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PRIME z-IDEALS OF $C(X)$ AND RELATED RINGS

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1. **Introduction.** Let $C(X)$ be the ring of continuous real-valued functions on a (completely regular) topological space X. The structure of the prime ideals and the prime z-ideals of $C(X)$ has been the subject of much investigation (see e.g. [1], [3], [5]). One of the surprising facts about $C(X)$ is that the sum of two prime ideals is again prime. The sum of two z-ideals is also a z-ideal. In this paper we show that the sum of two prime ideals is in fact a prime z-ideal, as is the sum of a z-ideal and a prime ideal. We also show that every ideal I contains a unique maximal z-ideal which is prime when I is prime. From these results we obtain information about the chains of prime (z-) ideals in $C(X)$. The "related rings" of the title are of two types. In the first place, all results in §3 hold at least for absolutely convex subrings of $C(X)$. Secondly, we consider in §4 those rings in which the prime ideals have the reverse order of the primes in $C(X)$.

2. **Preliminaries.** The key reference for results about $C(X)$ is of course [1], and unless otherwise indicated, the following results come from there.

Letting $M(f)$ denote the set of maximal ideals containing $f \in C(X)$, a z-ideal can be defined as an ideal I such that if $M(f) \supseteq M(g)$ and $g \in I$, then $f \in I$. The facts we need are

2.1. The sum of two z-ideals is a z-ideal.

2.2. The sum of two prime ideals is prime.

2.3. The prime ideals containing a given prime ideal form a chain.

2.4. Every z-ideal I is an intersection of prime z-ideals (the minimal primes containing I).

2.5. A z-ideal is prime iff it contains a prime ideal.

A subring $A \subset C(X)$ is called absolutely convex if $|f| \leq |g|$ and $g \in A \Rightarrow f \in A$. For some time it was believed that a proof for 2.1 and 2.2 depended on properties of βX but in [5] Rudd gave an algebraic proof showing that they, and 2.3, hold in absolutely convex subrings of $C(X)$. Since 2.4 holds in any commutative ring ([4]) and 2.5 follows from 2.3 and 2.4, we shall assume in §3 that all rings are absolutely convex subrings of $C(X)$.

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3. **Absolutely convex subrings of** $C(X)$**.** Every ideal I is contained in a least z-ideal, namely I_z = the intersection of all z-ideals containing I. In the notation of [1, 2.7], $I_z = Z^{\leftarrow}[Z[I]]$. I_z also has an elementwise characterization [4, 1.13] which we quote here for future reference:

$$I_z = \{a \in R \mid \exists b \in I \text{ with } M(b) \subseteq M(a)\}.$$

If I is not a z-ideal, and contains a prime ideal, it follows from 2.5 that I_z is a prime z-ideal (in fact the unique minimal prime ideal containing I). The next result holds in any ring.

PROPOSITION 3.1. (a) *If A is any index set and $\{I_\alpha\}_{\alpha\in A}$ is any family of ideals, then* $(\sum_{\alpha\in A} I_\alpha)_z = (\sum_{\alpha\in A} I_{\alpha z})_z$.

(b) *The following are equivalent:*

(i) *For all z-ideals I, J $I+J$ is a z-ideal.*

(ii) *For all ideals I, J $(I+J)_z = I_z + J_z$.*

(iii) *For every family $\{I_\alpha\}_{\alpha\in A}$ of z-ideals, $\sum_{\alpha\in A} I_\alpha$ is a z-ideal.*

(iv) *For every family $\{I_\alpha\}$ of ideals, $(\sum_{\alpha\in A} I_\alpha)_z = \sum_{\alpha\in A} I_{\alpha z}$.*

Proof. (a) Both $(\sum I_\alpha)_z$ and $(\sum I_{\alpha z})_z$ are z-ideals containing $\sum I_\alpha$ so by the minimality of the former, it is contained in the latter. Conversely $(\sum I_\alpha)_z$ is a z-ideal containing each I_α, therefore containing each $I_{\alpha z}$ and so contains $(\sum I_{\alpha z})_z$.

(b) (i)$\Rightarrow$(ii) By (i), $I_z + J_z$ is a z-ideal and so equals $(I_z + J_z)_z$.

(ii)$\Rightarrow$(i) If I, J are z-ideals then using (ii) we have $I+J = (I_z + J_z) = (I+J)_z$ which is a z-ideal.

