = T_m + S_m$. According to Lemma 1.1 there exists a linear extension of the above mapping from $\mathscr{L}_b{}^e(A, M)$ into $\mathscr{L}_b(L, M)$, namely, $T \to T_m = (T^+)_m - (T^-)_m$. This mapping is one-to-one. Indeed, if $T_m = (T^+)_m - (T^-)_m = \theta$ then $(T^+)_m = (T^-)_m$ and hence $T^- = T^+$ on A and this implies that $T = T^+ - T^- = \theta$. It is also true that $T \to T_m$ is a Riesz homomorphism from $\mathscr{L}_b{}^e(A, M)$ into $\mathscr{L}_b(L, M)$. Indeed, if $\theta \leqq T$, $S \in \mathscr{L}_b{}^e(A, M)$, then $T_m \vee S_m$ is a positive extension of $T \vee S$, so $(T \vee S)_m \leqq T_m \vee S_m$. On the other hand if U is a positive extension of $T \vee S$ then $U \geqq T_m \vee U_m$. This shows that $(T \vee S)_m \geqq T_m \vee S_m$, so $(T \vee S)_m = T_m \vee S_m$. Now assume T, $S \in \mathscr{L}_b{}^e(A, M)$. Then we have $\theta \leqq T + S^- + T^-, S + S^- + T^- \in \mathscr{L}_b{}^e(A, M)$. Thus

$$\begin{aligned}[][(T + S^- + T^-) \vee (S + S^- + T^-)]_m \\ &= (T + S^- + T^-)_m \vee (S + S^- + T^-)_m \\ &= [T_m + (S^- + T^-)_m] \vee [S_m + (S^- + T^-)_m].\end{aligned}$$

So, we get

$$\begin{aligned} T_m \vee S_m &= [T_m + (S^- + T^-)_m \vee [S_m + (S^- + T^-)_m] - (S^- + T^-)_m \\ &= [(T + S^- + T^-) \vee (S + S^- + T^-)]_m - (S^- + T^-)_m \\ &= (T \vee S)_m. \end{aligned}$$

Note also that $(T_m)_m = T_m$ for all $T \in \mathscr{L}_b{}^e(A, M)$. From this observation and Theorem 3.2 it follows that the range of the mapping $T \to T_m$ is the band $(A^\circ)^d$. Hence, we have proved the following theorem:

THEOREM 3.3. *Let L and M be two Riesz spaces with M Dedekind complete and let A be an ideal of L. Then we have:*

(i) *The mapping $T \to T_m$ from $\mathscr{L}_b{}^e(A, M)$ into $\mathscr{L}_b(L, M)$ is a one-to-one Riesz homomorphism.*

(ii) *The range of the mapping $T \to T_m$ is the band $(A^\circ)^d$.*

Note. The results of the section are generalizations of the corresponding results for l near functionals due to Luxemburg and Zaanen (see [**7**, Note IX]).

4. Integral and normal integral transformations. Let L be the Riesz space of all real valued, Lebesgue integrable functions defined on [0, 1] with ordering $f \leqq g$ whenever $f(x) \leqq g(x)$ for all $x \in [0, 1]$. Consider the linear functionals

$$\varphi(u) = \int_0^1 u(x)dx, \quad u \in L,$$

i.e., φ is the usual Lebesgue integral, and $\psi(u) = u(0)$, $u \in L$. We can verify easily that $u_n \downarrow \theta$ in L, implies $\varphi(u_n) \downarrow 0$ and $\psi(u_n) \downarrow 0$ in $\mathbf{R}$. Also $u_\alpha \downarrow \theta$ in L implies $\psi(u_\alpha) \downarrow 0$ in $\mathbf{R}$, but not necessarily $\varphi(u_\alpha) \downarrow 0$ as the following example shows: Let $u_\alpha = 1 - \chi_\alpha$, $\alpha \subseteq [0, 1]$; α finite. Then $u_\alpha \downarrow \theta$ in L, but $\varphi(u_\alpha) = 1$ for all α.

In the next definition we characterize the above properties.

Definition 4.1. *Let L and M be two given Riesz spaces. A transformation T of $\mathscr{L}(L, M)$ is called an integral (respectively, a normal integral) if $T(u_n) \xrightarrow{(o)} \theta$ in M (respectively $T(u_\alpha) \xrightarrow{(o)} \theta$ in M) whenever $u_n \xrightarrow{(o)} \theta$ in L (respectively $u_\alpha \xrightarrow{(o)} \theta$ in L).*

It is evident that a normal integral is an integral but the converse is not always true as the example preceding the definition shows.

