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CHARACTERISTIC CYCLES IN HERMITIAN SYMMETRIC SPACES

BRIAN D. BOE AND JOSEPH H. G. FU

ABSTRACT. We give explicit combinatorial expresssions for the characteristic cycles associated to certain canonical sheaves on Schubert varieties X in the classical Hermitian symmetric spaces: namely the intersection homology sheaves IH_X and the constant sheaves $\mathbb{C}_X$. The three main cases of interest are the Hermitian symmetric spaces for groups of type A_n (the standard Grassmannian), C_n (the Lagrangian Grassmannian) and D_n. In particular we find that $\mathrm{CC}(\mathrm{IH}_X)$ is irreducible for all Schubert varieties X if and only if the associated Dynkin diagram is simply laced. The result for Schubert varieties in the standard Grassmannian had been established earlier by Bressler, Finkelberg and Lunts, while the computations in the C_n and D_n cases are new.

Our approach is to compute $\mathrm{CC}(\mathbb{C}_X)$ by a direct geometric method, then to use the combinatorics of the Kazhdan-Lusztig polynomials (simplified for Hermitian symmetric spaces) to compute $\mathrm{CC}(\mathrm{IH}_X)$. The geometric method is based on the fundamental formula

$$\mathrm{CC}(\mathbb{C}_X) = \lim_{r \downarrow 0} \mathrm{CC}(\mathbb{C}_{X_r}),$$

where the $X_r \downarrow X$ constitute a family of tubes around the variety X. This formula leads at once to an expression for the coefficients of $\mathrm{CC}(\mathbb{C}_X)$ as the degrees of certain singular maps between spheres.

Introduction. In the chapter of representation theory pioneered by Kahzdan and Lusztig [KL1, 2], *Schubert varieties* in generalized flag manifolds play a fundamental role. In particular, the proof of the Kazhdan-Lusztig conjectures due to [BB] and [BK] has brought the *intersection homology* (IH) *sheaves* of these varieties, and the *characteristic cycles* of these sheaves, to the center of the subject.

A later conjecture of Kazhdan and Lusztig [KL3] states that the characteristic cycle of the IH sheaf associated to any Schubert variety in the manifold of complete flags in $\mathbb{C}^n$ (*i.e.* the flag manifold associated to the group of type A_{n-1}), is irreducible. A special case of this conjecture was proved by Bressler-Finkelberg-Lunts [BFL], who showed that the statement is true when the complete flag manifold is replaced by the corresponding maximally degenerate flag manifold (the Grassmannian). Tanisaki [Tan] was able to compute characteristic cycles in some other low-dimensional cases, and his examples show in particular that the conjecture is false for the (full) flag manifold associated to the group of type C_n. Quite recently, Kashiwara and Saito [Ka] have constructed counterexamples to the original (A_n) conjecture. Nevertheless, a complete determination of these characteristic cycles remains of serious interest to other areas of mathematics.

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In this paper we present algorithms for the characteristic cycles of the IH sheaves associated to the Schubert varieties lying in the Hermitian symmetric spaces of the classical groups; of these, the interesting cases are the spaces associated to the pairs (A_n, $A_m \times A_{n-m-1}$) (the standard Grassmannian), (C_n, A_{n-1}) (the Lagrangian Grassmannian) and (D_n, A_{n-1}). Thus our really new results concern the latter two cases—it is our understanding that before now no one had succeeded in making these calculations, despite several attempts (at least for C_n).

Our results may be summed up as follows:

THEOREM (THEOREM 7.1A). *Let H_G be a Hermitian symmetric space associated to a classical Lie group G. The characteristic cycles of the intersection homology sheaves associated to the Schubert varieties $X \subset H_G$ are all irreducible iff the Dynkin diagram of G is simply laced.*

In particular the characteristic cycles of the IH sheaves for Schubert varieties in the Hermitian symmetric space associated to the pair (D_n, A_{n-1}) are irreducible.

We also have the following positive result which includes the C_n case:

THEOREM (COROLLARY 7.1E). *Let H_G be a Hermitian symmetric space associated to a classical Lie group G. The multiplicities m_Y^X of the characteristic cycles of the intersection homology sheaves associated to Schubert varieties $Y \subset X \subset H_G$ are all either 0 or 1. Moreover, $m_Y^X = 0$ unless* $\operatorname{codim}_X Y$ *is even.*

Our approach to these problems seems very different from the usual one. In particular, intersection homology and sheaf theory are pushed off-stage and only enter the scene in the form of the Kazhdan-Lusztig polynomials, which by [KL2] are the Poincaré polynomials of the IH sheaves. We work instead in terms of the *normal cycle* $\mathbf{N}(X)$ [Fu1, 2, 3] of a Schubert variety X in one of the spaces under consideration. This cycle is equivalent to the characteristic cycle of the constant sheaf on X, but admits a simple geometric construction (*cf.* Section 2.2). Our first main result (Theorem 4.2E) gives an algorithm for the coefficients (which we call *MacPherson coefficients*) in the decomposition of $\mathbf{N}(X)$ in terms of the normal bundles of the strata of X. The main tool used in the proof is Theorem 2.2A, which expresses the MacPherson coefficients as the degrees of certain (singular) mappings of spheres.

As is well-known, the array of MacPherson coefficients (associated to pairs of subvarieties of X) determines the array of *Euler obstructions* associated to pairs of strata in X. In our second main result (Theorem 6.2A) we use this relation to produce from the algorithm of Theorem 4.2E an algorithm for the Euler obstructions for pairs of Schubert strata in the classical Hermitian symmetric spaces. The combinatorial structure of these algorithms is compatible with that of the formulas for the Kazhdan-Lusztig polynomials (in the Hermitian symmetric cases) due to [LS] and [Boe]. Thus we are able (in (7.1.1) and Theorems 7.1A and 7.1B) to produce from Theorem 6.2A a simple algorithm for the multiplicities of the IH sheaves.

In short, the method is to compute by geometric means the coefficients of the characteristic cycle of the constant sheaf, combining these with the Kazhdan-Lusztig polynomials to produce the desired multiplicities. It would of course in many ways be preferable to be able to compute these multiplicities by a direct geometric procedure, bypassing the combinatorics, as was done in [BFL] for the A_n case. Unfortunately our method does not seem to provide any hints in this direction.

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Contents

1. **Notation, conventions and background.**

1.1. *Generalities.* Let M be a Riemannian manifold. The tangent and cotangent bundles of M are denoted TM and T^*M respectively. The sphere bundle of unit tangents is written

$$\mathbb{U}M := \{\xi \in TM : |\xi| = 1\},$$

and $\mathbb{U}^*M$ is the bundle of unit cotangents. Given a submanifold $V \subset M$, the vector bundle of normals to V is denoted $\vec{\nu}V$ or $\vec{\nu}_M V$, and the unit normal bundle is written

$$\nu V = \nu_M V := \vec{\nu}_M V \cap \mathbb{U}M.$$

The corresponding conormal bundles are denoted $\vec{\nu}^*V$, $\nu^*V := \vec{\nu}^*V \cap \mathbb{U}^*M$.

Projection maps will sometimes be denoted by π_A, where A is the target of the projection. Thus the projection $TM \longrightarrow M$ is written π_M. In the case of a product $A \times B$ we will often write instead π_1, π_2 for the projections onto the first and second factors, respectively.

The topological closure of a set A is denoted $\bar{A}$.

If A is a finite set then we put $\mathbb{C}^A$ for the complex vector space of functions $A \longrightarrow \mathbb{C}$. We equip this space with the standard Hermitian inner product $\langle v, w\rangle := \sum_{\alpha\in A} v_\alpha \bar{w}_\alpha$. If $B \subset A$ we regard $\mathbb{C}^B$ as a subspace of $\mathbb{C}^A$ by extending $x \in \mathbb{C}^B$ by zero to all of A. If $x \in \mathbb{C}^A$ then x^B denotes the orthogonal projection of x into $\mathbb{C}^B$. The unit sphere of $\mathbb{C}^A$ is denoted $\mathbb{S}^A$, and the complex projective space on $\mathbb{C}^A$ by $\mathbb{P}^A$.

The set of regular points of a variety Y is denoted Y°. If $\mathcal{S}$ is a stratification of a space X and $x \in X$ then the stratum containing x is denoted Σ_x.

1.2. *Geometric measure theory.* Many constructions in this paper are based on the theory of *integral currents* of Federer and Fleming, [Fe, Chapter 4], and its specialization to semialgebraic currents (that is, currents given by integration over semialgebraic chains with $\mathbb{Z}$ coefficients). Formally, currents are linear operators on differential forms. The group of integral currents of dimension k in a manifold M is denoted $\mathbb{I}_k(M)$. If U is a semialgebraic open subset of an oriented submanifold $V \subset M$ of dimension k, then we denote the current given by integration over U by $[\![U]\!] \in \mathbb{I}_k(M)$.

Some important operations in this theory are as follows.

(1) *Push-forward.* If $f: M \longrightarrow N$ is a proper locally Lipschitz map and $T \in \mathbb{I}_k(M)$, then the current f_*T on N given by

$$(f_*T)(\phi) := T(f^*\phi)$$

for k-forms ϕ on N, is an integral current on N.
More generally, if T is semialgebraic and f is a semialgebraic function (*i.e.* a continuous function with semialgebraic graph) on the support of T then the push forward may be defined as follows. There is a unique semialgebraic current Γ supported on the graph of $f|\operatorname{spt} T$ such that $\pi_{M*}\Gamma = T$. Then $f_*T := \pi_{N*}\Gamma$.

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(2) *Restriction.* Given an integral current $T \in \mathbb{I}_k(M)$ on M and an open set $U \subset M$, we denote by $T|U$ the element of $\mathbb{I}_k(U)$ given by restricting the domain of the operator T to (the natural inclusion of) the space of differential forms supported in U. Putting $i_U \colon U \longrightarrow M$ for the inclusion map, we have also

$$T\lfloor U := i_{U*}(T|U).$$

(3) *Boundary.* $\partial T(\phi) := T(d\phi)$. Note that $f_*\partial = \partial f_*$ for f as in (1) above.

(4) *Slicing.* Given a locally Lipschitz function $f \colon M \longrightarrow \mathbb{R}$, the *slice* of T by f at $r \in \mathbb{R}$ is

$$\langle T, f, r\rangle := \partial\big(T\lfloor f^{-1}(-\infty, r)\big) - (\partial T)\lfloor f^{-1}(-\infty, r).$$

For T, f semialgebraic this defines a current-valued function of r that is continuous off of a finite set of jump discontinuities. If $T = [\![V]\!]$ is integration over a smooth oriented submanifold V and r is a regular value of $f|V$ then the slice $\langle T, f, r\rangle = [\![V \cap f^{-1}(r)]\!]$.

1.3. *Łojasiewicz inequality.* We will use the following fundamental inequality. Let $g \colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be an analytic function, and let $x_1, x_2, \cdots \longrightarrow x_0$ in $\mathbb{R}^n$, where $g(x_0) = 0$. Then there are constants $\alpha \in (0, 1), c > 0$ such that

$$|\nabla g(x_i)| \geq c g(x_i)^\alpha, \quad i = 1, 2, \ldots.$$

For a proof, see [KP].

In fact we will only need this in case g is polynomial, and moreover we will only use the weaker statement:

$$\lim_{i\to\infty} \frac{g(x_i)}{|\nabla g(x_i)|} = 0 \tag{1.3.1}$$

provided $\nabla g(x_i) \neq 0$ for $i = 1, 2 \ldots$.

2. Characteristic cycles and normal cycles.

2.1. CC(IH) *and* CC($\mathbb{C}$). In this subsection, we introduce the basic objects of study—characteristic cycles, (co)normal cycles, MacPherson coefficients, Euler obstructions, and Kazhdan-Lusztig numbers—as well as the relations between them.

Associated to any constructible sheaf (or complex of sheaves) $\mathcal{F}$ on a complex manifold H^n is its *characteristic cycle* CC($\mathcal{F}$), a conic Lagrangian cycle living in the cotangent bundle T^*H (*cf.* [KS]). The operation CC factors through the Grothendieck group $\mathbf{K}(\mathbf{D}^bH)$, which is isomorphic via the map $\mathcal{F} \longmapsto \chi_{\mathcal{F}}$, the fiberwise Euler characteristic of $\mathcal{F}$, to the group C-Func(H) of constructible functions on H. Thus for any subvariety $X \subset H$, the constant sheaf $\mathbb{C}_X$ corresponds to the characteristic function $1_X \in$ C-Func(H). The corresponding map from C-Func(H) to the additive group Lag(H) of conic Lagrangian cycles is a homomorphism of abelian groups. However we find it more convenient to work directly with varieties rather than constructible functions.

PROPOSITION 2.1A. *The constructible function operation* $\mathrm{CC}\colon \text{C-Func}(H) \longrightarrow \mathrm{Lag}(H)$ *is the unique group homomorphism satisfying*

$$\mathrm{CC}(1_X) = \vec{\mathbf{N}}^*(X)$$

for all closed varieties $X \subset H$, *where* $\vec{\mathbf{N}}^*$ *is the conormal cycle operation of [Fu3].*

PROOF. *Cf.* [Fu3, 4.2]. ∎

Recall that the conormal cycle $\vec{\mathbf{N}}^*(X)$ decomposes as

$$\vec{\mathbf{N}}^*(X) = \sum_{\Sigma\in S} d_\Sigma^X \vec{\mathbf{N}}^*(\Sigma) \tag{2.1.1}$$

where S is any Whitney stratification of X, $d_\Sigma^X \in \mathbb{Z}$ and each current $\vec{\mathbf{N}}^*(\Sigma)$ is given by integration over the manifold $\nu^*\Sigma$ of conormals to the stratum Σ. Note that the closure of $\vec{\nu}^*X \subset T^*H$ is a closed analytic subvariety; however, we give this variety a canonical orientation that is distinct from the orientation induced from the complex structure, *cf.* [Fu2] and Section 2.2 below. We will call d_Σ^X the *MacPherson coefficient of* Σ *in* X; abusing notation we put also $d_{\bar\Sigma}^X := d_\Sigma^X$. With this convention we have always $d_X^X = 1$.

Since any constructible function $f \in \text{C-Func}(H)$ may be expressed as a locally finite sum $f = \sum_X n_X 1_X$, where X ranges over the closed subvarieties of H and $n_X \in \mathbb{Z}$, the characteristic cycle $\mathrm{CC}(f)$ admits an expression

$$\mathrm{CC}(f) = \sum_{\Sigma\in S} \alpha_\Sigma \vec{\mathbf{N}}^*(\Sigma),$$

$\alpha_\Sigma = \sum_X n_X d_\Sigma^X$, for some Whitney stratification of H. If there is a stratum $\Sigma_0 \in S$ such that $\alpha_{\Sigma_0} = 1$ and $\alpha_\Sigma = 0$ for $\Sigma \neq \Sigma_0$, we say that $\mathrm{CC}(f)$ is *irreducible*.

Now let H be a (possibly degenerate) flag manifold and S the Whitney stratification of H by Schubert cells, with the usual partial order on S given by $\Sigma \succeq \Sigma' \Leftrightarrow \bar\Sigma \supset \Sigma'$. The subgroups of $\text{C-Func}(H)$ and $\mathrm{Lag}(H)$ generated respectively by $\{1_\Sigma \mid \Sigma \in S\}$ and $\{\vec{\mathbf{N}}^*(\Sigma) \mid \Sigma \in S\}$ are obviously both isomorphic to $\mathbb{Z}^S$ via the maps $F\colon \mathbb{Z}^S \longrightarrow \text{C-Func}(H)$ and $L\colon \mathbb{Z}^S \longrightarrow Lag(H)$ determined by $F(\delta^\Sigma) = 1_\Sigma$, $L(\delta^\Sigma) = \vec{\mathbf{N}}^*(\Sigma)$, where $\delta^\Sigma \in \mathbb{Z}^S$ is the Kronecker delta: $\delta^\Sigma_{\Sigma'} = 1$ if $\Sigma = \Sigma'$, and $= 0$ otherwise. Let $ch\colon \mathbb{Z}^S \longrightarrow \mathbb{Z}^S$ be the induced map

$$\begin{array}{ccc} \mathbb{Z}^S & \xrightarrow{ch} & \mathbb{Z}^S \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle L}\big\downarrow \\ \text{C-Func}(H) & \xrightarrow{\mathrm{CC}} & \mathrm{Lag}(H). \end{array} \tag{2.1.2}$$

If f is a constructible function in the span of $\{1_\Sigma \mid \Sigma \in S\}$ then $\mathrm{CC}(f)$ is irreducible iff $ch\big(F^{-1}(f)\big) = \delta^{\Sigma_0}$ for some $\Sigma_0 \in S$.

For each Schubert cell $\Sigma \in S$, define $\zeta^\Sigma, p^\Sigma, m^\Sigma \in \mathbb{Z}^S$ by

$$\begin{aligned} \zeta^\Sigma_{\Sigma'} &= \begin{cases} 1, & \Sigma' \preceq \Sigma \\ 0, & \text{otherwise,} \end{cases} \\ p^\Sigma_{\Sigma'} &= P^{\bar\Sigma}_{\Sigma'}(1), \\ m^\Sigma &= ch(p^\Sigma), \end{aligned} \tag{2.1.3}$$

where $P_Y^X(q)$ is the Kazhdan-Lusztig polynomial of the pair (X, Y) of Schubert varieties (*cf.* [KL1]). Recall that $P_Y^X = 0$ unless $Y \subset X$. Thus $F(p^{\Sigma})$ is the constructible function associated to the intersection homology sheaf $\mathrm{IH}_{\bar\Sigma}$ of $\bar\Sigma$ [KL2], and $m^{\Sigma}_{\Sigma'}$ is the coefficient of $\vec{\mathbf{N}}^*(\Sigma')$ in $\mathrm{CC}(\mathrm{IH}_{\bar\Sigma})$. Furthermore $\zeta^{\Sigma} = F^{-1}(1_{\bar\Sigma})$, with $ch(\zeta^{\Sigma}) = d^{\bar\Sigma}$, the vector of MacPherson coefficients of $\bar\Sigma$ (by (2.1.1)). Each of the collections $\{\zeta^{\Sigma}\}$, $\{d^{\bar\Sigma}\}$, $\{p^{\Sigma}\}$, $\{m^{\Sigma}\}$ is a basis for $\mathbb{Z}^S$.

Denote by $e = (e^{\Sigma}_{\Sigma'})$ the matrix expressing the ζ basis in terms of the d basis:

$$(e \cdot d^{\bar\Sigma})_{\Sigma''} := \sum_{\Sigma'} e^{\Sigma'}_{\Sigma''} d^{\bar\Sigma}_{\Sigma'} = \zeta^{\Sigma}_{\Sigma''} \tag{2.1.4}$$

(thus $e^{\Sigma}_{\Sigma'}$ is the Euler obstruction of Σ at a generic point of Σ', *cf.* [Mac, BDK, Gin]). Now

$$ch(e \cdot d^{\bar\Sigma}) = ch(\zeta^{\Sigma}) = d^{\bar\Sigma}, \quad \Sigma \in S, \tag{2.1.5}$$

and therefore

$$ch^{-1}(\alpha) = e \cdot \alpha \tag{2.1.6}$$

for any $\alpha \in \mathbb{Z}^S$. In particular,

$$ch^{-1}(\delta^{\Sigma}) = e \cdot \delta^{\Sigma} = e^{\Sigma},$$

and similarly

$$ch^{-1}(m^{\Sigma}) = e \cdot m^{\Sigma} = p^{\Sigma}, \tag{2.1.7}$$

for all Schubert cells Σ. The first relation here means: *the unique constructible function f with irreducible characteristic cycle* $\mathrm{CC}(f) = \vec{\mathbf{N}}^*(\Sigma)$ *is* $f = \sum_{\Sigma'} e^{\Sigma}_{\Sigma'} 1_{\Sigma'}$.

Our method for computing the $\mathrm{CC}(\mathrm{IH}_X)$ for Schubert varieties $X = \bar\Sigma$ is based on a computation of the MacPherson coefficients d^X_Y. From the d^X_Y we compute the Euler obstructions e using (2.1.4). If these are equal to the Kazhdan-Lusztig numbers then the characteristic cycles of the IH sheaves are irreducible, and we are done. If not, we solve the relation (2.1.7) to find the coefficients m.