(iii)$\Leftrightarrow$(iv) similarly and (iii)$\Rightarrow$(i) trivially.

(i)$\Rightarrow$(iii) If $M(a) \supseteq M(b)$ and $b \in \sum_{\alpha\in A} I_\alpha$ then there is a finite subset $B \subset A$ so that $b \in \sum_{\beta\in B} I_\beta$ and from (i) this is a z-ideal so $a \in \sum_{\beta\in B} I_\beta \subset \sum_{\alpha\in A} I_\alpha$. □

The next result shows that 2.2 follows from a slightly weaker form of 2.1, and in fact that the sum of two prime ideals not in a chain is not only prime but also a z-ideal. Recall that minimal primes are z-ideals and that every prime ideal is contained in a unique maximal ideal. Thus if P and Q are primes contained in distinct maximal ideals, $P+Q=R$ so we assume from now on that P and Q are in the same maximal ideal. The next result holds in any ring where 2.3 is true.

THEOREM 3.2. *If the sum of minimal prime ideals in R is a z-ideal then the sum of every two prime ideals which are not in a chain is a prime z-ideal. In fact if $\{P_\alpha\}_{\alpha\in A}$ is a family of prime ideals not all in a chain then $\sum_{\alpha\in A} P_\alpha$ is a prime z-ideal.*

Proof. P and Q contain distinct minimal prime ideals. Let I_P and I_Q be a choice for each. By hypothesis, $I_P + I_Q$ is a z-ideal and is prime by 2.5. Also

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$I_P+I_Q \subset P+Q$. On the other hand, I_P+I_Q is a prime ideal containing I_P and I_Q so is in a chain with both P and Q. Since P and Q are not themselves in a chain, both P and Q must be contained in I_P+I_Q whence $P+Q=I_P+I_Q$ is a prime z-ideal.

In the same way if I_α is a minimal prime ideal contained in P_α then $\sum I_\alpha \subseteq \sum P_\alpha$. Conversely if $x \in \sum P_\alpha$ then there is a finite subfamily $\{P_i\}_{i=1}^n$ of $\{P_\alpha\}$ such that $x \in \sum_{i=1}^n P_i$. Without loss of generality, no two P_i are in a chain, and $n>1$. Then I_1 is a minimal prime ideal contained in the prime (by induction) ideal $\sum_{i=1}^{n-1} P_i$ so $\sum_{i=1}^n P_i = I_1 + I_n$. Thus $x \in I_1+I_n \subset \sum I_\alpha$ as required (Note that, in fact, we have shown that every finite sum of prime ideals not in a chain is the sum of two minimal prime z-ideals.) □

We have seen that if P is a prime ideal which is not a z-ideal, then P_z is a prime z-ideal minimal over P. Dually the next result shows that there is a greatest prime z-ideal contained in P.

Theorem 3.3. *If I is an ideal which is not a z-ideal but which contains a z-ideal A, then there is a greatest element in $S_A = \{z\text{-ideals } J \mid A \subseteq J \subset I\}$. When I is prime this greatest element is also prime.*

Proof. In any commutative ring, since $S_A \neq \varnothing$ and is inductive, S_A has maximal elements by Zorn's lemma. If I is prime, all these elements will be prime. For suppose J is maximal in S_A. Then there is a prime ideal Q minimal with respect to containing J and contained in I. But because Q is minimal over the z-ideal J, it is a prime z-ideal (2.4). By the maximality of J, $J=Q$ is prime. Now in our rings, if J_1 and J_2 are maximal in S_A, then by 2.1 $J_1+J_2 \in S_A$ contradicting the maximality of each J_i. Therefore there is a unique maximal element in S_A as required (see also 4.4). □

It follows that the maximal element of S_A is independent of the choice of A (since 0 is a z-ideal it will do) and so we will denote it by I^z (and put $I^z=I$ if I is already a z-ideal). Recall that an intersection of z-ideals is a z-ideal.

Lemma 3.4. $\bigcap I_\alpha^z = (\bigcap I_\alpha)^z$.

Proof. $\bigcap I_\alpha^z$ is a z-ideal contained in $\bigcap I_\alpha$. If J is a z-ideal contained in $\bigcap I_\alpha$, then $J \subset I_\alpha$ for all α so $J \subset I_\alpha^z$ for all α and $J \subset \bigcap I_\alpha^z$. Thus $\bigcap I_\alpha^z$ is the greatest z-ideal contained in $\bigcap I_\alpha$. □

We can now give an elementwise characterization of I^z corresponding to that for I_z.