The next theorem is due to T. Ogasawara [**10**]. A proof can be found in [**13**, Theorem VIII 3.3, p. 216].

THEOREM 4.2. *Let L and M be two Riesz spaces with M Dedekind complete. Then we have:*

(i) *The set of all normal integrals of $\mathscr{L}_b(L, M)$ forms a band of $\mathscr{L}_b(L, M)$.*

(ii) *The set of all integrals of $\mathscr{L}_b(L, M)$ forms a band of $\mathscr{L}_b(L, M)$.*

Given $T \in \mathscr{L}_b(L, M)$ the ideal $N_T = \{u \in L : |T|(|u|) = \theta\}$ is called the null ideal of T and the band $C_T = N_T{}^d$ is called the carrier of T.

THEOREM 4.3. *Let L and M be two Riesz spaces with L σ-Dedekind complete and with M super Dedekind complete. Then we have:*

(i) *For every $\theta \leqq T \in \mathscr{L}_b(L, M)$, the band C_T is a projection band, i.e., $\{N_T\} \oplus C_T = L$. ($\{N_T\}$ denotes the band generated by N_T in L.)*

(ii) *If $T \in \mathscr{L}_b(L, M)$ is an integral then T is a normal integral if and only if N_T is a band of T.*

Proof. Repeat the proof of Theorem 31.15 of [**7**, Note X, p. 494].

Note: For the necessity of Theorem 4.3 (ii) we do not have to assume that L is σ-Dedekind complete.

The next example shows that the above statements may be false if L is Archimedean but no σ-Dedekind complete.

Example 4.4. Let $L = C(\mathbf{R}_\infty)$, where $\mathbf{R}_\infty$ is the one-point compactification of $\mathbf{R}$ considered with the discrete topology (see [**8**, Example (v), p. 140]) and let $M = \mathbf{R}$. Consider the positive linear functional $\theta \leqq \varphi \in L^\sim$ defined by

$$\varphi(u) = u(\infty) + \sum_{n=1}^{\infty} \frac{u(n)}{2^n}, \quad u \in L.$$

Note that φ is an integral but not a normal integral. Also

$$N_\varphi = \{u \in L : u(n) = 0, \text{ for } n = 1, 2, \ldots\}.$$

It is easily seen that N_φ is a band of L with the property $N_\varphi \oplus C_\varphi \neq L$.

More properties about integrals and normal integrals are included in the next theorems.

THEOREM 4.5. *Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Assume $\theta \leqq T \in \mathscr{L}_b(A, M)$ is an integral (respectively* *a normal integral) and assume further that T is an extendable transformation. Then the minimal extension T_m of T, determined by Theorem* 3.1 (ii) *is an integral* (*respectively, a normal integral*).

Proof. Assume that $\theta \leqq u_n \uparrow u$ in L, and assume $v \in A$; $\theta \leqq v \leqq u$. Then $\theta \leqq v \wedge u_n \uparrow v$ in L, and since A is an ideal of L we have also that $v \wedge u_n \uparrow u$ in A. Hence, $T(v \wedge u_n) \uparrow T(v)$ in M. But $T(v \wedge u_n) = T_m(v \wedge u_n) \leqq T_m(u_n) \leqq T_m(u)$ and this shows that $T(v) \leqq \sup \{T_m(u_n) : n = 1, 2, \ldots\} \leqq T_m(u)$. It follows now from Theorem 3.1 (ii) that $T_m(u_n) \uparrow T_m(u)$ and this shows that T_m is a normal integral of $\mathscr{L}_b(L, M)$. The proof for the normal integral is similar.

THEOREM 4.6. *Let L and M be two Riesz spaces with M super Dedekind complete. If $\theta \leqq T \in \mathscr{L}_b(L, M)$ is a strictly positive transformation which is an integral then T is a normal integral.*

Proof. Repeat the proof of Theorem 31.11 (ii), [**7**, Note X, p. 493].

A Theorem of H. Nakano [**9**, Theorem 20.1, p. 74] states that if L is σ-Dedekind complete and if φ and ψ are two order bounded normal integrals then $\varphi \perp \psi$ in $L^\sim$ if and only if $C_\varphi \perp C_\psi$.

This result was generalized by Luxemburg and Zaanen for Archimedean Riesz spaces (see [**7**, Note IV, Theorem 31.2 (ii), p. 373]). The following example due to W. A. J. Luxemburg shows that this Theorem cannot be further generalized.