2.2. *The normal cycle of a variety with conic singularities.* It is convenient to work not with the full conormal cycle $\vec{\mathbf{N}}^*(X)$ but with the unit *normal cycle* $\mathbf{N}(X) \in \mathbb{I}_{2n-1}(\mathbb{U}H)$, where $n = \dim_{\mathbb{C}} H$. Putting $\iota\colon T^*H \longrightarrow TH$ for the isomorphism induced by the metric and $\rho\colon TH \longrightarrow \mathbb{R}$ for the length function, this is given by

$$\mathbf{N}(X) = \langle \iota_* \vec{\mathbf{N}}^*(X), \rho, 1 \rangle. \tag{2.2.1}$$

Thus $\vec{\mathbf{N}}^*(X)$ is the cone over $\iota_*^{-1}\mathbf{N}(X)$. On the smooth part of X the normal cycle determines the standard orientation of the unit normal bundle νX, locally equal to the product of the orientation of X as a complex manifold and the orientation of the normal sphere $\nu_x X$ as the sphere of the complex vector space $\vec{\nu}_x X$.

The cycle $\mathbf{N}(X)$ admits the following geometric construction [Fu3]. Let $g: H \longrightarrow \mathbb{R}$ be a nonnegative, locally Lipschitz, subanalytic function with $g^{-1}(0) = X$; we call such a function g an *aura* for X. Let $[\![dg]\!] \in \mathbb{I}_{2n}(T^*H)$ be the differential current of g, as in [Fu1]. Let $U \subset H$ be a neighborhood of X, small enough that $\operatorname{spt}[\![dg]\!] \cap \pi_H^{-1}(U)$ does not intersect the zero-section of T^*H, and denote by $\tilde{G}$ the current representing the graph of the normalized gradient of g; *i.e.*, putting $u: TH - (\text{zero-section}) \longrightarrow \mathbb{U}H$ for the radial projection, $\tilde{G} = u_* \iota_* [\![dg]\!]$. Then

$$\mathbf{N}(X) = -\partial(\tilde{G} | \pi_H^{-1} U). \tag{2.2.2}$$

We will be particularly interested in the case where $X \subset \mathbb{C}^n$ is an algebraic cone over the origin. In this case it is clear that in the decomposition $\mathbf{N}(X) = \sum_Z d_Z^X \mathbf{N}(Z^\circ)$, all of the strata Z in the sum are themselves conic. For each $r > 0$, put $\mathbb{S}_r$ for the sphere of radius r about the origin. From the product formula for normal cycles ([Fu3], 4.5) the normal cycles of the real algebraic subvarieties $X \cap \mathbb{S}_r$, $r > 0$, satisfy

$$\langle \mathbf{N}(X), \delta \circ \pi_H, r \rangle = \mathbf{N}_{\mathbb{S}_r}(X \cap \mathbb{S}_r) = \sum_Z d_Z^X \mathbf{N}_{\mathbb{S}_r}(Z^\circ \cap \mathbb{S}_r) \tag{2.2.3}$$

for all sufficiently small $r > 0$, where $\delta(x) := |x|$ and $\mathbf{N}_{\mathbb{S}_r}$ denotes the normal cycle relative to the ambient manifold $\mathbb{S}_r$. Let $g, \tilde{G}$ be as in the last paragraph, and for $r > 0$ let $G_r := \langle \tilde{G}, \delta \circ \pi_H, r \rangle$ denote the current representing the restriction to $\mathbb{S}_r$ of the normalized gradient of g. Now if $U_r \subset \mathbb{S}_r$ is a suitably small neighborhood of $X \cap \mathbb{S}_r$, then slicing the relation (2.2.2) by the radial function $\delta \circ \pi_H$ we obtain from (2.2.3)

$$\partial(G_r | U_r) = -\mathbf{N}_{\mathbb{S}_r}(X \cap S_r). \tag{2.2.4}$$

Moreover, for cones the MacPherson coefficient of the origin may be computed by the following theorem. This result is the foundation of our approach to the calculations which are the main point of this paper: the theorem may be used to compute MacPherson coefficients of more general substrata. For if $X \subset H$ is a complex analytic variety, Σ a stratum of X, and $V \subset H$ is a submanifold transverse to Σ with $V \cap \Sigma = \{x_0\}$, then $d_\Sigma^X = d_{\{x_0\}}^{(V \cap X)}$. We call $V \cap X$ a *normal slice* for the pair (X, Σ) (or $(X, \bar{\Sigma})$). Although the normal slice is not usually well-defined as a variety, the associated MacPherson coefficient is. By abuse of language we will refer to *the* normal slice for (X, Y), denoted X_Y. In Section 3 we will construct natural models for normal slices of pairs of Schubert varieties.

THEOREM 2.2A. *Let $X \subset \mathbb{C}^n$ be an algebraic cone over 0, and let $g: \mathbb{C}^n \longrightarrow \mathbb{R}$ be a smooth aura for X. Define $\phi(x) := \frac{\nabla g(x)}{|\nabla g(x)|}$; this map is well defined for $x \in B(0, r_0) - X$ if $r_0 > 0$ is sufficiently small. Let $r \in (0, r_0)$ be fixed. Then*

$$\deg \phi := \sum_{x \in \mathbb{S}_r \cap \phi^{-1}(y)} \operatorname{sgn} \det D(\phi | \mathbb{S}_r)(x) = d_{\{0\}}^X \tag{2.2.5}$$

for almost every y in the unit sphere $\mathbb{S}^{2n-1}$.

PROOF OF 2.2A. We will prove the following more general and (from the current-theoretic point of view) formally more natural fact, which contains Theorem 2.2A as a

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special case. Let g be a (possibly nonsmooth) aura for X, and let $\tilde{G}$, G_r and δ be as in the preceding paragraphs. Let $\pi_2\colon \mathbb{U}\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}$ be the projection onto the fiber. Then for almost all $r > 0$,

$$\pi_{2*}G_r = d^X_{\{0\}}[\![\mathbb{S}^{2n-1}]\!]. \tag{2.2.6}$$

If g is smooth then G_r is a smooth graph, and the stated form of the theorem follows from (2.2.6) and the change of variables formula for multiple integrals.

Put $\pi_1\colon \mathbb{U}\mathbb{C}^n \simeq \mathbb{C}^n \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{C}^n$ for the projection of the bundle and $B_r = \delta^{-1}[0, r)$ for the open ball of radius r about the origin. Let $\mathcal{S}$ be a Whitney stratification of X by conic strata.

For almost every $r > 0$ we have by [Fe, 4.2.1]

$$\begin{aligned} \partial(\tilde{G}\lfloor\pi_1^{-1}B_r) &= (\partial\tilde{G})\lfloor\pi_1^{-1}B_r + G_r \\ &= -\mathbf{N}(X)\lfloor\pi_1^{-1}B_r + G_r \qquad \text{by (2.2.2)} \\ &= -\sum_{\Sigma\in\mathcal{S}} d^X_\Sigma \mathbf{N}(\Sigma)\lfloor\pi_1^{-1}B_r + G_r. \end{aligned} \tag{2.2.7}$$

Projecting onto the spherical factor, the support theorem [Fe, 4.1.20] gives

$$\pi_{2*}\partial(\tilde{G}\lfloor\pi_1^{-1}B_r) = \partial\pi_{2*}(\tilde{G}\lfloor\pi_1^{-1}B_r) = 0$$

since the $2n$-current under the boundary is supported on $\mathbb{S}^{2n-1}$. Therefore (2.2.7) becomes

$$\sum_{\Sigma\in\mathcal{S}} d^X_\Sigma \pi_{2*}\big(\mathbf{N}(\Sigma)\lfloor\pi_1^{-1}B_r\big) = \pi_{2*}G_r. \tag{2.2.8}$$

We claim that $\pi_{2*}\big(\mathbf{N}(\Sigma)\lfloor\pi_1^{-1}B_r\big) = 0$ for each stratum $\Sigma \neq \{0\}$. In fact, we will show that the set $\pi_2(\nu\Sigma)\cap\mathbb{S}^{2n-1}$ has real dimension at most $2n-3$, so by the support theorem it cannot support an integral current of dimension $2n-1$.

Each such stratum Σ is the cone over a projectivized stratum $\mathbb{P}\Sigma \subset \mathbb{P}^{n-1}$. The manifold $\mathbb{P}\nu^*(\mathbb{P}\Sigma) \subset \mathbb{P}T^*\mathbb{P}^{n-1}$ of projective conormals has a closure which is a Lagrangian subvariety of $\mathbb{P}T^*\mathbb{P}^{n-1}$, and therefore has complex dimension at most $n-1$. Since the projectivization of $\pi_2(\nu^*\Sigma) \subset \mathbb{S}^{(2n-1)*}$ is equal to the projection onto the second factor of $\mathbb{P}T^*\mathbb{P}^{n-1} \subset \mathbb{P}^{n-1} \times \mathbb{P}^{(n-1)*}$, we find that $\pi_2(\nu^*\Sigma)$ is the intersection with $\mathbb{S}^{(2n-1)*}$ of a cone of complex dimension at most $n-1$ in $\mathbb{C}^{n*}$. In particular, its real dimension is at most $2n-3$, and $\nu(\Sigma)\cap\mathbb{S}^{2n-1}$ is its image under ι.

Thus (2.2.8) reduces to

$$\begin{aligned} \pi_{2*}G_r &= d^X_{\{0\}}\pi_{2*}\mathbf{N}^*(\{0\}) \\ &= d^X_{\{0\}}[\![\mathbb{S}^{2n-1}]\!]. \end{aligned}$$

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3. **Hermitian Symmetric Spaces.** For the rest of this paper we will assume that H is an *irreducible Hermitian symmetric space of compact type*. These are the manifolds $K\backslash G_0$ where

(1) G_0 is a compact connected simple Lie group with finite center;
(2) K is a maximal connected proper subgroup of G_0;
(3) K has nondiscrete center

(*cf.* [Hel, Chapter 8]). Concretely, there are five classical families and two exceptional cases. In the present paper we restrict our attention to the classical families, which are shown in Table 3.1.

Type	G_0	K	$\dim_{\mathbb{R}} K\backslash G_0$	$\mathrm{rank}(K\backslash G_0)$
I	$SU(n+m)$	$S\big(U(n)\times U(m)\big)$	$2nm$	$\min(n,m)$
II	$Sp(n)$	$U(n)$	$n(n+1)$	n
III	$\mathrm{SO}(2n)$	$U(n)$	$n(n-1)$	$[n/2]$
IV	$\mathrm{SO}(n+2)$, n odd	$\mathrm{SO}(n)\times\mathrm{SO}(2)$	$2n$	2
V	$\mathrm{SO}(n+2)$, n even	$\mathrm{SO}(n)\times\mathrm{SO}(2)$	$2n$	2

TABLE 3.1. Compact Classical Hermitian Symmetric Spaces.

3.1. *Schubert varieties.* We now give explicit descriptions of these manifolds $K\backslash G_0$ and their Schubert varieties in types I, II, and III, which admit a parallel treatment. These descriptions are all well known (*cf.* [Tak]). We discuss the (much simpler) types IV and V at the end of this section.

CAUTION. In the sequel, we refer often to coordinates corresponding to certain entries of matrices. We view these matrices as embedded in a Cartesian plane with their lower left corner at the origin, with (i,j) coordinate referring to the entry in *column i* (counting from 1 at the left) and *row j* (counting from 1 at the *bottom*). For consistency, we follow this same convention when using matrix notation. Thus g^{ij} denotes the entry of g in the i-th column from the left and the j-th row from the bottom; an $n\times m$ matrix has n columns and m rows; etc.

We use the standard indexing scheme for matrix multiplication, which we denote by $\bullet$: thus if $y\in\mathbb{C}^{m\times n}$ and $v\in\mathbb{C}^n$ then $y\bullet v=(\sum_{j=1}^n y^{1j}v^j,\dots,\sum_{j=1}^n y^{mj}v^j)$.

1. $S\big(U(n)\times U(m)\big)\backslash SU(n+m)$. This is the familiar Grassmann manifold $G_{n,m}$ of n-planes in $\mathbb{C}^{n+m}$. Fixing a flag $\mathbb{C}^1\subset\mathbb{C}^2\subset\cdots\subset\mathbb{C}^{n+m}$, the Schubert cells in $G_{n,m}$ are the subsets

$$\{\Pi\in G_{n,m}\mid \dim(\Pi\cap\mathbb{C}^i)=r_i\} \tag{3.1.1}$$

corresponding to nondecreasing sequences $r_1\le\cdots\le r_{n+m}=n$. They may be described more economically as the subsets

$$\Sigma=[a_1-1,a_2-2,\dots,a_n-n]=\{\Pi\mid\dim(\Pi\cap\mathbb{C}^{a_i})=i\} \tag{3.1.2}$$

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where $0 \le a_1 - 1 \le a_2 - 2 \le \cdots \le a_n - n \le m$. The Schubert variety $Y = \bar{\Sigma}$ is given by changing the last "$=$" to "$\ge$" in (3.1.2), and will also be denoted $[a_1-1, a_2-2, \ldots, a_n-n]$. Thus the Schubert cells/varieties are classified by $m \times n$ *Young diagrams*: n rows of "boxes" (left justified), with no more than m boxes in each row, each row having no more boxes than the row below. (Note that this is the opposite of the usual convention—our Young diagrams are flipped, top to bottom; *cf.* Figure 3.1.) We view these Young diagrams as embedded in a fixed $m \times n$ rectangle $M := \{1, \ldots, m\} \times \{1, \ldots n\}$. Put $\Delta(\Sigma) = \Delta(Y)$ for the Young diagram associated to Σ or Y, and denote the complementary diagram by $\bar{\Delta}(\Sigma) = \bar{\Delta}(Y) := M - \Delta(\Sigma)$. Then an open dense subset of Y is identified via coordinates with $\mathbb{C}^{\Delta(Y)}$, and the normal slice $(G_{n,m})_Y$ of Y in the full Grassmannian will be identified below with $\mathbb{C}^{\bar{\Delta}(Y)} =: \mathbb{C}_0^{\bar{\Delta}(Y)}$.

2. $U(n)\backslash Sp(n)$. This is the "Lagrangian Grassmannian" Λ_n of n-planes in $\mathbb{C}^{2n}$ totally isotropic with respect to the symplectic form $\langle v|w\rangle = v^t J w$, $v, w \in \mathbb{C}^{2n}$, where

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

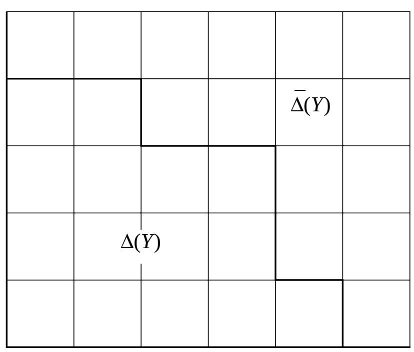

FIGURE 3.1. The Young diagram of $Y = [0, 2, 4, 4, 5]$ in $G_{5,6}$.

In particular Λ_n is naturally embedded in $G_{n,n}$. The Schubert varieties Y in Λ_n are the intersections with Λ_n of the Schubert varieties in $G_{n,n}$. Thus they may be classified by Young diagrams—in fact they correspond precisely to the Young diagrams which are symmetric about the diagonal $i = j$ in the $n \times n$ square M. As we shall see below, the full normal slice $(\Lambda_n)_Y$ is identified with the space $\mathbb{C}_+^{\bar{\Delta}(Y)}$ of *symmetric* matrices in $\mathbb{C}^{\bar{\Delta}(Y)}$.

3. $U(n)\backslash SO(2n)$. This is the connected component Λ_n^- of $\mathbb{C}^n \times 0$ in the space of n-planes in $\mathbb{C}^{2n}$, totally isotropic with respect to the quadratic form $\langle v \,|\, w\rangle = v^t K w$, v,

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$w \in \mathbb{C}^{2n}$, where

$$K = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{pmatrix}.$$

As in type II, the Schubert varieties Y in Λ_n^- are classified by symmetric Young diagrams, with the additional constraint that there be an *even* number of boxes along the diagonal. We will see that the normal slice $(\Lambda_n^-)_Y$ is represented by the space $\mathbb{C}_-^{\bar{\Delta}(Y)}$ of *antisymmetric* matrices in $\mathbb{C}^{\bar{\Delta}(Y)}$.

3.2. *Schubert slices.* This subsection provides explicit descriptions of the *Schubert slice* X_Y associated to a pair of Schubert varieties $X \supset Y$ in H of type I, II, or III. Put

$$\varepsilon = \begin{cases} 0 & \text{in type I} \\ + & \text{in type II} \\ - & \text{in type III.} \end{cases}$$

Let $\Omega(Y)$ denote the family of all rectangles $\mu \subset \bar{\Delta}(Y)$ with $(m,n) \in \mu$ and which touch (but do not cross) the boundary of $\Delta(Y)$; *cf.* Figure 3.2.

PROPOSITION 3.2A. *Suppose $X \supset Y$ are Schubert varieties in H of type I, II, or III.*

(1) *There is a function $\rho_Y^X: \Omega(Y) \longrightarrow \mathbb{Z}_{\geq 0}$ such that*

$$X_Y = \{x \in \mathbb{C}_\varepsilon^{\bar{\Delta}(Y)} \mid \operatorname{rank} x_\mu \leq \rho_Y^X(\mu) \text{ for all } \mu \in \Omega(Y)\}.$$

In types II and III, $\rho_Y^X(\mu) = \rho_Y^X(\mu^t)$, and in type III, $\rho_Y^X(\mu)$ is even if $\mu = \mu^t$. In fact, for $\mu \in \Omega(Y)$ the integer $\rho_Y^X(\mu)$ is the length of the diagonal line segment with one endpoint at the lower left corner of μ and the other endpoint on the boundary of $\Delta(X)$; cf. Figure 3.2.

(2) *Let X, Y be Schubert varieties of type I. For $x \in X_Y$ as in (1), the tangent space $T_x X_Y$ is spanned by vectors of the following three types:*

(i) *Truncated column operations: given indices $1 \leq i \leq j \leq m$, define $\kappa_{i\leftarrow j}(x)$ by*

$$\kappa_{i\leftarrow j}^{rs} = \begin{cases} x^{js} & \text{if } r = i \text{ and } (r,s) \in \bar{\Delta}(Y) \\ 0 & \text{otherwise;} \end{cases}$$

(ii) *Truncated row operations: given indices $1 \leq i \leq j \leq m$, define $\rho_{j\downarrow i}(x)$ by*

$$\rho_{j\downarrow i}^{rs} = \begin{cases} x^{rj} & \text{if } s = i \text{ and } (r,s) \in \bar{\Delta}(Y) \\ 0 & \text{otherwise;} \end{cases}$$

(iii) *Restricted tensor products: given $\mu \in \Omega$, let j be the row immediately below μ, let i be any column intersecting row j in $\bar{\Delta}(Y)$, and define $\tau_{i\times j}(x)$ by*

$$\tau_{i\times j}^{rs} = \begin{cases} x^{rj}x^{is} & \text{if } (r,s) \in \mu \\ 0 & \text{otherwise.} \end{cases}$$

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(3) The dimension of Y is given by

$$\dim Y = \begin{cases} \#\{(i,j) \mid (i,j) \in \Delta(Y)\} & \text{in type I} \\ \#\{(i,j) \mid i \le j,\ (i,j) \in \Delta(Y)\} & \text{in type II} \\ \#\{(i,j) \mid i < j,\ (i,j) \in \Delta(Y)\} & \text{in type III.} \end{cases}$$

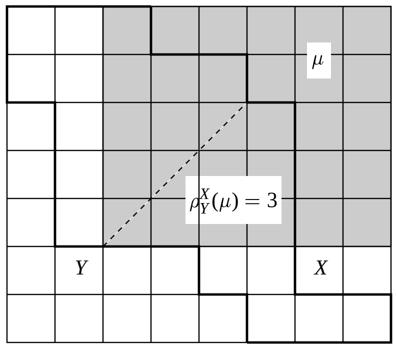

FIGURE 3.2. The rank function ρ_Y^X.

PROOF. For this proof, we switch temporarily from "Young diagram cartesian" coordinates back to *standard matrix coordinates*; that is, g^{ij} will denote the entry in row i and column j, with rows numbered from the *top* and columns numbered from the left.

Assume first that we are in type I. We may regard $G_{n,m}$ as a quotient of $G := GL(n+m,\mathbb{C})$ in the usual way as $P\backslash G$, where P is the "block lower triangular" parabolic subgroup $P = \{g \in G \mid g^{ij} = 0 \text{ if } i \le n \text{ and } j > n\}$; the correspondence with the geometric description above is $g \longmapsto \operatorname{span}\{g^{1\cdot}, \ldots, g^{n\cdot}\}$. The action of P is by row operations on $g \in G$, where it is illegal to move any row $i > n$ to any other row $i' \le n$. Let a Schubert cell Σ be given as in (3.1.2), set $Y = \overline{\Sigma}$, and put $S_1 := \{a_1 < \cdots < a_n\}$, $S_2 := \{1, \ldots, m+n\} - S_1 =: \{b_1 < b_2 < \cdots < b_m\}$. Given $g \in G$, the coset Pg belongs to Σ iff the rank of each submatrix $[g^{ij} \mid i \le n, j \ge k]$ is equal to $\#(S_1 \cap \{k, \ldots, m+n\})$ for $k = 1, \ldots, m+n$.

