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DENSITY OF NUMERICAL RADIUS ATTAINING OPERATORS ON SOME REFLEXIVE SPACES

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We show that for reflexive spaces X the density of numerical radius attaining operators in $L(X)$ is equivalent to the density of numerical radius attaining operators in $L(X^*)$. As a consequence of this fact and of a result of Berg and Sims, we prove that for uniformly smooth spaces X the numerical radius attaining operators are dense in $L(X)$.

If X is a Banach space, we define the numerical radius of a bounded linear operator $T : X \to X$, denoted by $v(T)$, by

$$v(T) = \sup\{|\langle x^*, Tx\rangle| : (x, x^*) \in \Pi(X)\} ,$$

where $\Pi(X) = \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = \langle x^*, x\rangle = 1\}$.

We say that T attains its numerical radius if there is $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |\langle x_0^*, Tx_0\rangle|$ and denote the set of numerical radius attaining operators by $\mathrm{NRA}(X)$.

We denote $L(X)$ the space of all bounded linear operators from X to X.

THEOREM 1. *Let X be a reflexive space. Then* $\overline{\mathrm{NRA}(X)} = L(X)$ *if*

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and only if $\overline{\mathrm{NRA}(X^*)} = L(X^*)$.

Proof. We will show first that $\overline{\mathrm{NRA}(X)} = L(X)$ implies $\overline{\mathrm{NRA}(X^*)} = L(X^*)$.

Let $R \in L(X^*)$ and $\varepsilon > 0$ be given. Since X is reflexive, there is $T \in L(X)$ such that $T^* = R$.

By hypothesis, there is $T_0 \in \mathrm{NRA}(X)$ such that $\|T-T_0\| < \varepsilon$. Let $(x_0, x_0^*) \in \Pi(X)$ be such that $v(T_0) = |\langle x_0^*, T_0x_0\rangle|$.

Then $(x_0^*, \hat{x}_0) \in \Pi(X^*)$, where ^ denotes the canonical injection from X into X^{**} , which in this case is also onto.

Since $v(T_0) = v(T_0^*)$ ([2]), we get $v(T_0^*) = |\langle \hat{x}_0, T_0^*x_0^*\rangle|$ and then $T_0^* \in \mathrm{NRA}(X^*)$.

But $\|R-T_0^*\| = \|T-T_0\| < \varepsilon$ and so we conclude that $\overline{\mathrm{NRA}(X^*)} = L(X^*)$.

For the reverse implication, note that X^* is reflexive and apply the first part of the proof to get that $\overline{\mathrm{NRA}(X^*)} = L(X^*)$ implies $\overline{\mathrm{NRA}(X^{**})} = L(X^{**})$.

Since X is reflexive, $\mathrm{NRA}(X^{**}) = \mathrm{NRA}(X)$ and $L(X^{**}) = L(X)$ and we have finished the proof.

COROLLARY 2. *Let* X *be a uniformly smooth space. Then* $\overline{\mathrm{NRA}(X)} = L(X)$.

Proof. Since X is uniformly smooth, X^* is uniformly convex. By [1], $\overline{\mathrm{NRA}(X^*)} = L(X^*)$. But X is also reflexive and applying Theorem 1 we get $\overline{\mathrm{NRA}(X)} = L(X)$.

We note that it is possible to give a direct proof of the result of Corollary 2, by means of a minor modification on Berg and Sims' argument ([1]).

Incidentally, we do not know any example of a reflexive space X for which $\overline{\mathrm{NRA}(X)} \neq L(X)$. Actually we do not know any example of a Banach space with this property.

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