Consider $L = M = L_1([0, 1])$. So, both L and M are super Dedekind complete Riesz spaces. Let $\theta \leqq S, T : L \rightarrow M$, $Su = u$,

$$Tu = \left(\int_0^1 u(x)dx \right) \cdot e \quad \text{for all } u \in L \qquad (e(x) = 1 \text{ for all } x \in [0, 1]).$$

Note that both S and T are normal integrals. Also $N_T = N_S = \{\theta\}$. So $C_T = C_S = L$. But $S \perp T$ as it is easily seen from Theorem 1.2 (ii). (Note that $(S \wedge T)(e) = \theta$ and this implies that $T \wedge S = \theta$, since $S \wedge T$ is a normal integral according to Theorem 4.2.)

Given two Riesz spaces L and M with M Dedekind complete we denote by $(\mathscr{L}_b)_n = (\mathscr{L}_b(L, M))_n$, $(\mathscr{L}_b)_c = (\mathscr{L}_b(L, M))_c$ the bands of the normal integrals and integrals, respectively of $\mathscr{L}_b(L, M)$.

It follows from the fact that $\mathscr{L}_b(L, M)$ is a Dedekind complete Riesz space that

$$\mathscr{L}_b(L, M) = (\mathscr{L}_b)_n \oplus ((\mathscr{L}_b)_n)^d = (\mathscr{L}_b)_c \oplus ((\mathscr{L}_b)_c)^d.$$

We shall denote the bands $((\mathscr{L}_b)_n)^d$, $((\mathscr{L}_b)_c)^d$ by $(\mathscr{L}_b)_{sn}$, $(\mathscr{L}_b)_s$, respectively. The band $(\mathscr{L}_b)_{sn} \cap (\mathscr{L}_b)_c$ is denoted by $(\mathscr{L}_b)_{sn,c}$. It is easily seen that

$$\mathscr{L}_b(L, M) = (\mathscr{L}_b)_n \oplus (\mathscr{L}_b)_{sn,c} \oplus (\mathscr{L}_b)_s.$$

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Thus every $T \in \mathscr{L}_b(L, M)$ has a unique decomposition $T = T_n + T_{sn,c} + T_s$ ($T_c = T_n + T_{sn,c}$) where the elements on the right are in $(\mathscr{L}_b)_n$, $(\mathscr{L}_b)_{sn,c}$ and $(\mathscr{L}_b)_s$, respectively. It is easy to see [**8**, Theorem 4.4 (iii)] that

$$T^+ = T_n^+ + T_{sn,c}^+ + T_s^+, \quad T^- = T_n^- + T_{sn,c}^- + T_s^-,$$
$$|T| = |T_n| + |T_{sn,c}| + |T_s|$$

are the decompositions of T^+, T^- and $|T|$, respectively. The operator T_n is called the normal component of T, $T_c = T_n + T_{sn,c}$ is the integral component of T, T_s is the singular integral component of T and the operator $T_{sn} = T_{sn,c} + T_s$ is called the singular normal integral component of $T(T_{sn} \in (\mathscr{L}_b)_{sn})$.

We shall investigate next some of the properties of the different components of T. We start with the following Lemma.

LEMMA 4.7. *Assume that L and M are two Riesz spaces with M Dedekind complete and that $\theta \leqq T \in \mathscr{L}_b(L, M)$. We consider the following mappings from L^+ into M^+:*

(i) $T_L(u) = \inf\{\sup\{T(u_n)\} : \theta \leqq u_n \uparrow u\}$,
(ii) $\bar{T}_L(u) = \inf\{\sup\{T(u_\alpha)\} : \theta \leqq u_\alpha \uparrow u\}$,
(iii) $\bar{T}(u) = \sup\{\inf\{T(u_\alpha)\} : u \geqq u_\alpha \downarrow \theta\}$,

for every $u \in L^+$. Then, T_L, $\bar{T}_L$ and $\bar{T}$ are additive on L^+.

Proof. The proof is a straightforward verification and so we omit it.