This description shows that each point $p \in \Sigma$ admits a unique representative $g = g_p \in G$ satisfying the conditions

$$g^{ij} = \begin{cases} 1, & 1 \le i \le n,\ j = a_i \\ 0, & 1 \le i \le n,\ \text{and } j > a_i \text{ or } j \notin S_2 \\ 1, & n < i \le m+n,\ j = b_{i-n} \\ 0, & n < i \le m+n,\ j \ne b_{i-n}. \end{cases}$$

We call this the *tier picture* of $G_{n,m}$, the *top tier* consisting of the rows $i \le n$ of g and the *bottom tier* the rows $i > n$. (In Figure 3.3 we show the tier picture for the cell $\Sigma \subset G_{5,6}$ whose Young diagram was given in Figure 3.1.) In particular Σ contains a unique point $p_\Sigma = p_Y$ for which this representative is a permutation matrix $g_\Sigma := g_{p_\Sigma}$. It is clear that the tangent space to the Grassmannian at $p = p_\Sigma$ is represented by $\mathbb{C}^{\{1,\ldots,n\}\times S_2}$. The last

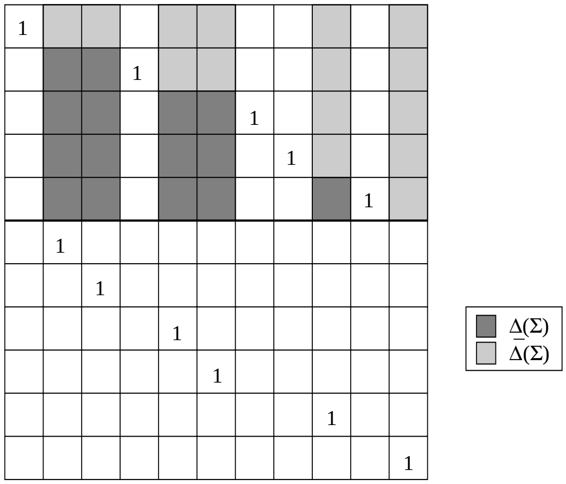

FIGURE 3.3. The tier picture.

paragraph implies that the subspace $T_p\Sigma$ is represented by the subspace spanned by the set of all coordinates $(i,j) \in \{1,\ldots,n\} \times S_2$ with $j < a_i$, and the normal space $\vec{\nu}_p\Sigma$ by the set of such (i,j) with $j > a_i$. Note that the Young diagram $\Delta(\Sigma)$ may be obtained from this diagram as the concatenation of all the blocks of coordinates corresponding to the tangent space $T_p\Sigma$, and $\overline{\Delta} = \overline{\Delta}(\Sigma)$ is obtained similarly by concatenating the blocks corresponding to the normal space. Thus $x \leftrightarrow g_\Sigma + x$ gives a linear isomorphism between $\mathbb{C}^{\overline{\Delta}}$ and $\vec{\nu}_p\Sigma$. In particular the dimension of Y is equal to the number of boxes in $\Delta(Y)$.

Now let $\Sigma' = [a_1' < a_2' < \cdots < a_n']$ be another Schubert cell incident to Y; *i.e.*, $X := \overline{\Sigma'} \supset Y$. From the descriptions above it is easy to see that $g_\Sigma + x \in \Sigma'$ iff $x \in \mathbb{C}^{\underline{\Delta}(Y)}$ satisfies $\operatorname{rank} x_\mu = \rho_Y^X(\mu)$ for all $\mu \in \Omega$. This implies (1) in type I.

We now wish to classify the tangent vectors to the normal slice Σ_Y' in type I. To this end we recall that right multiplication induces the standard action of G on the Grassmannian $P\backslash G$, where, putting B for the Borel subgroup of lower triangular elements of G, the Schubert cells are precisely the B-orbits in $P\backslash G$.

LEMMA 3.2B. *Let B be a Lie group and $\alpha\colon B \times M \longrightarrow M$ a smooth action of B on a manifold M. Let $p \in M$, $\Sigma_p := Bp$ the orbit of p, and $B_p \subset B$ the isotropy subgroup of p. Choose smooth transverse submanifolds $V \subset M$ to Σ_p at p and $F \subset B$ to B_p at e, $V \cap \Sigma_p = \{p\}$, $F \cap B_p = \{e\}$. For $q \in M$ sufficiently close to p, let $f = f(q)$ be the unique element of F such that $f \cdot q \in V$. For $b \in B$ and $q \in V$, put $\tilde{\alpha}(b,q) = f(b \cdot q) \cdot (b \cdot q)$, defined for (b,q) sufficiently close to (e,p). Then for $q \in V$ sufficiently close to p,*

$$\tilde{\alpha}_*(T_eB_p \oplus \{0_q\}) = T_q(V \cap \Sigma_q),$$

where 0_q is the origin of T_qV.

PROOF. The containment $\subset$ is clear, so we need only prove surjectivity. We first observe that, for $q \in V$ close to p, the map $(a,c) \longmapsto af(c \cdot q)c$ provides a local diffeomorphism of $F \times B_p$ with a neighborhood of e in B. In fact, if $\psi_q\colon B_p \longrightarrow B$, $\psi_q(c) = f(c \cdot q)c$, then the maps ψ_q vary continuously with q and approach the identity map as q tends to p along V. This shows that, for q close to p, $\psi_q(B_p)$ is transverse to F.

Now, given $b \in B$ close to e with $b \cdot q \in V \cap \Sigma_q$, write $b = af(c \cdot q)c$ as in the previous paragraph. Then $b \cdot q = af(c \cdot q)c \cdot q$. But by the uniqueness of $f(q')$ applied to $q' = f(c \cdot q)c \cdot q$, we conclude that $a = e$. Hence $b \cdot q = \tilde{\alpha}(c, q)$ with $c \in B_p$, which implies the lemma. ■

To apply the lemma in the present situation we observe that, given a point $p = p_\Sigma$ as above, the stabilizer B_p is given by

$$B_p = \{b \in B \mid b^{ij} = 0 \text{ if } (i,j) \in S_1 \times S_2\},$$

so that we may take

$$F = \{b \in B \mid b^{ij} = \delta^{ij} \text{ if } (i,j) \notin S_1 \times S_2\} \qquad \text{(Kronecker delta).}$$

Recalling that B is acting on row vectors $\xi \in \mathbb{C}^{n+m} = \mathbb{C}^{S_1} \oplus \mathbb{C}^{S_2}$, put

$$\begin{aligned} B_1 &:= \{b \in B_p \mid b|\mathbb{C}^{S_2} = \mathrm{id}\}, \\ B_2 &:= \{b \in B_p \mid b|\mathbb{C}^{S_1} = \mathrm{id}\}, \\ C &:= \{b \in B_p \mid \xi = (\xi b)^{S_i} \text{ for all } \xi \in \mathbb{C}^{S_i}, i = 1, 2\}; \end{aligned}$$

it is clear that if $b \in B_p$ is sufficiently close to the identity then there are unique $c \in C$ and $b_i \in B_i$, $i = 1, 2$ such that $b = b_1 b_2 c = b_2 b_1 c$.

The B_i are actually subgroups, and moreover stabilize the normal slice $(G_{n,m})_Y$. We show that the B_1 action corresponds to downward truncated row operations on the slice space. Let $x \in \mathbb{C}^{\bar{\Delta}}$ correspond to $g = g_\Sigma + x \in \vec{\nu}_p \Sigma$, and suppose we add (via B_1) a multiple of column a_j to column a_i in g ($i < j$). To find the corresponding element $g_\Sigma + x'$, subtract the same multiple of row i from row j (left P-action), and add appropriate multiples of column a_j to columns b_k, $a_i < b_k < a_j$ (F-action). This shows that x' is obtained from x by subtracting a multiple of row i from row j, truncated to $\mathbb{C}^{\bar{\Delta}}$. It is left as an exercise for the reader to check that B_2 gives leftward truncated column operations, and C gives restricted tensor products.

Now assume we are in type II. Recall that each Schubert cell is the intersection with Λ_n of a Schubert cell in $G_{n,n}$. In general, a Grassmannian Schubert cell $\tilde{\Sigma}$ will contain *no* isotropic planes *unless* the permutation matrix representing the point $p = p_{\tilde{\Sigma}}$ has the following property: if $j \in S_1, j \leq n$ then $2n+1-j \notin S_1$. Thus, given a type II Schubert cell (resp. variety) Σ (resp. Y), let $\tilde{\Sigma}$ (resp. $\tilde{Y}$) denote the associated type I Schubert cell (resp. variety) having the same Young diagram, where we may assume that p satisfies the above condition. Thus the normal slice X_Y is naturally embedded in $\mathbb{C}^{\bar{\Delta}(Y)}$. We claim that $\bar{\Delta}(Y)$ (and hence $\Delta(Y)$) is symmetric about the diagonal. Observe that $\bar{\Delta}(Y)$ is equivalent (in the slice picture) to the set $\{(i, b_j) \mid b_j > a_i, 1 \leq i, j \leq n\}$. But $a_i \in S_1$ iff $2n + 1 - a_i \in S_2$. Therefore $b_j = 2n + 1 - a_{n-j+1}$, $j = 1, \ldots, n$, so $b_j > a_i \Leftrightarrow b_{n-i+1} > a_{n-j+1} \Leftrightarrow (n - j + 1, b_{n-i+1}) \in \bar{\Delta}(Y)$; *i.e.*, $\bar{\Delta}(Y)$ is symmetric.

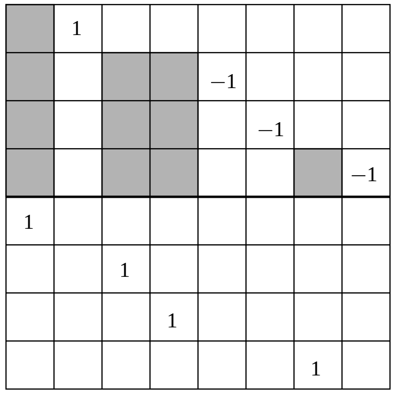

FIGURE 3.4. $\tilde{g}_\Sigma$ in type II.

It is convenient to modify the permutation matrix g_Σ by changing the (i, a_i) entry from 1 to -1 whenever $a_i > n$; call the resulting matrix $\tilde{g}_\Sigma$ (*cf.* Figure 3.4). For given a point in the normal slice $(\Lambda_n)_\Sigma$ represented by a matrix $\tilde{g}_\Sigma + x$, (where as before $x^{ij} = 0$ for $i \le n$ unless $j > a_i$ and $j \in S_2$) the isotropy condition gives

$$0 = J(x^{i\cdot}, x^{n-j+1,\cdot}) = x^{i,b_j} - x^{n-j+1,b_{n-i+1}},$$

$1 \le i, j \le n$. Thus we may regard the normal slices for the Lagrangian Grassmannian as algebraic cones in the space $\mathbb{C}_+^{\bar\Delta(Y)}$ of symmetric elements of $\mathbb{C}^{\bar\Delta(Y)}$.

Observe that the dimension of Y is equal to the number of boxes of $\Delta(Y)$ lying on or above the line of symmetry.

Finally, assume we are in type III. As in type II, it is easy to see that the type I variety $\tilde\Sigma$ intersects Λ_n^- iff the point $p_{\tilde\Sigma}$ is isotropic. Furthermore if $\tilde{Y}_0$ is the minimal type I cell (point) then an element x of the normal slice $(G_{n,n})_{\tilde{Y}_0}$ represents an isotropic plane iff x is antisymmetric. It follows that $\tilde{Y} \cap \Lambda_n^-$ contains the origin $\mathbb{C}^n \times 0$ in its closure iff $p_{\tilde{Y}_0}$ has an even number of 1's in the top tier in columns $n+1, \ldots, 2n$. In particular the Schubert varieties in Λ_n^- are parametrized by the Young diagrams which are symmetric about the diagonal, having an *even* number of boxes along the line of symmetry; the dimension of Y is equal to the number of boxes in $\Delta(Y)$ that lie strictly above the line of symmetry. Finally, the subspace of $(G_{n,n})_{\tilde{Y}} \simeq \mathbb{C}^{\bar\Delta(Y)}$ representing the normal slice $(\Lambda_n^-)_Y$ is precisely the space $\mathbb{C}_-^{\bar\Delta(Y)}$ of antisymmetric elements. ∎

DEFINITION. Given a point $x \in \mathbb{C}_\varepsilon^{\bar\Delta} \simeq H_Y$ we will denote the Schubert stratum of H_Y containing x by Σ_x.

3.3. *Auras for normal slices.* From the descriptions in Proposition 3.2A it is now easy to provide natural auras (in the sense of Section 2.2) for these normal slices. (Henceforth we return to our "Young diagram cartesian coordinates." Also, we will often drop the subscript ε to simplify the notation.) Given a Schubert pair (X, Y), abbreviate $\rho := \rho_Y^X$,

$\bar{\Delta} := \bar{\Delta}(Y)$, and put for $x \in \mathbb{C}^{\bar{\Delta}}$

$$g_0(x) := \sum_{\mu \in \Omega} \Big| \bigwedge{}^{\rho(\mu)+1} x_\mu \Big|^2.$$

This gives the required aura in type I. If $\bar{\Delta}$ is symmetric and $\rho(\mu) = \rho(\mu^t)$ for all $\mu \in \Omega$ then g_0 is invariant under both of the involutions

$$i_\pm(x) := \pm x^t,$$

from which we deduce that

$$\nabla g_0(\mathbb{C}^{\bar{\Delta}}_\pm) \subset \mathbb{C}^{\bar{\Delta}}_\pm. \tag{3.3.1}$$

Thus in types II and III we take $g_\pm := g_0|\mathbb{C}^{\bar{\Delta}}_\pm$. For $x \in \mathbb{C}^{\bar{\Delta}} - X_Y$ sufficiently close to 0 the gradient $\nabla g_0(x) \neq 0$; put $\phi_0(x) := \frac{\nabla g_0(x)}{|\nabla g_0(x)|}$ for such x. Similarly $\phi_\pm := \frac{\nabla g_\pm}{|\nabla g_\pm|} = \phi_0|\mathbb{C}^{\bar{\Delta}}_\pm$. Thus $d^X_Y = \deg \phi_\varepsilon|\mathbb{S}^{\bar{\Delta}}_r$ for small $r > 0$, by 2.2A. Fixing such r, we abbreviate $\mathbb{S}^{\bar{\Delta}} := \mathbb{S}^{\bar{\Delta}}_r$.

LEMMA 3.3A. *Let $x_1, x_2, \cdots \longrightarrow x_0 \in X_Y$, where $x_1, x_2, \cdots \notin X_Y$. Suppose $\phi(x_i) \longrightarrow n \in T_{x_0}\mathbb{C}^{\bar{\Delta}}$. Then $n \in \vec{\nu}_{x_0}\Sigma_{x_0}$.*

PROOF. In types II and III each stratum Σ of X_Y is the intersection with $\mathbb{C}^{\bar{\Delta}}_\pm$ of a stratum $\tilde{\Sigma}$ of the corresponding type I normal slice; moreover the rank conditions ρ determining the slice are symmetric in the sense that $\rho(\mu^t) = \rho(\mu)$ for each $\mu \in \Omega$. Thus $\tilde{\Sigma}$ is stabilized by the appropriate involution $i_\pm$, from which it follows that if $x_0 \in \Sigma$ then $T_{x_0}\tilde{\Sigma}$ decomposes as the orthogonal direct sum of its subspaces of symmetric and antisymmetric elements. Therefore $\vec{\nu}_{x_0}\Sigma = \mathbb{C}^{\bar{\Delta}}_\pm \cap \vec{\nu}_{x_0}\tilde{\Sigma}$, and by (3.3.1) it is enough to prove the lemma in type I.

By the Łojasiewicz inequality (1.3.1) it will be enough to show that for each $\tau_0 \in T_{x_0}\Sigma$ there is a sequence $T_{x_i}\mathbb{C}^{\bar{\Delta}} \ni \tau_i \longrightarrow \tau_0$ such that

$$|D_{\tau_i} g(x_i)| = |Re\langle \nabla g(x_i), \tau_i \rangle| = O\big(g(x_i)\big)$$

as $i \longrightarrow \infty$. By Proposition 3.2A(2) each such $\tau_0 = \gamma(x_0)$ for some $\gamma \in T_e B_0$, where B_0 is the isotropy subgroup of the special point p_Y. Therefore we may achieve our goal by proving that

$$|D_{z\gamma(x_i)} g(x_i)| = |z| O\big(g(x_i)\big) \qquad (z \in \mathbb{C}) \tag{3.3.2}$$

as $i \longrightarrow \infty$, as γ ranges through the three types of infinitesimal generators for B_0.

Suppose first that γ is of type (i) or type (ii), hence corresponds to a truncated row or column operation: suppose for definiteness that it corresponds to the operation of moving row k to row j, $j \leq k$. We further suppose that $j < k$—if $j = k$ then the argument is simpler. Thinking of $g(x)$ as a sum of squares $|\delta|^2$ of wedge products δ of partial rows of x, we decompose g as

$$g = g_{jk} + g_{\bar{j}k} + g_{j\bar{k}} + g_{\overline{jk}},$$

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where g_{jk} denotes the sum of those terms corresponding to wedge products δ involving both the j-th and the k-th rows, $g_{\bar{j}k}$ denotes the sum of the terms involving the k-th row but not the j-th, etc. Then

$$g\big(x + z\gamma(x)\big) - g(x) = g_{j\bar{k}}\big(x + z\gamma(x)\big) - g_{j\bar{k}}(x)$$

for any $x \in \mathbb{C}^{\bar{\Delta}}$ and $z \in \mathbb{C}$. Moreover, since $j < k$, the description of 3.2A implies that to every term $|\delta|^2$ of $g_{j\bar{k}}$ there corresponds a term $|\delta'|^2$ of $g_{\bar{j}k}$, with $|\delta\big(x + z\gamma(x)\big) - \delta(x)| \leq |z||\delta'(x)|$. Thus

$$\begin{aligned}
|g\big(x + z\gamma(x)\big) - g(x)| &= \sum \Big|\delta\big(x + z\gamma(x)\big)\Big|^2 - |\delta(x)|^2 \\
&\leq \sum 2|z||\delta(x)||\delta'(x)| + |z|^2|\delta'(x)|^2 \\
&\leq 2|z|\big(g_{\bar{j}k}(x)\big)^{\frac{1}{2}}\big(g_{j\bar{k}}(x)\big)^{\frac{1}{2}} + |z|^2 g_{\bar{j}k}(x) \\
&\leq (2|z| + |z|^2)g(x)
\end{aligned}$$

by the Schwartz inequality; putting $x = x_1, x_2, \ldots$, (3.3.2) follows in this case.

If γ is of type (iii) then the proof is similar: in this case γ corresponds to the tensor product of a row w delimiting some rectangle $\mu \in \Omega$ with some column v of μ. Note that for every rectangle $\mu' \in \Omega$, either $\mu' \cap w$ is a complete row of μ', or $\mu' \cap v$ is a complete column of μ' (but not both). Decompose $g = g^r + g^c$, where g^r is the sum of those terms corresponding to rectangles of the former kind and g^c is the sum of terms corresponding to rectangles of the latter kind. Restricting our attention to g^r (the required estimate for g^c is similar), we think of $g^r(x)$ as a sum of squares of wedge products δ of partial rows of x, then decompose further $g^r = g_w + g_{\bar{w}}$, according to whether the wedge products in question do or do not contain (part of) w. Then

$$g^r\big(x + z\gamma(x)\big) - g^r(x) = g_{\bar{w}}\big(x + z\gamma(x)\big) - g_{\bar{w}}(x),$$

and to each wedge product δ occurring in $g_{\bar{w}}$ there correspond k wedge products $\delta'_1, \ldots, \delta'_k$ from g_w, where $k = k(\delta)$ is the number of factors in δ. Therefore

$$\Big|\delta\big(x + z\gamma(x)\big)\Big| \leq |z||v| \sum_{i=1}^{k} |\delta'_i(x)|,$$

and

$$\begin{aligned}
\Big|g^r\big(x + z\gamma(x)\big) - g^r(x)\Big| &= \sum \Big|\delta\big(x + z\gamma(x)\big)\Big|^2 - |\delta(x)|^2 \\
&\leq 2|z||v| \sum_{\delta} |\delta(x)| \Big(\sum_{i=1}^{k(\delta)} |\delta'_i(x)|\Big) + |z|^2|v|^2 \sum_{\delta} \sum_{i=1}^{k(\delta)} |\delta'_i(x)|^2 \\
&\leq 2n|z|\big(g_{\bar{w}}(x)\big)^{\frac{1}{2}}\big(g_w(x)\big)^{\frac{1}{2}} + |z|^2 n^2 |v|^2 g_w(x) \\
&\leq (2n|z| + |z|^2 n^2 |v|^2) g(x),
\end{aligned}$$

and (3.3.2) follows for this case as well. ∎

3.4. *Types IV and V.* Here we give descriptions of the manifolds $K\backslash G_0$ and their Schubert varieties in the "easy" types IV and V.