Proposition 3.5.

$$I^z = \{a \in I_Z \mid M(y) \supseteq M(a) \Rightarrow y \in I\}.$$

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Proof. Denote the set on the right by S. S is an ideal, for if $a, b \in S$, then $a-b \in I_z$ and if $M(f) \supseteq M(a-b) \supseteq M(a) \cap M(b)$ then following [5, Lemma 3.1] $\exists h, k \in R$ such that (i) $f=h+k$ (ii) $|h|<|f|$, $|k|<|f|$ and (iii) $fa^2=h(a^2+b^2)$, $fb^2=k(a^2+b^2)$. We want to show $M(h) \supseteq M(a)$. If $a \in M$ then from (iii) either $h \in M$, or $a^2+b^2 \in M$ so $b \in M$ and so $f \in M$. But M is absolutely convex so $h \in M$ from (ii). Thus $M(h) \supseteq M(a)$ and similarly $M(k) \supseteq M(b)$. Since $a, b \in S$, therefore $h, k \in I$ and so from (i) $f \in I$. Thus $a-b \in S$ as required.

Also if $a \in S$, $r \in R$ then $ar \in I_z$ and $M(y) \supseteq M(ra) \supseteq M(a) \Rightarrow y \in I$ so $ar \in S$.

Moreover $I \supseteq S$ since $a \in S$ and $M(a)=M(a) \Rightarrow a \in I$; and S is a z-ideal for if $M(x) \supseteq M(a)$ with $a \in S$, then a is in the z-ideal I_z so $x \in I_z$. Also if $M(y) \supseteq M(x)$ then $M(y) \supseteq M(a)$ so $y \in I$.

Finally every z-ideal $J \subset I$ is contained in S for if $x \in J$, then $x \in I_z$ and $M(y) \supset M(x) \Rightarrow y \in J \subset I$ so $x \in S$. Thus S is the greatest z-ideal contained in I. □

Consider now how the prime and prime z-ideals occur in chains: We saw that if I is not a z-ideal and $I \supset Q$ a prime ideal then I_z is a *prime* z-ideal. Similarly I^z is a prime z-ideal by 2.5 since $I^z \supset Q^z$ and Q^z is prime. Suppose Q is not a z-ideal. Since I^z, Q and Q_z are primes containing Q^z, they are in a chain and the possible cases are:

(1) $Q^z \subset Q \subset Q_z \subseteq I^z \subset I$ (1(a)) $Q^z \subset Q \subset I^z \subseteq Q_z$; but this violates the minimality of Q_z unless $Q_z = I^z$ in which case this is a special case of (1).

(2) $Q^z \subseteq I^z \subset Q \subset Q_z$ which contradicts the maximality of Q^z unless $Q^z = I^z$.

Now we can show that each prime contained in a prime P is in a chain with P^z:

Proposition 3.6. *If $P \supset Q$ are primes which are not z-ideals then* (a) *either* (i) $Q \subset Q_z \subset P^z \subset P$ *or* (ii) $P^z \subset Q \subset P \subset Q_z$.

(b) *In case* (*i*) $Q_z = P^z = I$ *iff I is the unique z-ideal between Q and P.*

(c) *In case* (ii) *if J is any prime ideal with $P^z \subset J \subset Q_z$ then $P^z = J^z = Q^z$ and $P_z = J_z = Q_z$.*

Proof. (a) Case (i) is as above. Case (ii) comes from Case (2) above, noting that P and Q_z are primes containing Q and so must be in a chain. If $Q_z \subset P$ we would contradict the maximality of P^z, so we must have $P \subset Q_z$.

(b) If I is any z-ideal between P and Q then it must lie between P^z and Q_z. The result follows.

(c) This follows from the maximality of P^z and the minimality of Q_z. □

For example, if P contains a prime Q_1 that is not in a chain with Q then $Q+Q_1$ is a z-ideal between Q and P so Case (i) must hold. Then in fact $P^z \subset Q_1$. On the other hand in $C(X)$ itself if P has an immediate predecessor Q (P is "upper" and Q is "lower") then they are not z-ideals (see [1, Ch. 14])

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and since there are no primes between them, (ii) must hold. Then also Q is not upper and P is not lower, ([3, 2.11]) so there is an infinite chain of primes J between P^z and Q, and an infinite chain between P and Q_z all of which satisfy (c) of the proposition.