THEOREM 4.8. *Let L and M be as in Lemma 4.7. Assume further that $^\circ(M_n^\sim) = \{u \in M : \varphi(u) = \theta$ for all $\varphi \in M_n\} = \{\theta\}$. Then for every $\theta \leqq T \in \mathscr{L}_b(L, M)$ and for every $u \in L^+$ we have:*

(i) $T_c(u) = \inf\{\sup\{T(u_n)\} : \theta \leqq u_n \uparrow u\}$,
(ii) $T_n(u) = \inf\{\sup\{T(u_\alpha)\} : \theta \leqq u_\alpha \uparrow u\}$,
(iii) $T_{sn}(u) = \sup\{\inf\{T(u_\alpha)\} : u \geqq u_\alpha \downarrow \theta\}$.

Proof. According to Lemma 1.1 T_L, $\bar{T}_L$ and $\bar{T}$ are extendable to the whole L. Let $u_n \downarrow \theta$ in L. Then $T_L(u_n) \downarrow h \geqq \theta$ in M for some h of M^+. We show next that $h = \theta$. To this end let $\theta \leqq \varphi \in M_n^\sim$. Note that

$$\begin{aligned}(\varphi_0 T_L)(u) &= \varphi(T_L(u)) = \varphi(\inf\{\sup\{T(u_n)\} : \theta \leqq u_n \uparrow u\}) \\ &= \inf\{\sup\{\varphi(T(u_n))\} : \theta \leqq u_n \uparrow u\} \\ &= (\varphi_0 T)_L(u), \quad \text{for all } u \in L^+.\end{aligned}$$

Hence $\varphi_0 T_L = (\varphi_0 T)_L$. Now use Theorem 20.4 of [**7**, Note VI, p. 663] to get that $(\varphi_0 T)_L = (\varphi_0 T)_c$, i.e., that $(\varphi_0 T)_L$ is an integral. Thus $(\varphi_0 T_L)(u_n) \downarrow \theta$. But we also have $(\varphi_0 T_L)(u_n) = \varphi(T_L(u_n)) \downarrow \varphi(h)$. Thus $\varphi(h) = 0$ for all $\varphi \in M_n^\sim$ and so $h = \theta$ and hence T_L is an integral. Now, it follows from $\theta \leqq T_L \leqq T$ that $T_L = (T_L)_c \leqq T_c$. On the other hand we have $T_c \leqq T$ and so $(T_c)_L \leqq T_L$. But from the definition of $(T_c)_L$ it follows that $(T_c)_L = T_L$, thus $T_c \leqq T_L$. Hence $T_c = T_L$ and the proof of the first formula is finished. For the other two results use the same arguments in connect on with Luxemburg's Theorem 57.6, of [**5**, Note XV, p. 441].

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Example 4.9. Let L_1 be the Riesz space exhibited before the Definition 4.1 and let $L_2 = C_{[0,1]}$. Now let $L = L_1 \times L_2$ and $M = \mathbf{R}$. Define $\varphi \in L^\sim$ by

$$\varphi(f) = \int_0^1 u(x)dx + \int_0^1 v(x)dx \quad \text{for all } f = (u, v) \in L.$$

Now use the formulas of the previous theorem to get:

$$\varphi_c(f) = \int_0^1 u(x)dx, \quad \varphi_n(f) = 0, \quad \varphi_{sn,c}(f) = \varphi_c(f), \quad \text{and}$$

$$\varphi_s(f) = \int_0^1 v(x)dx, \quad \text{for all } f = (u, v) \in L.$$

THEOREM 4.10. *Let L and M be as in Lemma* 4.7. *Then we have:*

(i) *In the formula* $T_c(u) = \inf \{\sup \{T(u_n)\} : \theta \leqq u_n \uparrow u\}$, *the greatest lower bound is attained if, and only if,* N_{T_s} *is super order dense in L.*

(ii) *In the formula* $T_n(u) = \inf \{\sup \{T(u_\alpha)\} : \theta \leqq u_\alpha \uparrow u\}$, *the greatest lower bound is attained if, and only if,* $N_{T_{sn}}$ *is order dense in* $L(T_{sn} = T_{sn,c} + T_s)$.

Proof. (i) Let $\theta \leqq u \in L$ and let $T_c(u) = \sup \{T(u_n)\}$ where $\theta \leqq u_n \uparrow u$. Note that $T = T_c + T_s$. It follows from this that $T_s(u_n) = 0$ for all n, i.e., that $\{u_n\} \subseteq N_{T_s}$. This shows that N_{T_s} is super order dense in L. Now let N_{T_s} be super order dense in L and let $\theta \leqq u \in L$. Pick a sequence $\{u_n\} \subseteq N_{T_s}$ such that $\theta \leqq u_n \uparrow u$. But then $T(u_n) = T_c(u_n) \uparrow T_c(u)$, i.e., that $T_c(u) = \min \{\sup \{T(u_n)\} : \theta \leqq u_n \uparrow u\}$.