4. $\big(\mathrm{SO}(n)\times\mathrm{SO}(2)\big)\backslash\mathrm{SO}(n+2)$, n odd. This is the complex quadric R_n consisting of all $[z]\in\mathbb{C}P_{n+1}$ whose homogeneous components z_i $(0\le i\le n+1)$ satisfy

$$\sum_{i=0}^{n+1} z_i z_{n+1-i} = 0. \tag{3.4.1}$$

Set $\ell=(n+1)/2$, $P_i=\{[z]\in\mathbb{C}P_{n+1}\mid z_{i+1}=z_{i+2}=\cdots=z_{n+1}=0\}$, $0\le i\le n+1$, and $Q_i=P_{i+1}\cap R_n$, $\ell-1\le i\le n$. Notice that $P_i\subset R_n$ for $0\le i\le\ell-1$, and $Q_{\ell-1}=P_{\ell-1}$. The Schubert varieties for Q_n are

$$P_0\subset P_1\subset\cdots\subset P_{\ell-1}\subset Q_\ell\subset Q_{\ell+1}\subset\cdots\subset Q_n=R_n \tag{3.4.2}$$

(where in each case the subscript equals the complex dimension).

5. $\big(\mathrm{SO}(n)\times\mathrm{SO}(2)\big)\backslash\mathrm{SO}(n+2)$, n even. This is again the complex quadric R_n, defined by (3.4.1). Set $\ell=(n+2)/2$, and define P_i, Q_i as above. Put $P'_{\ell-1}=\{[z]\in\mathbb{C}P_{n+1}\mid z_\ell=z_{\ell+2}=z_{\ell+3}=\cdots=z_{n+1}=0\}$. The Schubert varieties are

$$\begin{array}{llll} P_0\subset P_1\subset\cdots\subset & P_{\ell-2}\subset & P_{\ell-1} & \\ & \cap & \cap & \\ & P'_{\ell-1}\subset & Q_\ell\subset Q_{\ell+1}\subset\cdots\subset Q_n. \end{array} \tag{3.4.3}$$

4. **MacPherson coefficients for Schubert varieties.** In this section we state our algorithm for computing the MacPherson coefficients d_Y^X for pairs $X\supset Y$ of Schubert varieties in types I, II and III. In fact the algorithm yields a polynomial $D_Y^X(q)$, and the MacPherson coefficients are given by $d_Y^X=D_Y^X(1)$. The algorithm is stated in terms of rooted weighted trees constructed from the Young diagrams of X and Y. These trees are in turn abstracted from a certain combinatorial diagram Γ_0 inside the diagram $\bar\Delta=\bar\Delta(Y)$ of the slice space. This and a related diagram will be important for the linear algebra involved in the proof of the algorithm.

4.1. *The dot configurations.* We construct certain subsets Γ_0, Γ_+ and Γ_- of $\bar\Delta$, which will be crucial for all of our subsequent constructions. They may be thought of as generalizing to the deleted matrix $\bar\Delta$ the diagonal of a complete matrix.

Consider the partial order on $\bar\Delta$ defined by: $(i,j)\succeq(i',j')$ iff $i\ge i'$ and $j\ge j'$. Thus there is a unique maximal box, namely the upper right corner, and the minimal boxes are those adjacent to the indentations of the boundary of $\Delta(Y)$.

To construct Γ_0 we mark (with dots) certain entries of $\bar\Delta$ by the following inductive procedure (see Figure 4.1). As the initial step, mark all entries that are minimal with respect to the order $\succ$. For the inductive step, consider all entries of $\bar\Delta$ that share neither a row nor a column with any previously marked entry; of these, again mark all of the minimal ones. Continue until every box of $\bar\Delta$ shares either a row or a column (or both)

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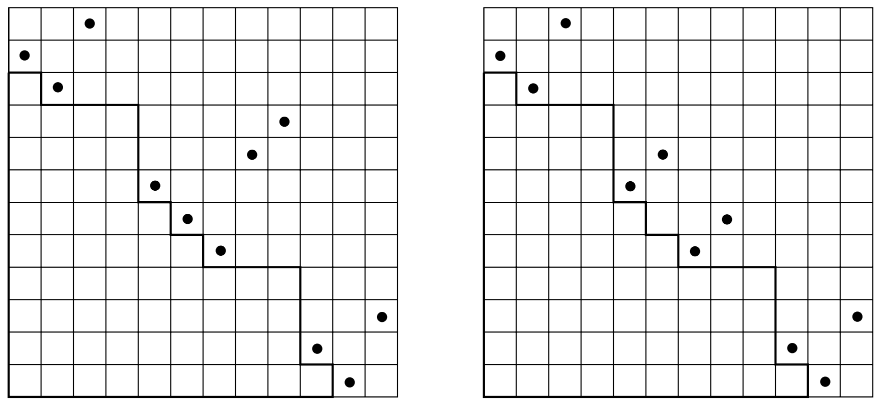

$\Gamma_0 = \Gamma_+$ Γ_-

FIGURE 4.1. The dot configurations $\Gamma_0 = \Gamma_+$ and Γ_-.

with some marked box. Now take Γ_0 to be the set of all marked entries. It is clear that if $\bar{\Delta}$ is symmetric about the diagonal then so is Γ_0. In this case we set $\Gamma_+ = \Gamma_0$.

We construct Γ_- only when $\bar{\Delta}$ is symmetric. Consider the partial order above restricted to the *off-diagonal boxes*, then mark the entries of $\bar{\Delta}$ following the procedure above, marking at each stage all those off-diagonal entries which are minimal with respect to the restriction of $\succeq$. Clearly $\#(\Gamma_-)$ is even. Observe that one could also construct Γ_- from Γ_+ as follows. If the number of diagonal elements of Γ_+ is odd, unmark the greatest of these. Then replace each pair (i,i), (j,j) of adjacent diagonal elements by the pair (i,j), (j,i).

A less invariant but more practical way to perform the marking process above is to proceed from left to right in $\bar{\Delta}$, marking with a dot the lowest entry in each column which does not lie in a row containing any previously placed dot. In the construction of Γ_- the diagonal entries are also excluded. If it is not possible to place a dot in any given column, the column is left blank and the process continues with the next column.

We will refer also to the result of the marking procedure above as the *dot configuration* of $\bar{\Delta}$ (or of Y). In general, Γ_0, Γ_+, Γ_- will be used in connection with Schubert varieties in types I and II, and III respectively; note, however, that the combinatorial construction of Section 4.2 below is based on Γ_0 in all three cases.

REMARK 4.1A. The significance of the dot configurations Γ_ε is illustrated by the following observations. Let A denote the set of all subrectangles $\alpha \subset \bar{M}$ containing the lower left corner $(1,1)$. Consider the set $U \subset \mathbb{C}^{\bar{\Delta}}_{\varepsilon}$ of all elements $y \in \mathbb{C}^{\bar{\Delta}}_{\varepsilon}$ such that $\operatorname{rank} y^\alpha = \#(\alpha \cap \Gamma_\varepsilon)$ for all $\alpha \in A$. Then U is open in $\mathbb{C}^{\bar{\Delta}}_{\varepsilon}$. Moreover, if $y \in U$ then $y \notin \vec{\nu}_x \Sigma_x$ for any $x \neq 0$. This may be deduced from the classification of tangent vectors to Σ_x at x given in Proposition 3.2A(2). For (using notation from the proof of that proposition, and letting * denote adjoint) it is clear that there are elements $b_i \in B_i$, $i = 1, 2$, such that $y_0 := b_1^* y b_2^*$ lies in the complex torus $(\mathbb{C}^*)^\Gamma_\varepsilon$ (if $\varepsilon = \pm$ then $b_1 = b_2$). Now regarding $x \in \Sigma_x = Z_Y$ as an element of the appropriate type I Schubert slice $\tilde{Z}_{\tilde{Y}}$, by the construction of the dot configurations either there exists $\beta \in B_1$ for which

$$0 \neq \langle \beta b_1 x b_2 - b_1 x b_2, y_0 \rangle = \langle b_1(b_1^{-1}\beta b_1 x - x) b_2, y_0 \rangle$$
$$= \langle b_1^{-1}\beta b_1 x - x, y \rangle,$$

or else there exists $\gamma \in B_2$ for which the corresponding inequality with right multiplication holds. But by 3.2A(2), $b_1^{-1}\beta b_1 x - x \in T_x \tilde{Z}_{\tilde{Y}}$. This proves the assertion in type I, and also in types II and III if we recall that $\tilde{Z}_{\tilde{Y}}$ is invariant under the appropriate involution $i_{\pm}$.

REMARK 4.1B. The following observation is easy to check, and will be useful: if Γ_0 has a unique maximal element then either every row or every column of $\overline{\Delta}$ contains an element of Γ_0. Therefore the same is true of Γ_- if the number of diagonal elements of Γ_0 is even.

4.2. *Statement of the theorem.* Given a pair $X \supset Y$ of Schubert varieties, recall the description of X_Y given in 3.2A(1). We produce the polynomial D_Y^X from the rank function ρ defining the normal slice X_Y, together with the dot configuration Γ_0.

Our construction is carried out in terms of certain connected rooted trees associated to Γ_0 and ρ. In type I we use the so-called *Hasse diagram* of the poset of dots Γ_0; *i.e.*, the connected rooted tree isomorphic to it as a poset, edges corresponding to dots. (See Figure 4.2.)

In types II and III we use the Hasse diagram of the sub-poset consisting of all dots of Γ_0 lying on or above the diagonal. Furthermore we distinguish the edges which correspond to the diagonal dots from the off-diagonal dots by drawing the former vertically, and the latter obliquely. The vertical ones will be called *central* edges, the oblique ones *side* edges. A central edge is *odd* if the number of central edges below it is even.

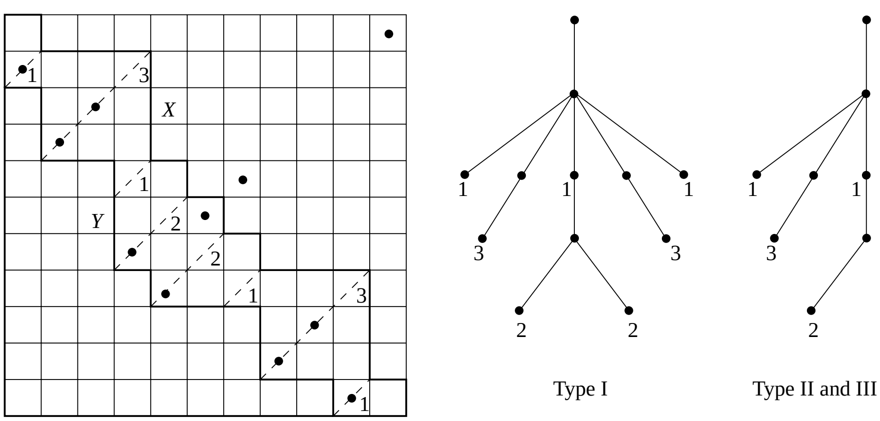

FIGURE 4.2. The trees T_Y^X.

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To each vertex v of this tree (except the top one) we now associate a non-negative integer *capacity* $c(v)$ as follows. Let (i,j) be the dot corresponding to the (unique) edge directly above v. Then

$$c(v) := \min\{\rho(\mu) \mid (i,j) \in \mu \in \Omega\}. \tag{4.2.1}$$

It is of course easy to read this off from the Young diagrams $\Delta(X)$, $\Delta(Y)$, using Proposition 3.2A. The capacity of an *edge* is defined to be the minimum of the capacities of all vertices lying below it. We denote the resulting tree, with the assigned capacities, by $\mathcal{T}_Y^X$.

REMARKS 4.2A. 1. In drawing the trees we omit all capacities (on non-minimal vertices) which are consequences of others.

2. It is easy to see that if $\bar{\Delta}(Y)$ is symmetric, then no diagonal element of Γ_0 is dominated by an off-diagonal element. Therefore, in types II and III any edge above a central edge is itself central.

We now construct the polynomial $D_Y^X(q)$ from $\mathcal{T} = \mathcal{T}_Y^X$. Put $\Lambda = \Lambda(\mathcal{T})$ for the set of all *labelings* $\lambda\colon \{\text{edges } e \text{ of } \mathcal{T}\} \longrightarrow \{\pm 1\}$ satisfying the conditions:

(4.2.2)

(1) If v is a vertex with capacity c, then the number of edges e above v with $\lambda(e) = -1$ is less than or equal to c; and

(2) In type III, the labels on each pair of central edges must be the same (where the central edges are numbered beginning with 1 at the bottom, and 1 is paired with 2, 3 with 4, etc.) If there is an odd number of central edges, the label on the top one must always be +1.

The *sign* $\sigma(\lambda)$ is defined by

(1) In type I, $\sigma(\lambda) := \prod_{\text{edges } e} \lambda(e)$;

(2) In types II and III, $\sigma(\lambda) := \prod_{\substack{\text{side}\\ \text{edges } e}} \lambda(e) \cdot \prod_{\substack{\text{odd central}\\ \text{edges } e}} \lambda(e)$.

The *weight* $|\lambda|$ is defined by

(1) In types I and II, $|\lambda| := \sum_{\lambda(e)=-1} (\text{number of edges below } e)$;

(2) In type III, $|\lambda| := \sum_{\lambda(e)=-1} \{\text{number of side edges below } e) + \text{number of even central edges below } e)\}$.

(We make the convention that if $\mathcal{T}_Y^X$ is the empty tree, then it has only the "empty" labeling λ with $\sigma(\lambda) = 1$ and $|\lambda| = 0$.) Finally, we define

$$D_Y^X(q) = \sum_{\lambda \in \Lambda} \sigma(\lambda) q^{|\lambda|}. \tag{4.2.3}$$

When $Y \not\subset X$ we set $D_Y^X(q) \equiv 0$.

More generally, if $\mathcal{T}$ is any tree-with-capacities of this form (but not necessarily arising from a pair of Schubert varieties $Y \subset X$), we define $\Lambda(\mathcal{T})$ by (4.2.2) and $D(\mathcal{T})$ to be the polynomial in q determined by the rules above. We call two such trees-with-capacities *equivalent* if they have the same underlying trees and the same collections of allowed labelings.

EXAMPLE 4.2B. Consider the tree T_Y^X of Figure 4.3. Below it are all possible labelings λ which meet condition (4.2.2(1)). (We write the labels simply as + or − instead of +1 or −1.) Below each labeling λ we write the associated monomial $\sigma(\lambda)q^{|\lambda|}$ in each of types I, II, or III; in type III the designation "NA" means that the labeling is not allowed by condition (4.2.2(2)).

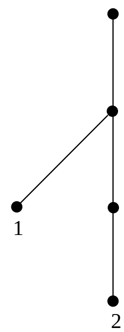

FIGURE 4.3. A sample tree T_Y^X.

	+	+	+	+	+	+	+	+	−	−	−
	+ +	+ +	− +	− +	+ −	+ −	− −	− −	+ +	+ +	+ −
	+	−	+	−	+	−	+	−	+	−	+
I:	1	−1	−1	1	$-q$	q	q	$-q$	$-q^3$	q^3	q^4
II:	1	−1	−1	1	q	$-q$	$-q$	q	$-q^3$	q^3	$-q^4$
III:	1	NA	−1	NA	NA	−1	NA	1	NA	NA	NA

Thus, according to (4.2.3), we obtain

$$D_Y^X = \begin{cases} q^4 & \text{in type I} \\ -q^4 & \text{in type II} \\ 0 & \text{in type III.} \end{cases}$$

As illustrated in the example there is a great deal of cancellation in (4.2.3). The procedure may be streamlined as follows. Let us say that an edge e of T_Y^X is *special* if e is a minimal central edge in type III having at least one side edge emanating from its upper vertex. Similarly, a vertex v is *special* if it is the lower vertex of a special edge. An edge (resp. vertex) is called *ordinary* if it is not special.

LEMMA 4.2C. *$D_Y^X(q)$ may be computed by the formula (4.2.3), where the summation is over the set $\Lambda_0 \subset \Lambda$ of all labelings λ satisfying, in addition to (4.2.2), the conditions:*

(3) $\lambda(e) = 1$ for every ordinary minimal edge e; and

(4) if v is an ordinary minimal vertex with capacity c, the number of edges e above v with $\lambda(e) = -1$ is equal to c.

In the example above, only the very last labeling satisfies (3) and (4) (and there is no such labeling in type III, so $D_Y^X \equiv 0$).

PROOF. Let $\lambda \in \Lambda$ and suppose $\lambda(e_1) = -1$ for some minimal edge e_1. If e_1 is not a type III central edge, then we obtain another allowed labeling λ^- by

$$\lambda^-(e) = \begin{cases} \lambda(e), & e \neq e_1 \\ -\lambda(e), & e = e_1. \end{cases} \tag{4.2.4}$$

If e_1 is an ordinary central edge in type III, then there is a central edge e_2 directly above e_1 (recall Remark 4.2A), and we define λ^- by

$$\lambda^-(e) = \begin{cases} \lambda(e), & e \neq e_1, e_2 \\ -\lambda(e), & e = e_1 \text{ or } e_2. \end{cases} \tag{4.2.5}$$

In either case, $|\lambda^-| = |\lambda|$ while $\sigma(\lambda^-) = -\sigma(\lambda)$, so that $\sigma(\lambda)q^{|\lambda|} + \sigma(\lambda^-)q^{|\lambda^-|} = 0$.

Suppose e_1 is a minimal edge whose lower vertex v has capacity c, λ is an allowed labeling with $\lambda(e_1) = +1$, and there are fewer than c edges e above e_1 with $\lambda(e) = -1$. If e_1 is not a type III central edge, then we obtain another allowed labeling λ^- by (4.2.4). If e_1 is an ordinary central edge in type III, then c must be even, and because the labels -1 above e_1 occur in pairs (by (4.2.2(2)) and Remark 4.2A), there can be at most $c-2$ edges e above e_1 with $\lambda(e) = -1$. Thus (4.2.5) defines an allowed labeling λ^-. In either case, the terms in D coming from λ and λ^- again cancel. ∎

REMARK 4.2D. If $D^X_Y \neq 0$, then it is not difficult to see that there is a unique labeling $\lambda_{\max}$ having maximal weight, and hence D^X_Y has leading coefficient ± 1. (In fact, $\lambda_{\max}$ is obtained by proceeding from top to bottom in the tree, labeling each edge with -1 if this is permitted by (4.2.2).) Other properties of the D polynomials are more elusive. For example, the signs of the coefficients need not all be the same, nor do they usually alternate, and coefficients other than $0, \pm 1$ do occur. Since (as the theorem below implies) the D polynomials are a sort of "quantized" version of the MacPherson coefficients, it is natural to seek a geometric interpretation of the polynomials themselves. However we have been unable to find such an interpretation.

Now we may state our result on the MacPherson coefficients in types I, II and III.

THEOREM 4.2E. *Given a Schubert variety X in a Hermitian symmetric space of type I, II or III, the normal cycle of X is given by*

$$\mathbf{N}(X) = \sum_{Y \subset X} D^X_Y(1)\mathbf{N}(Y^\circ),$$

where the sum is over all Schubert varieties $Y \subset X$, and $D^X_Y(q)$ are the polynomials constructed in this subsection. In particular, $d^X_Y = D^X_Y(1)$.