Now Rudd has shown ([5]) that the sum of a semiprime ideal I and a prime ideal P is prime (we assume I and P are not in a chain). The next result shows that this sum is sometimes a z-ideal. Recall (2.4) that a z-ideal is semiprime.

THEOREM 3.7. (a) *If I is a z-ideal and P is prime, $I+P$ is a z-ideal.*

(b) *If I is semi-prime and P is prime, then $I+P=I+(I+P)^z$. If moreover $I+P$ is not a z-ideal then $I+P$ is a minimal prime containing I and I contains no prime ideal.*

Proof. (a) If I is prime, we are done by 3.2. If I is not prime, then since I is a z-ideal in $I+P$, $(I+P)^z \supsetneqq I$. Then $I+P=P+(I+P)^z$ is a z-ideal.

(b) If I is semiprime write $I=\bigcap Q_i$ where the Q_i are minimal primes containing I. Since $I+P$ and P are primes in a chain, we have by Proposition 3.6 either: (i) $P\subseteq(I+P)^z\subset I+P$ or (ii) $(I+P)^z\subseteq P\subset I+P$. In case (i), if we add I to each term we get $I+P=I+(I+P)^z$. In case (ii), suppose first that I and $(I+P)^z$ are in a chain. If $I\subset(I+P)^z$ then $I\subset P$, a contradiction. If $(I+P)^z\subset I=\bigcap Q_i$ then the Q_i contain the prime ideal $(I+P)^z$ so form a chain, which is a contradiction. Now suppose I and $(I+P)^z$ are not in a chain. Then $I+(I+P)^z$ is a prime ideal (by Rudd's result) containing $(I+P)^z$ so is in a chain with P. If $P\subset I+(I+P)^z\subset I+P$ we again have $I+P=I+(I+P)^z$ directly; and if $I+(I+P)^z\subset P$, we have $I\subset P$, a contradiction.

If I contains a prime, it *is* prime (since the primes containing it form a chain) and we are done by 3.2.

Finally, if there is a prime Q with $I\subset Q\subset I+P$ then $I+P=Q+P$ is a z-ideal. Therefore if $I+P$ is not a z-ideal, no such Q exists so $I+P$ is a minimal prime containing I. □

REMARKS. In view of (b), in order to prove $I+P$ is a z-ideal it suffices to assume that P is a z-ideal; in fact it suffices to assume that $P=(I+P)^z$ is the largest z-ideal in $I+P$. We also have seen that if $I=\bigcap Q_i$, Q_i minimal over I, then we can assume $Q_1=I+P$. Since $I+P=(\bigcap Q_i)+P\subseteq\bigcap(Q_i+P)$ is always true, we then have $I+P=\bigcap(Q_i+P)$. Now Q_1 is the only Q_i containing P so each $Q_i+P(i\neq 1)$ is a z-ideal by 3.2. Thus if $I+P$ could be represented as $\bigcap_{i\neq 1}(Q_i+P)$, it would be a z-ideal. For example, suppose $I=\bigcap_{i=2}^n Q_i$ can be written as a finite intersection of minimal primes $Q_i\neq Q_1$. Then $x\in\bigcap_2^n(Q_i+P)\Rightarrow\exists q_i\in Q_i$, $p_i\in P$ with $x=q_i+p_i$ $i=2,\ldots,n$, so $\prod_2^n q_i\in\prod_2^n Q_i\subset\bigcap_2^n Q_i=I$ so $\pi q_i\in Q_1$. Hence $q_i\in Q_1$ for some i and $x\in Q_1=I+P$ so $I+P=\bigcap_2^n(Q_i+P)$ is a z-ideal. There are z-ideals of this type. For if every finite representation of I as $\bigcap Q_i$ requires Q_1, let $J=\bigcap_{i=2}^n Q_i$. Then $J\supsetneqq I$. J is semiprime so either $J+P$ is a z-ideal or $J+P$ is a minimal prime containing J.

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But $J+P\neq Q_i$ for $i=2,\ldots,n$ (or else $Q_i=J+P\supsetneqq I+P=Q_1$) so $J=\bigcap_2^n Q_i$ is written as finite intersection not involving $J+P$ and hence $J+P$ is a z-ideal as above.