(ii) A similar argument proves (ii).

THEOREM 4.11. *Let L and M be two Riesz spaces with M Dedekind complete and let* $T \in \mathscr{L}_b(L, M)$. *Then the largest ideal on which T is an integral is* N_{T_s} *and the largest ideal on which T is normal is the ideal* $N_{T_{sn}} = N_{T_s} \cap N_{T_{sn,c}}$.

Proof. Note first that T restricted to N_{T_s} is an integral. Now assume that A is an ideal such that T is an integral when restricted to A. Then T_s restricted to A is also an integral. But T_s restricted to A has an extension to all of L (namely T_s). Thus T_s has a minimal positive extension $(T_s)_m$ which according to Theorem 4.5 is an integral. But $\theta \leqq (T_s)_m \leqq T_s \in ((\mathscr{L}_b)_c)^d$, so $(T_s)_m \in ((\mathscr{L}_b)_c)^d$. Hence $(T_s)_m = 0$ and this implies $A \subseteq N_{T_s}$. A similar argument proves the second part.

THEOREM 4.12. *Let L and M be two Riesz spaces with M super Dedekind complete. Then we have:*

(i) *For every* $T \in (\mathscr{L}_b)_{sn,c}$ *the null ideal* N_T *is quasi-order dense in L. In particular, if L is Archimedean,* N_T *is order dense in L.*

(ii) *The largest ideal on which an integral* $T \in (\mathscr{L}_b)_c$ *is normal is the* σ*-quasi order dense ideal* $N_{T_{sn,c}}$.

Proof. (i) Let $\theta \leqq T \in (\mathscr{L}_b)_{sn,c}$ and let $N_T{}^d \neq \{\theta\}$. Then T restricted to $N_T{}^d$ is a strictly positive integral and hence it is normal on $N_T{}^d$ by Theorem

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4.6. Note that the restriction of T on $N_T{}^d$ has a smallest positive extension T_m, which according to Theorem 4.5 is a normal integral of $\mathscr{L}_b(L, M)$. It follows from $\theta \leqq T_m \leqq T$ on L that $T_m \in (\mathscr{L}_b)_{sn}$ and hence $T_m = \theta$. This implies that $T = \theta$ on $N_T{}^d$, i.e., $N_T{}^d = \{\theta\}$, a contradiction. Thus $N_T{}^d = \{\theta\}$ and so $N_T{}^{dd} = L$.

(ii) If $T_{sn,c} \in (\mathscr{L}_b)_{sn,c} = (\mathscr{L}_b)_{sn} \cap (\mathscr{L}_b)_c$, $N_{T_{sn,c}}$ is a quasi order dense ideal in L, according to the previous statement and since $T_{sn,c}$ is in $(\mathscr{L}_b)_c$, it is evident that $N_{T_{sn,c}}$ is a σ-ideal of L. Now let $\theta \leqq T \in (\mathscr{L}_b)_c$ and let A be an ideal of L on which T is normal. Write $T = T_n + T_{sn,c} + T_s$ and note that $T_s = \theta$. Thus $T = T_n + T_{sn,c}$ and so T is normal restricted on $N_{T_{sn,c}}$. Note also that $T_{sn,c} = T - T_n$ is normal if restricted to A. Theorem 4.5 shows that the minimal extension of $T_{sn,c}$ (from A to L) is also normal. Since $\theta \leqq (T_{sn,c})_m \leqq T_{sn,c}$ we get that $(T_{sn,c})_m \in (\mathscr{L}_b)_{sn}$, i.e., $(T_{sn,c})_m = \theta$. So, $T_{sn,c} = \theta$ on A, i.e., $A \subseteq N_{T_{sn,c}}$ and the proof is finished.

COROLLARY 4.13. *Let L and M be as in the previous theorem. Then we have:*

(i) *For every $T \in (\mathscr{L}_b)_c$ we have $N_T \oplus N_T{}^d \subseteq N_{T_{sn,c}}$.*

(ii) *If L is Archimedean, then $T \in (\mathscr{L}_b)_{sn,c}$ if, and only if, $T \in (\mathscr{L}_b)_c$ and $N_T{}^d = \{\theta\}$.*

Note. The last three results are generalizations of the corresponding results for $L^\sim$ due to W. A. J. Luxemburg (see [**5**, pp. 417–420]).

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