5. **Proof of the main algorithm.** This section is devoted to the Proof of Theorem 4.2E. The idea is to conform as closely as possible to the following outline, which, sadly, we were unable to execute in its full simplicity. For clarity we restrict the discussion to the A_n case, the standard Grassmannian. We make the hypothesis that $\deg \phi = \deg(\phi|\mathbb{S}^{\Gamma_0})$, where $\Gamma_0 \subset \bar{\Delta}$ is the dot configuration defined in Section 4. (Our inability to prove this hypothesis accounts entirely for the failure of the present account to yield a rigorous proof.) The hypothesis holds if there is an open subset $U \subset \mathbb{S}^{\Gamma_0}$ such that $\phi^{-1}(U) \subset U$. Put $\Sigma_+ \subset \mathbb{S}^{\Gamma_0}$ for the curved simplex consisting of all points in the sphere

with all coordinates real and nonnegative. Then ϕ is well-defined on the interior of Σ_+ and it is easy to prove that $\deg(\phi|\mathbb{S}^{\Gamma_0}) = \deg(\phi|\Sigma_+) =: d$.

To evaluate (inductively) this last degree we distinguish two cases. Let $\gamma_1, \ldots, \gamma_k \in \Gamma_0$ denote the maximal elements of Γ_0. If $k > 1$ then we are in the so-called *decomposable* case. In this case Σ_+ is the join of the faces $\Sigma_1, \ldots, \Sigma_k$ corresponding to the subsets of Γ_0 dominated by the respective γ_i, and the map $\phi|\Sigma_+$ is the join of the maps $\phi|\Sigma_i\colon \Sigma_i \longrightarrow \Sigma_i$, $i = 1, \ldots, k$. In this situation it is clear that $d = \prod_{i=1}^{k} \deg(\phi|\Sigma_i)$, where these last factors are equal to the MacPherson coefficients of simpler Schubert singularities.

In the complementary *indecomposable* case we use the elementary equation of currents

$$d[\![\partial\Sigma_+]\!] = d\partial[\![\Sigma_+]\!] = \partial\, d[\![\Sigma_+]\!] = \partial\phi_*[\![\Sigma_+]\!]. \tag{5.0.1}$$

If ϕ were defined throughout $\partial\Sigma_+$ then this would tell us that d is equal to the degree of the restriction of ϕ to the boundary of Σ_+. As it is, ϕ is in all cases of interest well-defined on the faces Σ_i of codimension 1, and even stabilizes these faces, but may not be defined on faces of higher codimension. Let v^* denote the distinguished vertex of Σ_+, associated to the maximal coordinate $\gamma = \gamma_1$. The face Σ^* opposite v^* has special properties: if $\phi(x_i) \longrightarrow \Sigma^*$ then either $x_i \longrightarrow \Sigma^*$ or $x_i \longrightarrow v^*$. It turns out that we may think of the preimage of Σ^* under ϕ as consisting of Σ^* itself together with a virtual copy of Σ^*, with the opposite orientation, at v^*. Thus the multiplicity of Σ^* in (5.0.1)—and therefore the degree d that we seek—is the difference of the degrees of these two self maps of Σ^*, and we again obtain a recurrence relation for the desired MacPherson coefficient.

It is not difficult to prove the hypothesis above if X and Y are "determinantal" Schubert varieties (*i.e.* varieties for which the defining sequences (3.1.2) assume only two distinct values). The computation of the array of these MacPherson coefficients by the method above is a useful exercise.

In the honest proof of the formula given below, the subsphere $\mathbb{S}^{\bar{\Delta}'}$ (defined in Section 5.3) corresponds to the face Σ^* and Z_Y (defined in Section 5.2) corresponds to the vertex v^*. The vestiges of the idea of the virtual face at v^* appear in the blowing up process used in Section 5.3.

5.1. *The decomposable case.* Let us now begin the formal proof. We use induction on the number of edges in the tree T_Y^X. If the tree is empty then both X and Y coincide with the ambient Hermitian symmetric space and $D_Y^X = 1 = d_Y^X$, as required.

For the inductive step, note first that we may assume that some maximal edge of T_Y^X has positive capacity. Otherwise, the description 3.2A of the normal slice X_Y implies that by shaving the Young diagrams of X and Y appropriately we may produce a new Schubert pair (X', Y') such that $(X')_{Y'} \simeq X_Y$ and where the tree $T_{Y'}^{X'}$ is equal to T_Y^X with one or more maximal edges of capacity zero deleted.

As in the discussion above, if Γ_0 contains a unique maximal element then we say that the pair (X, Y) is *indecomposable*, otherwise *decomposable*. Thus (X, Y) is indecomposable iff T_Y^X has a unique maximal edge, which in types II and III is required additionally to be central.

Suppose first that (X, Y) is decomposable. If, in type II or III, the tree T_Y^X consists entirely of side edges, then we will identify a type I Schubert pair (X_1, Y_1) such that

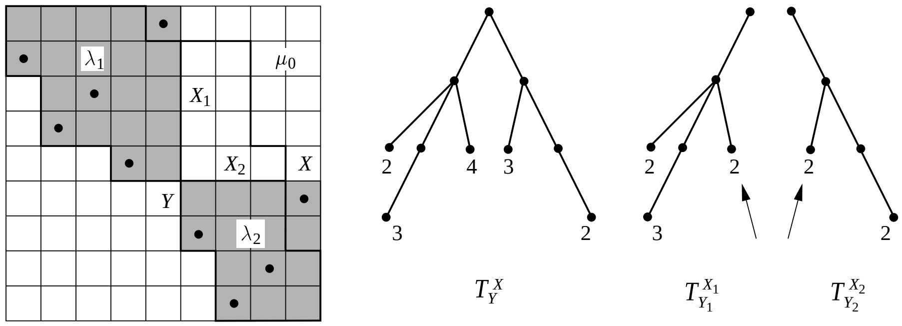

FIGURE 5.1. Decomposable case, type I.

$D_Y^X = D_{Y_1}^{X_1}$ and $d_Y^X = d_{Y_1}^{X_1}$. If on the other hand T_Y^X has more than one maximal edge we will find Schubert pairs (X', Y'), (X'', Y'') such that i) $d_Y^X = d_{Y'}^{X'} \cdot d_{Y''}^{X''}$ and ii) T_Y^X is the join of $T_{Y'}^{X'}$ and $T_{Y''}^{X''}$ at their top vertex. Condition ii) clearly implies that $D_Y^X = D_{Y'}^{X'} \cdot D_{Y''}^{X''}$.

The condition that (X, Y) be decomposable is equivalent to the existence of a rectangle $\mu_0 \in \Omega$ with $\mu_0 \cap \Gamma_0 = \mu_0 \cap \Gamma_- = \emptyset$. Define the modified aura

$$\tilde{g}(x) := (1 + |x^{\mu_0}|^2) g(x),$$

with

$$\nabla \tilde{g}(x) = (1 + |x^{\mu_0}|^2) \nabla g(x) + 2g(x) x^{\mu_0}.$$

Since for any $\mu \in \Omega$ the rectangle $\mu_0 \cap \mu$ is a block either of complete rows or of complete columns of μ, for $t \geq 0$ we have

$$\begin{aligned} g(x + tx^{\mu_0}) &= \sum_{\mu \in \Omega} \Big| \bigwedge\nolimits^{\rho(\mu)+1} (x^{\mu} + tx^{\mu_0 \cap \mu}) \Big|^2 \\ &\geq \sum_{\mu} \Big| \bigwedge\nolimits^{\rho(\mu)+1} x^{\mu} \Big|^2 \\ &= g(x). \end{aligned}$$

Therefore $\mathrm{Re}\langle \nabla g(x), x^{\mu_0} \rangle = D_{x^{\mu_0}} g(x) \geq 0$, whence $\big(\nabla \tilde{g}(x)\big)^{\mu_0} = 0$ iff $x^{\mu_0} = 0$ or $g(x) = 0$. If $y \in \mathbb{C}^{\bar{\Delta} - \mu_0}$ belongs to the open set U defined in Remark 4.1A, then y does not occur as a normal to any nontrivial Schubert stratum. Therefore, by Lemma 3.3A, $y \neq \lim \phi(x_i)$ for any sequence of points x_i converging to a nonzero point of X_Y. Moreover, if $x^{\mu_0} = 0$ and $x \notin X_Y$ then the restriction to $\mathbb{C}^{\mu_0}$ of the Hessian form $D^2 \tilde{g}(x)$ is positive definite. It follows that if we put $\tilde{\phi}_0 := \frac{\nabla \tilde{g}}{|\nabla \tilde{g}|}$ then in type I

$$d_Y^X = \deg(\tilde{\phi}_0 | \mathbb{S}^{\bar{\Delta}}) = \deg(\phi | \mathbb{S}^{\bar{\Delta} - \mu_0}).$$

It is also easy to deduce that if $\bar{\Delta}$ and the rank function ρ are both symmetric then, putting $\tilde{g}_\pm := \tilde{g} | \mathbb{C}_\pm^{\bar{\Delta}}$ and $\tilde{\phi}_\pm$ for the corresponding normalized gradients,

$$d_Y^X = \deg(\tilde{\phi}_\pm | \mathbb{S}_\pm^{\bar{\Delta}}) = \deg(\phi_\pm | \mathbb{S}_\pm^{\bar{\Delta} - (\mu_0 \cup \mu_0^t)}).$$

In type I we now proceed as follows. Denote the components of $\overline{\Delta} - \mu_0$ by λ_1, λ_2. Then $\phi|\mathbb{S}^{\overline{\Delta}-\mu_0}$ is the join of the maps $\phi_i := \phi|\mathbb{S}^{\lambda_i}$, $i = 1, 2$. Thus the degree of ϕ is the product of the degrees of the ϕ_i. But $\phi|\mathbb{C}^{\lambda_i}$ is the normalized gradient of $g|\mathbb{C}^{\lambda_i}$, which in turn is an aura for a Schubert slice $(X_i)_{Y_i}$ such that the weighted tree T_Y^X is equivalent to the join of the weighted trees $T_{Y_i}^{X_i}$ at their top vertex (see Figure 5.1). Therefore

$$\begin{aligned} d_Y^X &= d_{Y_1}^{X_1} \cdot d_{Y_2}^{X_2} \\ &= D_{Y_1}^{X_1}(1) \cdot D_{Y_2}^{X_2}(1) \qquad \text{by induction} \\ &= D_Y^X(1). \end{aligned}$$

In types II and III the argument is similar. In this case $\overline{\Delta} - (\mu_0 \cup \mu_0^t)$ has two symmetrically related components λ_1, λ_1^t, and possibly a third component λ_2 with $\lambda_2 = \lambda_2^t$. We express $\phi_\pm|\mathbb{S}_\pm^{\overline{\Delta}-(\mu_0\cup\mu_0^t)}$ as the join of $\phi_\pm|\mathbb{S}_\pm^{\lambda_1\cup\lambda_1^t}$ and $\phi_\pm|\mathbb{S}_\pm^{\lambda_2}$ (see Figure 5.2). The former map corresponds to a Schubert pair (X_1, Y_1) which is isomorphic to the type I Schubert pair (X_1', Y_1') determined by $\Delta(X_1') = \Delta(X) \cap \lambda_1$, $\Delta(Y_1') = \Delta(Y) \cap \lambda_1$. In the associated subtree $T_{Y_1}^{X_1}$ of T_Y^X all of the edges are side edges. Therefore the (logically prior) analysis of type I yields $d_{Y_1}^{X_1} = d_{Y_1'}^{X_1'} = D_{Y_1'}^{X_1'}(1) = D_{Y_1}^{X_1}(1)$. The latter map corresponds to a pair (X_2, Y_2) of type II or III, for which the result holds by induction. The proof now proceeds as in type I.

5.2. *The indecomposable case: combinatorial part.* Suppose now that (X, Y) is indecomposable, and let $n := \#(\Gamma_0)$. Let M_0 denote the rectangle spanned by $\overline{\Delta}(Y)$. In types II and III, M_0 is a square. By Remark 4.1B, either every row or every column of M_0 contains an element of Γ_0. Suppose for definiteness that every row does, and let μ^* denote the union of all the columns which do not; it is clear that $\mu^* \in \Omega$. Consider the modified aura $\tilde{g}(x) := g(x) + |x^{\mu^*}|^2$. Putting $\tilde{\phi} := |\nabla\tilde{g}|^{-1}\nabla\tilde{g}$, an argument similar to that of Section 5.1 shows that $d_Y^X = \deg\tilde{\phi} = \deg(\tilde{\phi}|\mathbb{S}^{\overline{\Delta}-\mu^*})$. This shows that d_Y^X depends only on the combinatorial data $(M_0 - \mu^*) \cap \Delta(Y)$, $(M_0 - \mu^*) \cap \Delta(X)$. As the same is true of the weighted tree T_Y^X (up to equivalence), we may assume even in type I that M_0 is a square of side n. We change coordinates so that the lower left corner of M_0 is $(1, 1)$ and the upper right corner is (n, n) (see Figure 5.3).

Let e_0 denote the unique maximal edge of T_Y^X, which (as noted in the introduction to Section 5) we may assume to have positive capacity. Suppose W is a Schubert variety with $X \supset W \supset Y$ and such that the weighted tree T_Y^W is equivalent to T_Y^X with e_0 removed. If in type III the number of central edges is odd, then obviously $D_Y^X = D_Y^W$. We therefore assume in type III that n is even. Suppose Z is a Schubert variety, $X \supset Z \supset Y$, such that:

in types I and II, T_Z^X is equivalent to T_Y^X with e_0 removed and the capacities of all remaining edges diminished by 1;

in type III, letting e_1 denote the central edge directly below e_0, T_Z^X is equivalent to T_Y^X with both e_0 and e_1 removed and the capacity of each remaining edge e diminished by 1 if $e < e_0$ but $e \not< e_1$ and by 2 if $e <$ both e_0 and e_1.

Put k for the cardinality of the set of diagonal elements of Γ_0 (= the number of central edges of T_Y^X). Thus in types II and III the cardinality of the set of side edges of T_Y^X is

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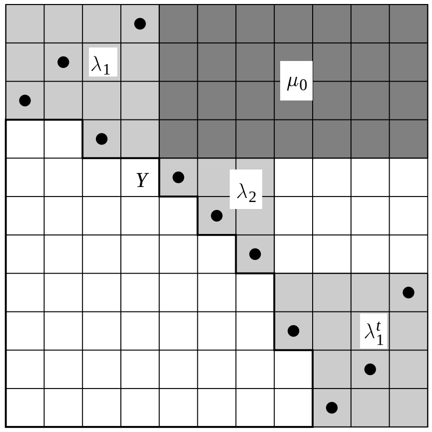

FIGURE 5.2. Decomposable case, types II and III.

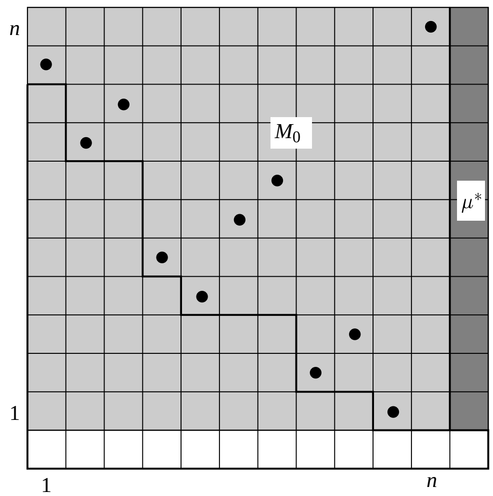

FIGURE 5.3. M_0, type I.

$\frac{n-k}{2}$; in particular, n and k must have the same parity. In type III put ℓ for the number of side edges $e < e_1$. Using (4.2.3) it is straightforward to verify that

$$D_Y^X(q) = \begin{cases} D_Y^W(q) - q^{n-1} D_Z^X(q) & \text{in type I} \\ D_Y^W(q) + (-1)^k q^{\frac{n+k}{2}-1} D_Z^X(q) & \text{in type II} \\ D_Y^W(q) - q^{\frac{n+k}{2}+\ell-2} D_Z^X(q) & \text{in type III,} \end{cases} \tag{5.2.1}$$

where the first (resp. second) term arises from those labelings with $\lambda(e_0) = +1$ (resp. -1). To complete the proof of the theorem it will therefore be enough to construct such Z and W and to show that

$$d_Y^X = \begin{cases} d_Y^W - d_Y^Z & \text{in types I and III} \\ d_Y^W + (-1)^n\, d_Y^Z & \text{in type II} \end{cases} \tag{5.2.2}$$

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(where we have exploited the fact that $(-1)^k = (-1)^n$ in type II).

We now identify the varieties W, Z. Put

$$\omega_\varepsilon := \begin{cases} \{n\} \times \{1, \ldots, n\}, & \varepsilon = 0 \\ \{n\} \times \{1, \ldots, n\} \cup \{1, \ldots, n\} \times \{n\}, & \varepsilon = \pm. \end{cases}$$

Then W is determined by the relation (*cf.* Figure 5.5)

$$\Delta(W) \cap M_0 = \Delta(X) \cap M_0 - \omega.$$

Assuming in type III that k is even, we describe the variety Z by means of the rank function ρ_Y^Z defining the normal slice Z_Y. Let $\gamma_0 = \gamma_+ = \{(n, n)\}$ denote the singleton consisting of the maximal dot of Γ_0, and $\gamma_- = \{(r, n), (n, r)\}$ the symmetric pair of maximal dots in Γ_-. Then Z is determined by the relation

$$\rho_Y^Z(\mu) \equiv \#(\gamma_\varepsilon \cap \mu), \quad \mu \in \Omega.$$

It is straightforward to check that the Young diagram $\Delta(Z)$ is constructed as follows. In types I and II, adjoin to $\Delta(Y)$ the "ribbon" consisting of all boxes adjacent to the boundary of $\Delta(Y)$, beginning at $(1, n)$ and ending at $(n, 1)$; see Figure 5.4. Therefore the dimension formulae 3.2A(3) gives

$$\text{(5.2.3)} \qquad \dim(Z_Y) = \dim(Z) - \dim(Y) = \begin{cases} 2n - 1 & \text{in type I,} \\ n & \text{in type II.} \end{cases}$$

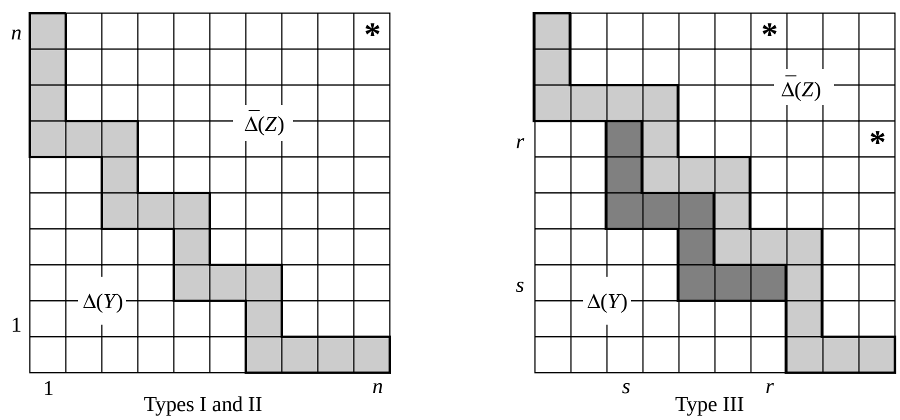

FIGURE 5.4. The Young diagram $\Delta(Z)$.

In type III, observe first that by indecomposability $(r, r) \in \overline{\Delta}(Y)$. Let $(r, s), (s, r)$ denote the minimal elements of $\overline{\Delta}$ lying in column r and row r of Δ, respectively; thus $s \leq r$. To obtain $\Delta(Z)$, first adjoin to $\Delta(Y)$ the short ribbon extending from (s, r) to (r, s), then adjoin to the resulting diagram the ribbon from $(1, n)$ to $(n, 1)$ (see Figure 5.4). The dimension formula now gives

$$\text{(5.2.4)} \qquad \dim(Z_Y) = n - 1 + r - s.$$

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This expression is necessarily *odd*, *i.e.* $r-s$ is even. For, columns s through $r-1$ of $\bar{\Delta}$ cannot contain any dots on or above row r: this is guaranteed by the placement of the dot at (r,n) in Γ_-. Similarly, rows s through $r-1$ of $\bar{\Delta}$ cannot contain any elements of Γ_- on or to the right of column r. Thus all of the dots in these rows and columns, which must be even in number, must lie within the square $\{(i,j) \mid (s,s) \preceq (i,j) \preceq (r-1,r-1)\}$, and since every row or column of $\bar{\Delta}$ contains an element of Γ_- (Remark 4.1B), the size $r-s$ of this square is even.

It is straightforward to verify that T_Z^X and T_Y^W are as stated.