4. **Other rings.** There is a class of rings related to the absolutely convex subrings of $C(X)$ whose prime (z-) ideals have an interesting structure. In [2] Hochster has shown that, given any commutative ring R, there are rings whose prime ideals have precisely the reverse order of the primes in R. Let $\tilde{R}$ denote one of these. Thus if A is an absolutely convex subring of $C(X)$, $\tilde{A}$ has the property:

4.1. The prime ideals contained in any prime ideal form a chain.

This property is shared by other classes of rings (e.g. Prufer domains) and from it alone follow:

4.2. Every prime contains a unique minimal prime.

4.3. The primes contained in any proper ideal from a chain.

4.4. Each prime ideal P which contains a z-ideal contains a unique maximal prime z-ideal P^z.

4.5. If P, Q are primes not in a chain, they are contained in different maximal ideals and by 4.3 are co-maximal. Thus $PQ=P\cap Q$, a property shared by $C(X)$.

4.6. A z-ideal which is contained in a unique maximal ideal is prime.

Now from 4.1 we know that if $\tilde{P}, \tilde{Q}$ are primes in $\tilde{A}$ (corresponding to the primes P and Q of A) and if they are not in a chain and contain the same minimal prime ideal then they contain a largest prime ideal—denote it by $\tilde{P}*\tilde{Q}$. In fact, it corresponds to $P+Q$.

Proposition 4.7. *If A is semisimple and if $P+Q$ is maximal in A then $\tilde{P}*\tilde{Q}$ is a prime z-ideal in $\tilde{A}$.*

Proof. If $\tilde{P}*\tilde{Q}$ is not a z-ideal then by 4.4 it contains a prime z-ideal; hence $P+Q$ is contained in the corresponding prime ideal, which is not possible if $P+Q$ is maximal. □

Consider now some special cases. The following are equivalent: (a) R is π-regular, (b) all prime ideals of R are maximal, (c) $\tilde{R}$ is π-regular. For $C(X)$ (π-) regularity is equivalent to a host of conditions [1, Ch. 14] and X is then called a P-space. In the same vein X is called an F-space iff every finitely generated ideal of $C(X)$ is principal ($C(X)$ is "Bezout") and this is equivalent to the condition that the primes contained in each maximal ideal form a chain i.e. the primes between each minimal prime ideal and the unique maximal ideal containing it form a chain. Clearly this happens precisely when the primes of $\widetilde{C(X)}$ have the same property. Then by applying 4.6 we have that a z-ideal I is prime iff I contains a prime ideal, which is condition 2.5, satisfied in $C(X)$. In particular for each prime $\tilde{P}$ of $\widetilde{C(X)}$, $(\tilde{P})_z$ is a prime z-ideal. In arbitrary $\tilde{A}$ for

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A an absolutely convex subring of $C(X)$ we have the following:

THEOREM 4.8. *If P is a prime ideal of A which is not a z-ideal and $\tilde{P}$ is the prime of $\tilde{A}$ corresponding to it then either $(\tilde{P})_z$ is prime or $\tilde{P}$ is not a z-ideal.*

Proof. If $(\tilde{P})_z$ is not prime, it is an intersection of prime z-ideals $\tilde{P}_i$. The prime ideal $Q = P^z \subsetneqq P$ corresponds to a prime ideal $\tilde{Q}$ in $\tilde{A}$ with the property (by 3.6(a)) that any other prime containing $\tilde{P}$ is in a chain with $\tilde{Q}$. In particular this is true of the $\tilde{P}_i$ and since we can assume no two of the $\tilde{P}_i$ are in a chain, either all $\tilde{P}_i \subset \tilde{Q}$ or all $\tilde{P}_i \supset \tilde{Q}$. But the primes in $\tilde{Q}$ form a chain by 4.1 so the first case is impossible and in the second case $(\tilde{P})_z \supseteq Q \supsetneqq \tilde{P}$ so $\tilde{P}$ is not a z-ideal.. □

REFERENCES

1. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York, 1960.
2. M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math Soc. **149** (1969), 43–60.
3. C. W. Kohls, *Prime ideals in rings of continuous functions*, Illinois J. Math **2** (1958), 505–536.
4. G. Mason, *z-ideals and prime ideals*, J. Algebra **26** (1973) 280–297.
5. D. Rudd, *On two sum theorems for ideals of $C(X)$*, Michigan Math. J. **17** (1970) 139–141.

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