We will need also the following result. Suppose that $m = n$ (*i.e.* M_0 is square), and in type III that k is even. Let $\mathbf{det}_\varepsilon$ denote the cone of all $y \in \mathbb{C}_\varepsilon^{\bar{\Delta}} \subset \mathbb{C}^{M_0}$ such that $\det y = 0$. Let $\mathbb{P}Z_Y \subset \mathbb{P}_\varepsilon^{\bar{\Delta}}$ denote the projective variety corresponding to the cone Z_Y, and $(\mathbb{P}Z_Y)^* \subset \mathbb{P}_\varepsilon^{\bar{\Delta}}$ its dual variety, where we have identified the dual projective space $(\mathbb{P}_\varepsilon^{\bar{\Delta}})^*$ with $\mathbb{P}_\varepsilon^{\bar{\Delta}}$ via the standard Hermitian metric. Put $\Gamma'_\varepsilon := \Gamma_\varepsilon - \omega_\varepsilon$.

LEMMA 5.2A.

$$(\mathbb{P}Z_Y)^* = \mathbb{P}\mathbf{det}_\varepsilon.$$

Moreover, there is a nonempty Zariski open subset U', $\mathbb{P}^{\Gamma'} \subset U' \subset \mathbb{P}\mathbf{det}_\varepsilon$ such that

(1) $U' \cap (\mathbb{P}Z'_Y)^* = \emptyset$ for all Schubert varieties $Z' \neq Z$, $Z' \supset Y$, and

(2) for each $\eta \in U'$ there is a unique $\xi \in \mathbb{P}Z_Y$ such that $\eta \in \mathbb{P}\nu_\xi \mathbb{P}Z_Y$.

PROOF. Given any $x \in \mathbb{C}^{\bar{\Delta}}$ and $y \in \nu\Sigma_x$, the classification 3.2A(2) of the tangent vectors to the stratum Σ_x shows that the matrix product $y \bullet \bar{x}^{n\cdot} = 0$. Thus if $x^\omega \neq 0$ then $\det y = 0$. Since the set of such points x is dense in Z_Y, it follows that $(\mathbb{P}Z_Y)^* \subset \mathbb{P}\mathbf{det}$.

Conversely, the classification 3.2A(2) implies that $(\mathbb{C}^*)_\varepsilon^{\Gamma'} \subset \nu_x Z_Y$ for $x \in \mathbb{S}_\varepsilon^\gamma$. Let A denote the set of all lower-left justified rectangles $\alpha \subset M_0$, *i.e.* the set of all α of the form $\alpha = \{1,\ldots,k\} \times \{1,\ldots,l\}$, $1 \leq k, l \leq n$. Putting

$$U' := \{y \in C_\varepsilon^{\bar{\Delta}} \mid \operatorname{rank} y^\alpha = \#(\Gamma'_\varepsilon \cap \alpha) \text{ for all } \alpha \in A\}$$

(see Figure 5.5), the lemma now follows by an argument similar to that given in Remark 4.1A. ∎

REMARKS 5.2B. 1. Denoting by H the ambient Hermitian symmetric space, the first statement of the lemma implies that the conormal varieties $\nu^*Z, \nu^*Y \subset T^*H$ intersect in a subvariety of codimension one. In fact, in types I and II the poset of Schubert varieties Z dominating Y with this property is isomorphic to the poset of edges of the tree T_Y^X: if $\alpha_e \subset \Gamma_0$ corresponds to the edge e then the corresponding Schubert variety $Z = Z_e$ is determined by the rank function $\rho_Y^Z(\mu) = \#(\mu \cap \alpha_e)$ (*cf.* Proposition 3.2A).

2. Conclusion (2) of 5.2A implies that the correspondence $\mathbb{P}\mathbf{det} \leftrightarrow \mathbb{P}\nu^*Z_Y$ is almost a birational equivalence—this becomes precisely true if we replace $\mathbb{P}\mathbf{det}$ by $\mathbb{P}\mathbf{det}^*$, the subvariety of the dual projective space $(\mathbb{P}^{\bar{\Delta}})^*$ corresponding to $\mathbb{P}^{\bar{\Delta}}$ under the diffeomorphism induced by the standard Hermitian structure.

3. It is not difficult to generalize 5.2A and its proof to obtain a characterization of the dual variety of any projectivized Schubert slice.

5.3. *The indecomposable case: analytic part.* We can now complete the Proof of Theorem 4.2E. We begin by stating the following lemma. The idea is that the map ϕ *almost*

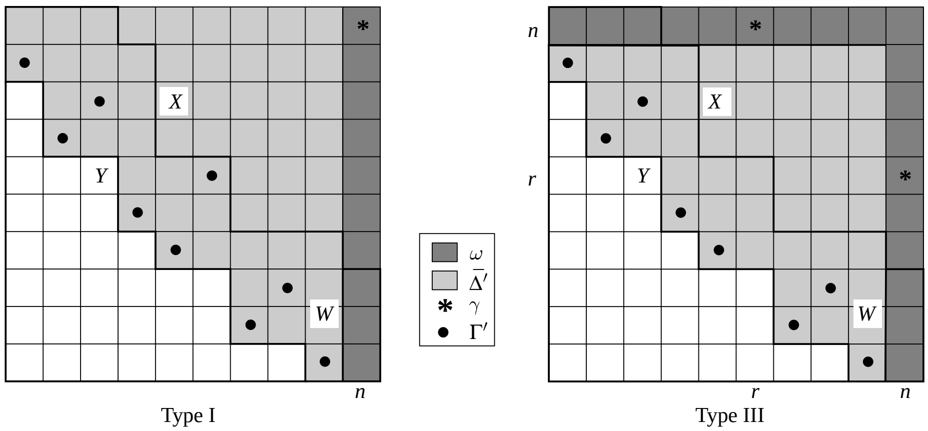

FIGURE 5.5. Various subsets of $\bar{\Delta}(Y)$.

enjoys the properties (3) and (4) below of the map ψ: ϕ is good enough that it may be perturbed to ψ without altering the degree. These properties are sufficient to prove the recursive formula (5.2.2), as we will show presently. The proof of the lemma is postponed to the next section.

We define

$$\bar{\Delta}'_\varepsilon := \bar{\Delta}_\varepsilon - \omega_\varepsilon,$$

and, for $y \in \mathbb{C}^{\bar{\Delta}}_\varepsilon - \mathbf{det}$ put

$$\theta(y) := \begin{cases} \frac{\det y}{|\det(y)|} & \text{in types I and II} \\ \frac{\operatorname{Pf} y}{|\operatorname{Pf}(y)|} & \text{in type III,} \end{cases}$$

where Pf is the Pfaffian (the algebraic square root of the determinant).

LEMMA 5.3A. *There is a continuous semialgebraic map* $\Psi\colon [0,1] \times \overline{\operatorname{graph}\phi} \longrightarrow \mathbb{S}^{\bar{\Delta}}_\varepsilon \times \mathbb{S}^{\bar{\Delta}}_\varepsilon$ *with the following properties.*

(1) *For each* $t \in [0,1]$, *there is a continuous semialgebraic map* $\psi_t\colon \mathbb{S}^{\bar{\Delta}} - X_Y \longrightarrow \mathbb{S}^{\bar{\Delta}}$ *such that* $\Psi_t(x,y) := \Psi(t,x,y) = \big(x, \psi_t(x)\big)$ *for* $x \in \mathbb{S}^{\bar{\Delta}} - X_Y$.

(2) *If* $t = 0$ *or* $x \in X_Y$ *then* $\Psi(t,x,y) = (x,y)$.

(3) *Put* $\psi := \psi_1$. *Then* $\psi|\mathbb{S}^{\bar{\Delta}'}_\varepsilon = \phi|\mathbb{S}^{\bar{\Delta}'}_\varepsilon$ *and* $\psi^{-1}(\mathbb{S}^{\bar{\Delta}'}_\varepsilon) \subset \mathbb{S}^{\bar{\Delta}'}_\varepsilon$. *If* $x \in \mathbb{S}^{\bar{\Delta}'}_\varepsilon - X_Y$ *then the transverse Jacobian* $\langle w, D\psi(x)\cdot v\rangle$ *is a positive definite form on* $v, w \in \mathbb{C}^{\omega}_\varepsilon = (T_x\mathbb{S}^{\bar{\Delta}'})^\perp \subset T_x\mathbb{S}^{\bar{\Delta}}_\varepsilon$.

(4) *There are open sets* $V, Q \subset \mathbb{S}^{\bar{\Delta}}_\varepsilon$ *with* $V \supset \mathbb{S}^{\gamma}_\varepsilon$ *and* $Q \cap \mathbb{S}^{\Gamma'_\varepsilon}_\varepsilon \neq \emptyset$, *such that if* $x \in \psi^{-1}(Q) \cap V$ *then* $\det \psi(x) \neq 0$ *and*

$$\theta\big(\psi(x)\big) = \theta\big(x^{\gamma_\varepsilon} + \psi(x)\big). \tag{5.3.1}$$

CONCLUSION OF THE PROOF OF THEOREM 4.2E. Put G for the current representing the graph of ϕ. Then

$$\begin{aligned}\partial \pi_{2*}\Psi_*([\![s,t]\!]\times G) &= \pi_{2*}\Psi_*\partial([\![s,t]\!]\times G)\\ &= \pi_{2*}\Psi_*\big(([\![t]\!]-[\![s]\!])\times G-[\![s,t]\!]\times\partial G\big)\\ &= (\psi_{t*}-\psi_{s*})[\![\mathbb{S}^{\bar\Delta}]\!]-0\end{aligned}$$

—here the second term vanishes since it is an integral current of dimension $2\#\bar\Delta-1$ supported on the set $\pi_2\circ\Psi([s,t]\times\operatorname{spt}\partial G)=\pi_2\circ\Psi_0(\operatorname{spt}\partial G)$, which has dimension at most $2\#\bar\Delta-2$ by the Proof of Theorem 2.2A. Similarly, the current under the boundary on the left-hand side has dimension $2\#\bar\Delta$, hence vanishes, so $(\psi_*-\phi_*)[\![\mathbb{S}^{\bar\Delta}]\!]=(\psi_{1*}-\psi_{0*})[\![\mathbb{S}^{\bar\Delta}]\!]=0$. Thus the degree of ψ is well-defined, with $\deg\psi=\deg\phi=d_Y^X$. Thus we must show that $\deg\psi=d_Y^W\pm d_Z^X$, as in (5.2.2).

By 5.2A, given any element $y\in\mathbb{S}_\varepsilon^{\Gamma'}\cap U'$ we have $\bigcap_{r>0}\overline{\phi^{-1}\big(B(y,r)\big)}-\phi^{-1}(y)\subset\mathbb{S}_\varepsilon^{\gamma}$. Therefore 5.3A (2) and (3) imply that $\bigcap_{r>0}\overline{\phi^{-1}\big(B(y,r)\big)}\subset\mathbb{S}^{\bar\Delta'}\cup\mathbb{S}^{\gamma}$. Thus given any open set $U\supset\mathbb{S}_\varepsilon^{\bar\Delta'}$, disjoint from V, the open set Q of 5.3A (4) above may be taken so small that $\psi^{-1}(Q)\subset U\cup V$. For such Q we have

$$d_Y^X[\![Q]\!]=(\psi_*[\![\mathbb{S}_\varepsilon^{\bar\Delta}]\!])|Q=(\psi_*[\![U]\!])|Q+(\psi_*[\![V]\!])|Q=:d_1[\![Q]\!]+d_2[\![Q]\!]$$

for some $d_1,d_2\in\mathbb{Z}$. It is clear that $d_1=d_Y^W$ since by (3) any regular value of $\phi_\varepsilon|\mathbb{S}_\varepsilon^{\bar\Delta'}$ is a regular value of $\psi|U$, and the local degrees of these two maps coincide on $\mathbb{S}_\varepsilon^{\bar\Delta'}$.

To evaluate d_2, observe that by 5.3A (2) and the basic relations (2.2.3), (2.2.4), the current H representing the graph of ψ has a nonzero boundary satisfying

$$(\partial H)|V\times Q=(\partial G)|V\times Q=-d_Z^X\mathbf{N}_{\mathbb{S}}(Z_Y\cap\mathbb{S}^{\bar\Delta})|V\times Q. \tag{5.3.2}$$

It is clear that $\psi_*[\![V]\!]|Q=\pi_{2*}(H|V\times Q)$. We wish to exploit the fact that $\partial H\neq0$ by using the fact that if $f\colon M\longrightarrow N$ is a map between manifolds with $f(\partial M)\subset\partial N$ then the degree of f is equal to the degree of $f|\partial M$. The problem is of course that the target submanifold Q has no boundary and that π_2 takes ∂H to zero—indeed this fact is central to our method, guaranteeing that the degree of ψ is well-defined. However we can introduce additional boundary by "blowing up" Q over the codimension 2 submanifold $\mathbf{det}\cap Q$. More precisely, consider the "blow-up"

$$\tilde{\mathbb{S}}:=\overline{\big\{\big(y,\theta(y)\big)\mid y\in\mathbb{S}_\varepsilon^{\bar\Delta}-\mathbf{det}\big\}}\subset\mathbb{S}_\varepsilon^{\bar\Delta}\times\mathbb{S}^1$$

of the image sphere. Put $\sigma\colon\tilde{\mathbb{S}}\longrightarrow\mathbb{S}_\varepsilon^{\bar\Delta}$ for the projection and $\tilde A:=\sigma^{-1}(A)$ for $A\subset\mathbb{S}_\varepsilon^{\bar\Delta}$. We now show that $\pi_2|(\operatorname{graph}\psi\cap V\times Q)$ lifts and extends to a map $\Theta\colon\overline{\operatorname{graph}\psi}\cap V\times Q\longrightarrow\tilde{\mathbb{S}}$, and examine the boundary behavior of Θ.

Put $\bar\theta:=\theta\circ\pi_2|\mathbb{S}^{\bar\Delta}\times(\mathbb{S}^{\bar\Delta}-\mathbf{det})$. Condition (5.3.1) may be restated as

$$\bar\theta(x,y)=\theta(x^\gamma+y)\text{ for }(x,y)\in\operatorname{graph}\psi\cap V\times Q. \tag{5.3.3}$$

Thus $\bar\theta$ admits a Lipschitz extension, again denoted $\bar\theta$, to all of $\overline{\operatorname{graph}\psi}\cap(V\times Q)$, satisfying the same relation (5.3.3). Therefore $\sigma^{-1}\circ\pi_2$ also extends to a Lipschitz map $\Theta\colon\operatorname{spt}H\cap(V\times Q)\longrightarrow\tilde{\mathbb{S}}$, given by $\Theta(x,y)=\big(y,\bar\theta(x,y)\big)\in\tilde{\mathbb{S}}\subset\mathbb{S}^{\bar\Delta}\times\mathbb{S}^1$ for $(x,y)\in\operatorname{spt}H\cap(V\times Q)$.

The natural orientation on $\mathbf{det}$ induces a natural orientation on the circle bundle $\widetilde{\mathbf{det}}$, opposite to the orientation of $\widetilde{\mathbf{det}}$ as the boundary of $\tilde{Q}$:

$$\partial[\![\tilde{Q}]\!] = -[\![\widetilde{\mathbf{det}}]\!]|\tilde{Q}. \tag{5.3.4}$$

Therefore

$$\begin{aligned} d_2[\![\widetilde{\mathbf{det}}]\!] &= -d_2\partial[\![\tilde{Q}]\!] \\ &= -\partial\Theta_*(H|V\times Q) \\ &= -\Theta_*(\partial H|V\times Q) \\ &= d_Z^X\Theta_*[\![\nu(Z_Y\cap\mathbb{S}^{\bar\Delta})\cap(V\times Q)]\!] \end{aligned}$$

by (5.3.2). Hence

$$d_2 = d_Z^X \deg(\Theta),$$

where we regard Θ as a map from $\nu(Z_Y\cap\mathbb{S}^{\bar\Delta})$ to $\widetilde{\mathbf{det}}$.

To evaluate this degree, let $\mathbb{P}\mathbf{det}\subset\mathbb{P}^{\bar\Delta}$ denote the projective variety corresponding to $\mathbf{det}$, so $\mathbf{det}$ is a principal $\mathbb{S}^1$ bundle over $\mathbb{P}\mathbf{det}$. Thus $\widetilde{\mathbf{det}}$ is an $\mathbb{S}^1\times\mathbb{S}^1$ bundle over $\mathbb{P}\mathbf{det}$. Similarly we may regard $\nu(Z_Y\cap\mathbb{S}^{\bar\Delta})$ as an $\mathbb{S}^1\times\mathbb{S}^1$ bundle over the projectivized normal bundle $\mathbb{P}\nu\mathbb{P}Z_Y\subset\mathbb{P}^{\bar\Delta}\times\mathbb{P}^{\bar\Delta}$ over $\mathbb{P}Z_Y$, and Θ covers the natural projection π_2:

$$\begin{array}{ccc} \nu(Z_Y\cap\mathbb{S}^{\bar\Delta}) & \xrightarrow{\Theta} & \widetilde{\mathbf{det}} \\ \downarrow & & \downarrow \\ \mathbb{P}\nu\mathbb{P}Z_Y & \xrightarrow{\pi_2} & \mathbb{P}\mathbf{det} \end{array}$$

The degree of the map on the bottom may be evaluated by factoring through the dual projective space $(\mathbb{P}^{\bar\Delta})^*$:

$$\begin{array}{ccc} \mathbb{P}^{\bar\Delta}\times(\mathbb{P}^{\bar\Delta})^*\supset\mathbb{P}\nu^*\mathbb{P}Z_Y & \xrightarrow{\pi_2} & \mathbb{P}\mathbf{det}^*\subset(\mathbb{P}^{\bar\Delta})^* \\ \downarrow & & \downarrow \\ \mathbb{P}^{\bar\Delta}\times\mathbb{P}^{\bar\Delta}\supset\mathbb{P}\nu\mathbb{P}Z_Y & \xrightarrow{\pi_2} & \mathbb{P}\mathbf{det}\subset\mathbb{P}^{\bar\Delta} \end{array}$$

where $\mathbb{P}\mathbf{det}^*$ is the subvariety corresponding to $\mathbb{P}\mathbf{det}$ under the antiholomorphic isomorphism $\mathbb{P}^{\bar\Delta}\longrightarrow(\mathbb{P}^{\bar\Delta})^*$ induced by the Hermitian metric. By Lemma 5.2A, the top line is a birational equivalence of algebraic varieties, hence has degree +1, while the two vertical maps have degrees $(-1)^{\operatorname{codim}\mathbb{P}Z_Y-1}=(-1)^{\#(\bar\Delta)-\dim\mathbb{P}Z_Y}$ and $(-1)^{\dim\mathbb{P}\mathbf{det}}=(-1)^{\#(\bar\Delta)-2}$ respectively. Therefore the degree of the bottom map is $(-1)^{\dim\mathbb{P}Z_Y}$.

We complete the computation by evaluating the degree of the induced map on the fiber $\mathbb{S}^1\times\mathbb{S}^1$. The orientation of $\mathbb{S}^1\times\mathbb{S}^1$ induced by the canonical orientations of $\widetilde{\mathbf{det}}$ and $\mathbb{P}\mathbf{det}$ is such that the first factor comprises the fiber of the Hopf fibration $\mathbf{det}\longrightarrow\mathbb{P}\mathbf{det}$ and the second factor the fiber of the blowup $\widetilde{\mathbf{det}}\longrightarrow\mathbf{det}$. The orientation of $\mathbb{S}^1\times\mathbb{S}^1$ induced by the canonical orientation of $\mathbb{P}\nu\mathbb{P}Z_Y$ is given in a similar way, the first factor comprising the fiber of the Hopf fibration $Z_Y\longrightarrow\mathbb{P}Z_Y$ and the second factor giving the fibration of the normal bundle. Now if $(x,y)\in\nu Z_Y$ and $(z,w)\in\mathbb{S}^1\times\mathbb{S}^1$, with $\Theta(x,y)=$

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$\big(y, \overline{\theta}(x, y)\big) \in \widetilde{\mathbf{det}} \subset \mathbf{det} \times \mathbb{S}^1$, then using 5.3A (4) we obtain

$$\begin{aligned}\Theta\big((z, w) \cdot (x, y)\big) &= \Theta(zx, wy)\\ &= \big(wy, \overline{\theta}(zx, wy)\big)\\ &= \big(wy, \theta(zx^{\gamma} + wy)\big)\\ &= \big(wy, zw^{n-1}\theta(x^{\gamma} + y)\big)\\ &= (w, zw^{n-1}) \cdot \Theta(x, y).\end{aligned}$$

Therefore the degree of the induced map on the fibers is -1, so the total degree is

$$\begin{aligned}(-1)^{\dim \mathbb{P}Z_Y + 1} &= (-1)^{\dim Z_Y}\\ &= \begin{cases} -1 & \text{in types I and III}\\ (-1)^n & \text{in type II}\end{cases}\end{aligned}$$

(by (5.2.3) and (5.2.4)) as required. ∎

5.4. *Proof of Lemma 5.3A.* To complete the proof we establish the existence of the deformation ψ of the lemma. Given $x \in \mathbb{C}^{\bar{\Delta}} \subset \mathbb{C}^{M_0} \simeq \mathbb{C}^{n\times n}$, let

$$V_\varepsilon(x) := \begin{cases} (x^{n\cdot})^{\perp}, & \varepsilon = 0 \text{ or } +\\ (x^{\cdot r})^{\perp} \cap (x^{\cdot n})^{\perp}, & \varepsilon = - \end{cases} \quad \text{and} \quad U_0(x) := (x^{\cdot n})^{\perp}.$$

Given vectors $v, w \in \mathbb{C}^n$, let $v \otimes w$ denote the $n \times n$ matrix $[v^i w^j]_{ij}$. Thus

$$(v \otimes w) \bullet \bar{u} = \langle w, u\rangle v.$$

Put $v \odot w := \frac{1}{2}(v \otimes w + w \otimes v)$ and $v \oslash w := (v \otimes w - w \otimes v)$. It is clear that if $x \in \mathbb{C}^{M_0}_\varepsilon$ is sufficiently close to $\mathbb{S}^\gamma_\varepsilon$ then $\nabla g(x)$ (or indeed any element of $\mathbb{C}^{M_0}$) may be expressed uniquely as

$$\text{(5.4.1)} \quad \nabla g(x) = \begin{cases} c(x)(x^{nn})^{-1}x^{\cdot n} \otimes x^{n\cdot} + v(x) \otimes x^{n\cdot} + x^{\cdot n} \otimes u(x) + R(x) & \text{in type I}\\ c(x)(x^{nn})^{-1}x^{\cdot n} \odot x^{n\cdot} + v(x) \odot x^{n\cdot} + R(x) & \text{in type II}\\ c(x)(x^{rn})^{-1}x^{\cdot n} \oslash x^{r\cdot} + v(x) \oslash x^{n\cdot} + R(x) & \text{in type III}\end{cases}$$

where $c(x) \in \mathbb{C}$, $v(x) \in V_\varepsilon(x)$, $u(x) \in U_0(x)$, and

$$R(x) \in \begin{cases} V_0(x) \otimes U_0(x) & \text{in type I}\\ V_+(x) \odot V_+(x) & \text{in type II}\\ U_0(x) \oslash U_0(x) & \text{in type III.}\end{cases}$$

STEP 1.

$$\text{(5.4.2)} \qquad \lim_{x\to\mathbb{S}^\gamma_\varepsilon} c(x) g(x)^{-1} = 2,$$

$$\text{(5.4.3)} \qquad \lim_{x\to\mathbb{S}^\gamma_\varepsilon} g(x)^{-1} v(x) = \lim_{x\to\mathbb{S}^\gamma_\varepsilon} g(x)^{-1} u(x) = 0.$$

PROOF OF STEP 1. We abbreviate $v(x)$ as v, etc. Since $x \longrightarrow \mathbb{S}^\gamma$, the decomposition (5.4.1) gives

$$\nabla g(x) \bullet \overline{x^{n\cdot}} = c\mathbf{e}_n + o(c) + |x^{n\cdot}|^2 v,$$

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where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{C}^n$, $|x^{n\cdot}| \longrightarrow 1$ and $\langle v, \mathbf{e}_n \rangle \longrightarrow 0$. In type I, there is a similar formula involving u arising from left multiplication. It is enough therefore to prove that

$$g(x)^{-1}\nabla g(x) \bullet \overline{x^{n\cdot}} \longrightarrow 2\mathbf{e}_n \tag{5.4.4}$$

(in type I there is a symmetric argument for the estimate on u).

We may think of each term $|\bigwedge^{\rho(\mu)+1} x^\mu|^2$, $\mu \in \boldsymbol{\Omega}$, as a sum of squares of wedge products $\delta(x)$ of partial columns of x. In type III, note that by symmetry we may write

$$g(x) = \left|\bigwedge^{\rho(\mu_0)+1} x^{\mu_0}\right|^2 + 2 \sum_{\mu \in \Omega_1} \left|\bigwedge^{\rho(\mu)+1} x^\mu\right|^2, \tag{5.4.5}$$

where μ_0 is the unique square in Ω and $\Omega_1 \subset \Omega$ consists of all rectangles with lower left corner below the diagonal. The indecomposability hypothesis implies that $(n, r) \in \mu$ for every $\mu \in \boldsymbol{\Omega}_1 \cup \{\mu_0\}$. Thus in each type the n-th column of each x^μ entering in the sum is bounded away from zero as $x \longrightarrow \mathbb{S}^\gamma$.

Write $g = g_n + g_{\bar{n}}$, where g_n is the sum of all such wedge products involving the n-th column. We claim that

$$g_{\bar{n}}(x) g_n(x)^{-1} \longrightarrow 0. \tag{5.4.6}$$

For, given a nonzero wedge product $\delta(x)$ occurring in $g_{\bar{n}}(x)$, by replacing each factor in turn by the (partial) n-th column, we obtain $\rho(\mu) + 1$ wedge products $\delta_i(x)$ occurring in $g_n(x)$. Since $x^{jk} \longrightarrow 0$ for $(j, k) \notin \gamma_\varepsilon$ and $x^\gamma \not\longrightarrow 0$, the last paragraph implies that there is $i(x) \in \{1, \ldots, \rho(\mu) + 1\}$ such that $\delta(x)\delta_{i(x)}(x)^{-1} \longrightarrow 0$. This implies (5.4.6).

Recalling the classification 3.2A(2), let $\kappa_j \in T_e B_2$ denote the infinitesimal isotropy element corresponding to the operation of moving the n-th column to the j-th, $j = 1, \ldots, n$. Then for each $z \in \mathbb{C}$ and $x \in \mathbb{C}^\Delta$ the vector of directional derivatives of g in the directions $z\kappa_j(x)$ may be obtained by matrix multiplication of the matrix $\nabla g(x) \in \mathbb{C}^\Delta \subset \mathbb{C}^{M_0}$ with the complex conjugate of the column vector $zx^{n\cdot}$:

$$\left(D_{z\kappa_1(x)} g(x), \ldots, D_{z\kappa_n(x)} g(x)\right) = \mathrm{Re}\left[\bar{z}\left(\nabla g(x)\right) \bullet \overline{x^{n\cdot}}\right].$$

For $j < n$, an argument similar to the Proof of Lemma 3.3A shows that there is a constant C such that

$$|D_{z\kappa_j(x)} g(x)| \leq C|z| g_n(x)^{\frac{1}{2}} g_{\bar{n}}(x)^{\frac{1}{2}},$$

while

$$D_{\kappa_n(x)} g(x) = 2g_n(x),$$
$$D_{\sqrt{-1}\kappa_n(x)} g(x) = 0.$$

Thus (5.4.6) implies (5.4.4). ∎

REMARK. From these estimates and the Łojasiewicz inequality it follows that as $x \longrightarrow \mathbb{S}^\gamma$ most of $\nabla g(x)$ is contained in $R(x)$, with a secondary contribution from the first term of (5.4.1). In particular, if $\phi(x)$ converges to some point $y_0 \in \mathbf{det}^\circ$ then rank $R(x) = n-1$ eventually, and det $\phi(x)$ is close to the determinant of the sum of the first and last terms. The first estimate above implies that the first term is almost a positive multiple of x^γ.

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Therefore, in order to achieve the deformation ψ of the lemma, and in particular the key relation (5.3.1), we would like to suppress the middle terms in (5.4.1). However we cannot do this directly since in general the sum of the remaining terms belongs only to the full matrix space $\mathbb{C}^{M_0}$: we cannot ensure that the $M_0 - \bar{\Delta}$ components are all zero. This problem is addressed in Step 2.

STEP 2. Let $\lambda = \lambda_\varepsilon$ denote the nearest point projection onto $\mathbf{det}_\varepsilon$, well-defined on a neighborhood of $\mathbf{det}^\circ_\varepsilon$ in $\mathbb{C}^{\bar{\Delta}}_\varepsilon$. Suppose $x_1, x_2, \cdots \longrightarrow x_0 \in \mathbb{S}^\gamma$ and $\phi(x_i) = |\nabla g(x_i)|^{-1}\nabla g(x_i) \longrightarrow y_0 \in (\mathbb{C}^*)^{\Gamma'}_\varepsilon$. Then

$$\Big|\lambda\big(\phi(x_i)\big) - |R(x_i)|^{-1}R(x_i)\Big| = o\big(g(x_i)|\nabla g(x_i)|^{-1}\big)$$

as $i \longrightarrow \infty$.

PROOF OF STEP 2. We estimate

$$\begin{aligned}\Big|\lambda\big(\phi(x_i)\big) - |R(x_i)|^{-1}R(x_i)\Big| &\le \Big|\lambda\big(\phi(x_i)\big) - |\nabla g(x_i)|^{-1}R(x_i)\Big| \\ &\quad + |R(x_i)|\Big||\nabla g(x_i)|^{-1} - |R(x_i)|^{-1}\Big|.\end{aligned}$$

The second term is $o\big(g(x_i)|\nabla g(x_i)|^{-1}\big)$ by Step 1, since the first and last terms of (5.4.1) are orthogonal. The first term is dominated by

$$(5.4.7)\qquad \Big|\lambda\big(\phi(x_i)\big)^\gamma - |\nabla g(x_i)|^{-1}R(x_i)^\gamma\Big| + \Big|\lambda\big(\phi(x_i)\big)^{M_0-\gamma} - |\nabla g(x_i)|^{-1}R(x_i)^{M_0-\gamma}\Big|.$$

Obviously each $R(x_i)$ belongs to the full determinant variety $\det^{-1}(0) \subset \mathbb{C}^{M_0}_\varepsilon$. Therefore the first term of (5.4.7) is small compared to the second, since $\phi(x_i), R(x_i) \in \det^{-1}(0)$ are close to y_0 and $T_{y_0}\det^{-1}(0) = \mathbb{C}^{M_0-\gamma}$.

On the other hand the second term of (5.4.7) is dominated by

$$\begin{aligned}\left|\Big(\lambda\big(\phi(x_i)\big) - \phi(x_i)\Big)^{M_0-\gamma}\right| &+ \left|\Big(|\nabla g(x_i)|^{-1}R(x_i) - \phi(x_i)\Big)^{M_0-\gamma}\right| \\ &= o\Big(\big|\lambda\big(\phi(x_i)\big) - \phi(x_i)\big|\Big) + o\big(g(x_i)|\nabla g(x_i)|^{-1}\big),\end{aligned}$$

where the estimate on the second term follows from Step 1. To estimate the first term here, note that $\nu_{y_0}\mathbf{det} = \nu_{y_0}\det^{-1}(0) = \mathbb{C}^\gamma_\varepsilon$, so $\det^{-1}(0)$ meets $\mathbb{C}^{\bar{\Delta}}$ orthogonally at y_0. Therefore $\operatorname{dist}(y, \mathbf{det}) = O\Big(\operatorname{dist}\big(y, \det^{-1}(0)\big)\Big)$ for $y \in \mathbb{C}^{\bar{\Delta}}$ near y_0 and we may estimate

$$\begin{aligned}\Big|\lambda\big(\phi(x_i)\big) - \phi(x_i)\Big| &= \operatorname{dist}\big(\phi(x_i), \mathbf{det}\big) \\ &= O\Big(\operatorname{dist}\big(\phi(x), \det^{-1}(0)\big)\Big) \\ &\le O\Big(\big||\nabla g(x_i)|^{-1}R(x_i) - \phi(x_i)\big|\Big) \\ &= O\big(|\nabla g(x_i)|^{-1}g(x_i)\big).\end{aligned}$$

∎

We now construct ψ. We extend the function $|\nabla g(x)|^{-1}g(x)$ to all of $\mathbb{S}^{\bar{\Delta}}$ by setting it equal to 0 for $x \in X_Y$; so the Łojasiewicz inequality implies that this extension is a continuous semialgebraic function.

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STEP 3. Fix $y_0 \in (\mathbb{C}^*)^{\Gamma'}_\varepsilon \cap \mathbb{S}^{\bar{\Delta}}$ and an open set $Q_0 \subset \mathbb{S}^{\bar{\Delta}}$ such that the projection λ is well-defined on Q_0. Define $m\colon [0,1] \times [0,1] \times \mathbb{S}^{\bar{\Delta}} \times Q_0 \longrightarrow \mathbb{C}^{\bar{\Delta}}$ by

$$\begin{aligned} m(t,h,x,y) = (1-h)\big(y + t|\nabla g(x)|^{-1} g(x) x^\omega_\varepsilon\big) \\ + h\Big[(1-t)y + t\Big(|\nabla g(x)|^{-1} g(x) x^\gamma + \lambda(y)\Big)\Big]. \end{aligned}$$

Let there be given open sets $V, Q, V', Q' \subset \mathbb{S}^{\bar{\Delta}}$ such that $\mathbb{S}^\gamma_\varepsilon \subset V \subset\subset V'$, $y_0 \in Q \subset\subset Q' \subset Q_0$. Let $h\colon \mathbb{S}^{\bar{\Delta}}_\varepsilon \times \mathbb{S}^{\bar{\Delta}}_\varepsilon \longrightarrow [0,1]$ be a continuous semialgebraic function such that $h \equiv 1$ on $V \times Q$ and $\equiv 0$ outside $V' \times Q'$, and put

$$\Psi(t,x,y) := \begin{cases} u\big(y + t|\nabla g(x)|^{-1} g(x) x^\omega_\varepsilon\big) & \text{for } (x,y) \notin V' \times Q' \\ u\Big(m\big(t, h(x,y), x, y\big)\Big) & \text{for } (x,y) \in V' \times Q', \end{cases}$$

where $x^\omega_\varepsilon := \pi_{\mathbb{C}^{\bar{\Delta}}_\varepsilon}(x^{\omega_0}) = \begin{cases} x^{\omega_0} & \text{if } \varepsilon = 0 \\ (x^{\omega_0} + x^{\omega_0^t})/2 & \text{if } \varepsilon = \pm \end{cases}$ and $u(x) := |x|^{-1}x$ is the normalization map. Then Ψ is well-defined for $t \in [0,1]$ and $(x,y) \in \overline{\operatorname{graph}\phi}$, and satisfies the conditions (1)–(4) of the lemma.

PROOF OF STEP 3. We prove first that if V', Q' are small enough then

$$\det m\big(t,h,x,\phi(x)\big) \neq 0, \quad 0 \leq h,t \leq 1 \tag{5.4.8}$$

whenever $x \in V'$ and $\phi(x) \in Q'$. To this end we expand $m\big(t,h,x,\phi(x)\big)$ as in (5.4.1). Let us denote $c(x)^{-1}$ times the first term of (5.4.1) by $x^* = x^*_\varepsilon$. Observe that as $x \longrightarrow \mathbb{S}^\gamma$, both $x^\gamma - x^*$ and $x^\omega - x^* \longrightarrow 0$. Now a straightforward computation using Steps 1 and 2 gives

$$m\big(t,h,x,\phi(x)\big) = A(x)|\nabla g(x)|^{-1} g(x) x^* + B(x)|R(x)|^{-1} R(x) + o\big(g(x)|\nabla g(x)|^{-1}\big)$$

as $\big(x,\phi(x)\big)$ approaches $\mathbb{S}^\gamma \times \{y_0\}$, where A, $B \geq 1$ and the bound on the last term is independent of t and h. Therefore if V', Q' are chosen small enough then the determinant (respectively the Pfaffian, in type III) of the sum of the first two terms is clearly of the order $g(x)|\nabla g(x)|^{-1}$, and as all terms are bounded it follows that $\det m\big(t,h,x,\phi(x)\big)$ (or $\operatorname{Pf} m\big(t,h,x,\phi(x)\big)$) is of the same order. In particular (5.4.8) holds.

Thus for V', Q' small enough it is clear that $\Psi\big(t,x,\phi(x)\big)$ is well-defined for $\big(x,\phi(x)\big) \in V' \times Q'$. On the other hand, if $\big(x,\phi(x)\big) \notin V' \times Q'$ then the first expression for Ψ applies. But

$$\begin{aligned} \operatorname{Re}\langle \phi(x), x^\omega_\varepsilon \rangle &= |\nabla g(x)|^{-1} D_{x^{\omega_0}} g(x) \\ &= |\nabla g(x)|^{-1} \frac{d}{dt}\Big|_{t=0} g(x + t x^{\omega_0}) \\ &\geq 0, \end{aligned} \tag{5.4.9}$$

as in Section 5.1. In particular $\phi(x) + t x^\omega_\varepsilon \neq 0$ for $t \neq 0$ so Ψ is well-defined here too.

Now 5.3A(1) is immediate. To prove (2) we only need to check the case $(x,y) \in (V' \cap X_Y) \times Q$. Take Q to lie within the open subset U of Remark 4.1A. By Lemma 3.3A,

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$(x, y) \in (V' \cap X_Y) \times Q$ only if $y \in \mathbf{det}$. Thus $\lambda(y) = y$, and it is now straightforward to check (2).

To prove (3) we observe that since $\mathbb{S}^{\gamma} \cap \mathbb{S}^{\bar{\Delta}'} = \emptyset$, we may choose V' so that $V' \cap \mathbb{S}^{\bar{\Delta}'} = \emptyset$. Since $\mathbb{S}^{\bar{\Delta}'} \subset \mathbf{det}$ the relation (5.4.8) implies that if $\big(x, \phi(x)\big) \in V' \times Q'$ then $\psi(x) \notin \mathbb{S}^{\bar{\Delta}'}$. Now (3) follows from (5.4.9).

The first assertion of (4) is contained in (5.4.8). To prove the remaining assertion (5.3.1) we note that if $\big(x, \phi(x)\big) \in V \times Q$ then $\psi(x) = |\nabla g(x)|^{-1} g(x) x^{\gamma} + \lambda\big(\phi(x)\big)$. If Q is small enough then $\lambda\big(\phi(x)\big) \in \mathbf{det}^{\circ}$; *i.e.*, rank $\lambda\big(\phi(x)\big) = n - 1$ in types I and II and $n - 2$ in type III. Since rank $x^{\gamma} = n - \operatorname{rank} \lambda\big(\phi(x)\big)$, (5.3.1) is clear. ∎

This completes the proof of Lemma 5.3A and hence Theorem 4.2E.

6. **Inversion: Euler Obstruction Algorithms.** For most of this section we continue to assume that $Y \subset X$ are Schubert varieties in a Hermitian symmetric space of type I, II, or III. We will define "Euler obstruction polynomials" $E_Y^X(q)$ such that $E_Y^X(1) = e_Y^X$ (the Euler obstruction numbers of (2.1.4)), and also satisfying the inversion relation

$$\sum_{Y \subset Z \subset X} E_Y^Z(q) D_Z^X(q) \equiv 1 \tag{6.0.1}$$

analogous to (2.1.4). In the final subsection, we treat types IV and V.

For this section, we revert to cartesian coordinates based on the original rectangle M, rather than on $M_0 = \operatorname{span} \bar{\Delta}(Y)$.

6.1. *The E polynomials.* Given Schubert varieties $Y \subset X$, we define a new tree-with-capacities $\tilde{T}_Y^X$. Let $\Gamma_0(X)$ be the dot configuration as defined in Section 4.1, but constructed using $\bar{\Delta}(X)$ instead of $\bar{\Delta}(Y)$. The tree $\tilde{T}$ is then constructed from $\Gamma_0(X)$ as in Section 4.2. That is, $\tilde{T}$ is the Hasse diagram of the full poset of dots in type I; in types II and III it is the Hasse diagram of the subposet of dots lying on or above the diagonal, with diagonal dots corresponding to vertical "central" edges and above-diagonal dots corresponding to oblique "side" edges. In type III we make the following additional convention: if (i_1, j_1), $(i_2, j_2) \in \Gamma_0(X)$ correspond to edges e_1 and e_2 having the same parent in the tree, and if $i_1 < i_2$, then e_1 must be positioned to the left of e_2 in the tree. In this case we say that e_2 is a *little brother* of e_1. (See Figure 6.1, in which e_2 is a little brother of both e_8 and e_9, and e_8 is a little brother of e_9, but e_6 and e_7 are only "cousins", etc.)

We next define a capacity for each minimal vertex of the tree. We refer to the minimal elements of $\bar{\Delta}(X)$ as *indentations* of $\Delta(X)$. Thus each indentation of $\Delta(X)$ corresponds to a minimal edge $(a, b) \in \Gamma_0(X)$. Denote the lower vertex of this edge by v. Let $(a-c, b-c) \in \bar{\Delta}(Y)$ be the corresponding box adjacent to the boundary of $\Delta(Y)$. Then the capacity of v is defined to be c. (More generally, we define the capacity of any vertex of the tree to be the smallest of the capacities of the minimal vertices below it.) The tree endowed with these capacities is denoted $\tilde{T}_Y^X$.

Now consider the set $\tilde{\Lambda} = \tilde{\Lambda}(\tilde{T}_Y^X)$ of all *labelings* $\tilde{\lambda}\colon \{\text{edges}\, e \text{ of } \tilde{T}_Y^X\} \longrightarrow \mathbb{Z}_{\geq 0}$ subject to these conditions:

(1) The labels are non-increasing from bottom to top of the tree;

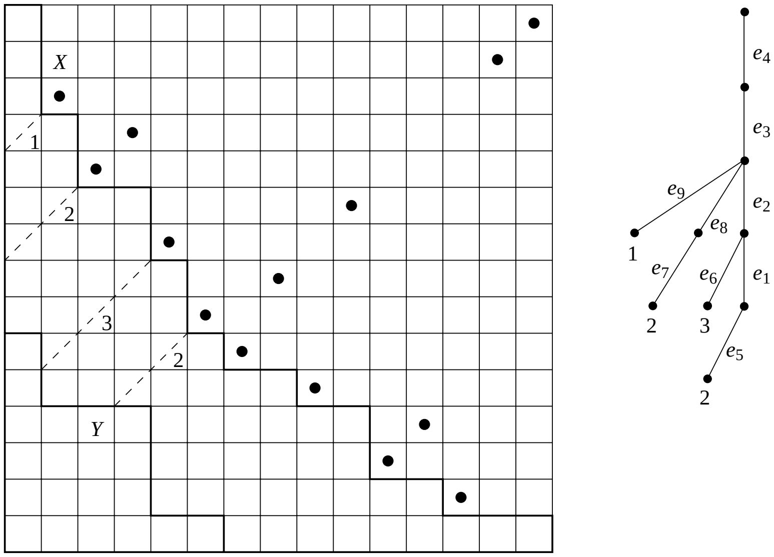

FIGURE 6.1. T_Y^X for types II and III.

(2) If v is a vertex with capacity c, then the label on any edge above v must be $\leq c$;

(3) In type III, the labels on each pair of central edges must be equal and even; if there is an odd number of central edges, the label on the top one must always be 0;

(6.1.1)

(4) In type III, if

  (a) e is a side edge,

  (b) e is maximal among the set of side edges of the tree, and

  (c) the number of central edges below the top vertex of e is even (possibly 0), then $\tilde{\lambda}(e)$ must be either

  (i) even or

  (ii) greater than $\tilde{\lambda}(e')$ for some little brother e' of e.

(In particular, if e satisfies (a), (b) and (c), but e has no little brother—there can be at most one such edge in the entire tree—then $\tilde{\lambda}(e)$ must be even.)

(In the example of Figure 6.1, rule (4) applies to edges e_8, e_9, and "in particular" to e_5; but not to e_6.)

The *sign* $\sigma(\tilde{\lambda})$ is defined by

(1) In types I and III, $\sigma(\tilde{\lambda}) = 1$;

(2) In type II, $\sigma(\tilde{\lambda}) = (-1)^b$, where $b = \sum\limits_{\substack{\text{central}\\ \text{edges } e}} \tilde{\lambda}(e)$.

The *weight* $|\tilde{\lambda}|$ is defined by

(1) In types I and II, $|\tilde{\lambda}| = \sum\limits_{e} \tilde{\lambda}(e)$;

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(2) In type III, $\quad |\tilde{\lambda}| = \sum\limits_{\substack{\text{side}\\ \text{edges } e}} \tilde{\lambda}(e) + \sum\limits_{\substack{\text{even central}\\ \text{edges } e}} \tilde{\lambda}(e).$

(As before if $\widetilde{T}_Y^X$ is the empty tree, then it has only the "empty" labeling $\tilde{\lambda}$ with $\sigma(\tilde{\lambda}) = 1$ and $|\tilde{\lambda}| = 0$.) Finally, we define

$$E_Y^X(q) = \sum_{\tilde{\lambda} \in \tilde{\Lambda}} \sigma(\tilde{\lambda}) q^{|\tilde{\lambda}|}. \tag{6.1.2}$$

When $Y \not\subset X$ we set $E_Y^X(q) \equiv 0$.

EXAMPLE 6.1A. Suppose the tree $\widetilde{T}_Y^X$ is the same as in Figure 4.3. Below we give all possible labelings $\tilde{\lambda}$ of this tree which satisfy conditions (6.1.1(1), (2)). Beneath each labeling $\tilde{\lambda}$ we write the associated monomial $\sigma(\tilde{\lambda}) q^{|\tilde{\lambda}|}$ in of each types I, II, or III; in type III the notation "NA" means that the labeling is not allowed according to conditions (6.1.1(3), (4)).

	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1
	0 0	0 0	1 0	1 0	0 1	1 1	1 1	0 0	1 0	0 1	1 1	1 1	0 2	1 2	1 2
	0	1	0	1	1	1	1	2	2	2	2	2	2	2	2
I:	1	q	q	q^2	q^2	q^3	q^4	q^2	q^3	q^3	q^4	q^5	q^4	q^5	q^6
II:	1	$-q$	q	$-q^2$	q^2	q^3	$-q^4$	q^2	q^3	$-q^3$	$-q^4$	q^5	q^4	q^5	$-q^6$
III:	1	NA	q	NA	NA	NA	NA	NA	NA	NA	NA	NA	q^2	NA	NA

Thus, according to (6.1.2), we obtain

$$E_Y^X = \begin{cases} 1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6 & \text{in type I} \\ 1 + q^2 + q^3 - q^4 + 2q^5 - q^6 & \text{in type II} \\ 1 + q + q^2 & \text{in type III.} \end{cases}$$

REMARK 6.1B. We note that, in types I and III, the algorithm for E_Y^X is equivalent to the algorithm for the Kazhdan-Lusztig polynomials $P_Y^X = Q_Y^X$ given in [LS] and [Boe, (3.10) and (4.1)], respectively. In type I, the two algorithms are literally identical. In type III, some of our present conventions are different than they were in [Boe], but these differences are only cosmetic; the two algorithms do produce the same polynomials. In other words, $E_Y^X = P_Y^X$ in types I and III.

In type II, however, the Euler obstruction polynomials are different from the Kazhdan-Lusztig polynomials. The simplest example occurs already in Λ_2, with $X = [1, 2]$, $Y = [0, 0]$ (notation from Section 3.1). The reader might find it instructive to check that $E_Y^X = 1 - q$ while $P_Y^X = 1$ (so $E_Y^X(1) = 0$, while $P_Y^X(1) = 1$).

6.2. *The fundamental inversion theorem for types I, II, and III.*

THEOREM 6.2A. *Fix Schubert varieties $Y \subset X$ in a compact Hermitian symmetric space of type I, II, or III. Define polynomials D_Y^X as in (4.2.3), and E_Y^X as in (6.1.2). Then*

$$\sum_{Y \subset Z \subset X} E_Y^Z(q) D_Z^X(q) \equiv 1.$$

Before beginning the proof, we need some further notation and two lemmas. Set $N = n + m$ in type I, $N = n$ in types II and III. Following [LS] and [Boe], we associate to any Young diagram $\Delta(Z) \subset M$ a word z of length N in two symbols α and β. Beginning at the upper left corner $(1, n)$, follow the boundary of $\Delta(Z)$, and write α for each length one vertical segment on the boundary and β for each length one horizontal segment. In type I, continue to the lower right corner $(m, 1)$; in types II and III, stop when the boundary reaches the diagonal. For example, the word associated to the Young diagram of Figure 3.1 is $\alpha\beta\beta\alpha\beta\beta\alpha\alpha\beta\alpha\beta$. (Similarly, the word associated to any Young diagram $\Delta(X)$, $\Delta(Y), \ldots$ will be denoted by the corresponding lower case letter $x, y, \ldots$. We will henceforth freely interchange upper and lower case letters without further comment.) We may view z as representing an element of the *Weyl group* of G (*cf.* [LS], [Boe]). Let $\ell(z)$ be the *length* of z, so $\ell(z) = \dim Z$. In particular, if $s = s_i$ is a "simple reflection" ($1 \le i \le N-1$ in type I, $1 \le i \le N$ in types II and III), s acts on z on the right. Then zs is obtained from z as follows: for $1 \le i \le N-1$, interchange the i-th and $(i+1)$-st symbols; for $i = N$ in type II, "reverse" the last symbol (where β is the reverse of α and vice versa); for $i = N$ in type III, interchange the last 2 symbols, and then reverse each of them.

The words inherit a partial order $\le$ corresponding to the inclusion order of the associated varieties and their Young diagrams (and to the restriction of the *Bruhat order* on the Weyl group). In particular, the following ordering relations hold (for appropriate s as above):

$$\begin{gathered} z = z_1\alpha\beta z_2 < zs = z_1\beta\alpha z_2 \\ z = z_1\alpha < zs = z_1\beta \\ z = z_1\alpha\alpha < zs = z_1\beta\beta. \end{gathered} \tag{6.2.1}$$

It will be useful to have a description of the action of the simple reflections in terms of Young diagrams. Suppose $z < zs =: z'$, and let Z, Z' be the corresponding Schubert varieties. Then $\Delta(Z')$ is formed by adjoining to $\Delta(Z)$ an indentation (a, b) of $\Delta(Z)$. In types II and III, if the new box is above the diagonal, a corresponding box is also added below the diagonal to preserve the symmetry. The reflection $s = s_n$ in type III is a special case: here $a = b$ and a 2×2 block of boxes $[a, a+1] \times [a, a+1]$ is added (ensuring that $\Delta(Z')$ has an even number of boxes on the diagonal). In any case, (a, b) will be a minimal element of $\Gamma_0(Z)$. We call the corresponding minimal edge in the tree associated to Z the *edge corresponding to s*.

Thus it is easy to see that if $Y \subsetneq X$ are Schubert varieties in type I, II, or III, corresponding to words $y < x$, then there exists at least one simple reflection s such that $y < ys \le x$.

LEMMA 6.2B. *Let $y < x$ correspond to $Y \subset X$, and assume there is a simple reflection s such that $y < ys$ but $x \not< xs$. Then $D^x_y(q) \equiv 0$.*

PROOF. Exclude for the moment the case $s = s_n$ in type III. Then in terms of Young diagrams, the condition of the lemma is that there exists an indentation (a, b) of $\Delta(Y)$, such

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that the corresponding point $(a+c, b+c) \in \overline{\Delta}(X)$ adjacent to $\Delta(X)$ is *not* an indentation of $\Delta(X)$ (see Figure 6.2). Thus at least one of $(a+c,\, b+c-1)$ or $(a+c-1,\, b+c)$ belongs to $\overline{\Delta}(X)$. Assume the former, for definiteness. Let e be the minimal edge of T_Y^X corresponding to (a, b), and let v_0 and v_1 be the lower and upper vertices of e, respectively. Then v_0 has capacity c. The rectangle $\mu \in \Omega$ indicated in Figure 6.2 with $\rho(\mu) = c-1$ shows that the capacity of v_1 is at most $c-1$.

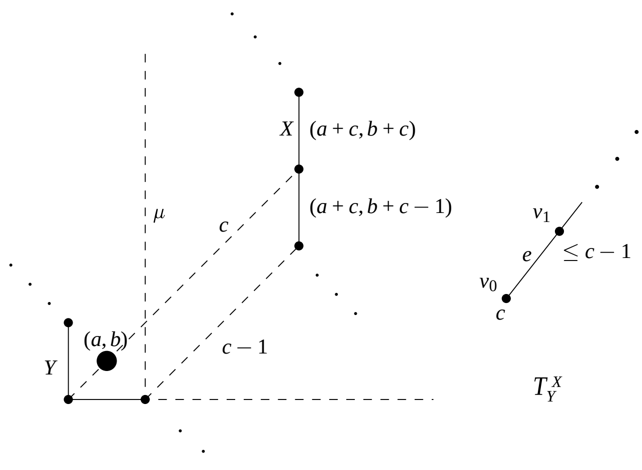

FIGURE 6.2. Portions of Y, X, and T_Y^X (generic case).

Therefore if $\lambda \in \Lambda$ is any labeling of T_Y^X with $\lambda(e) = +1$, there is a corresponding (allowed) labeling λ^-, identical to λ except that $\lambda^-(e) = -1$. Clearly $\sigma(\lambda^-) = -\sigma(\lambda)$ while $|\lambda^-| = |\lambda|$, so that the monomials in $D_y^x(q)$ coming from λ and λ^- cancel. Since all labelings pair up in this way, $D_y^x = 0$.

Now if $s = s_n$ in type III, the above argument is correct except when $y = y_1\alpha\alpha$ and $x = x_1\beta\alpha$. In this case both $\Delta(Y)$ and $\Delta(X)$ have indentations on the diagonal (see Figure 6.3). Recall from Section 3.1(3) that c must be even. Thus the tree has a minimal central edge e_1 whose parent is a central edge e_2, and e_1 is the only child of e_2. The capacity of the minimal central vertex is c, while the capacity of the upper vertex of e_2 is $\leq c-1$ (via the submatrix μ indicated in the figure). By Remark 4.2A(2), (4.2.2(3)) and the fact that $c-1$ is odd, there can be at most $c-2$ labels "-1" above e_2. Thus for any labeling λ with $\lambda(e_1) = \lambda(e_2) = +1$, there is a corresponding labeling λ^- with $\lambda^-(e_1) = \lambda^-(e_2) = -1$. Now the argument proceeds as before. ∎

LEMMA 6.2C. *Let $y < x$ correspond to $Y \subset X$, and assume that s is a simple reflection such that $x \not< xs$. Then $E_y^x = E_{ys}^x$.*

PROOF. In types I and III, $E_y^x = P_y^x$ (Remark 6.1B), so the lemma follows from [LS, (7.2.3)] and [Boe, (3.5)], respectively.

In type II, we may assume that $y < ys$. Then we have an indentation (a, b) of $\Delta(Y)$ such that the corresponding point $(a+c, b+c)$ of $\overline{\Delta}(X)$ is *not* an indentation of $\Delta(X)$. Changing y to ys simply adjoins the box (a, b) to $\Delta(Y)$, and this does not affect the capacity of any

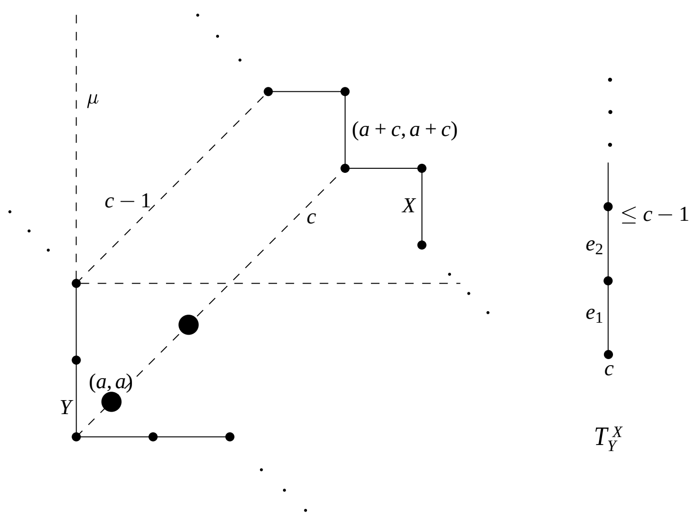

FIGURE 6.3. Portions of Y, X, and T_Y^X (type III, $s = s_n$).

minimal vertex of $\widetilde{T}_Y^X$: recall that to compute $E_\cdot^X$ we use the dot configuration $\Gamma_0(X)$, and only the capacities of the minimal vertices are relevant. Thus $E_y^x = E_{ys}^x$. ■

PROOF OF THEOREM 6.2A. Assume inductively that the result is true for all Schubert varieties in the Grassmannian $G_{n-1,m-1}$ (respectively, Λ_{n-1}, Λ_{n-1}^-), and that the result is true for all Y' such that $Y \underset{\neq}{\subset} Y' \subset X$. The initial cases $G_{n,0} \simeq G_{0,n}$, Λ_1, Λ_1^-, and $Y = X$ are easy.

Fix $y < x$ and a simple reflection s such that $y < ys \leq x$. Set

$$\mathbf{S}_y^x = \sum_{y \leq z \leq x} E_y^z(q) D_z^x(q). \tag{6.2.2}$$

The plan is to show that

$$\mathbf{S}_y^x - \mathbf{S}_{ys}^x = 0, \tag{6.2.3}$$

where $\mathbf{S}_{ys}^x = 1$ by induction.

Decompose the left side of (6.2.3) into several terms, the first and fourth of which are equal, by Lemma 6.2C:

$$\begin{aligned}
&\mathbf{S}_y^x - \mathbf{S}_{ys}^x \\
&\quad = \sum_{\substack{ys \leq z \leq x \\ z \not< zs}} E_y^z D_z^x + \sum_{\substack{ys \leq z \leq x \\ z < zs}} E_y^z D_z^x + \sum_{\substack{y \leq z \leq x \\ ys \not\leq z}} E_y^z D_z^x - \sum_{\substack{ys \leq z \leq x \\ z \not< zs}} E_{ys}^z D_z^x - \sum_{\substack{ys \leq z \leq x \\ z < zs}} E_{ys}^z D_z^x \\
&\quad = \sum_{\substack{ys \leq z \leq x \\ z < zs}} (E_y^z - E_{ys}^z) D_z^x + \sum_{\substack{y \leq z \leq x \\ ys \not\leq z}} E_y^z D_z^x.
\end{aligned}$$

Notice that $y \leq z$, $ys \not\leq z$ implies that $z < zs$; this is easy to see by considering the corresponding Young diagrams. This, together with the fact that $E_{ys}^z = 0$ when $ys \not\leq z$,

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means that we can combine the two terms. Therefore (6.2.3) is equivalent to proving:

$$\sum_{\substack{y \le z \le x \\ z < zs}} (E_y^z - E_{ys}^z) D_z^x = 0. \tag{6.2.4}$$

Now if $x \not< xs$ then $D_z^x = 0$ for $z < zs$ by Lemma 6.2B, and we are done. So it remains to prove (6.2.4) in the case $x < xs$. We digress to prove two more lemmas, which contain the important "recursion on rank" relations satisfied by the E and D polynomials.

LEMMA 6.2D. *Let $H = K\backslash G_0$ be a Hermitian symmetric space of type I, II, or III. Fix a simple reflection s, and suppose that $y < ys$, $z < zs$. Let $c = c(y,z)$ be the capacity of the minimal vertex of $\widetilde{T}_y^z$ corresponding to s. Then there are parameters $\overline{y}$, $\overline{z}$ for a Hermitian symmetric space $\overline{K}\backslash\overline{G}_0$ of the same type but lower rank, such that*

$$E_y^z - E_{ys}^z = (-1)^r q^c E_{\overline{y}}^{\overline{z}},$$

where

$$r = r(y,z) = \begin{cases} c, & \text{for } s = s_n \text{ in type II} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the map $z \longmapsto \overline{z}$ is an order-preserving bijection from the set of parameters z for $K\backslash G_0$ satisfying $z < zs$, onto the set of all parameters for $\overline{K}\backslash\overline{G}_0$.

PROOF. Given $z < zs$, define $\overline{z}$ as follows: if $s = s_i$, $i < N$, so that $z = z_1\alpha\beta z_2$, then $\overline{z} = z_1 z_2$; if $s = s_n$ in type II, so that $z = z_1\alpha$, then $\overline{z} = z_1$; and if $s = s_n$ in type III, so that $z = z_1\alpha\alpha$, then $\overline{z} = z_1$. (When $i < n$ in type III, the map $z \longmapsto \overline{z}$ takes the set of parameters having an *even* number of β's, to a set of parameters having an *odd* number of β's. But we may equally well use either set of parameters.) Now the validity of the last sentence of the lemma is clear.

In types I and III, since our E polynomials are the same as the Kazhdan-Lusztig polynomials (Remark 6.1B), the desired recursions follow from [LS, Lemma 6.6] and [Boe, Proposition 3.14]. (Note that, when $i = n-1$ or n in type III, $c = c(y,z)$ must be even.)

There remains the case of type II. Let e be the minimal edge of $\widetilde{T}_y^z$ corresponding to s. The labelings $\tilde{\lambda} \in \tilde{\Lambda}$ of $\widetilde{T}_y^z$ split up into two families $\tilde{\Lambda}^=$ and $\tilde{\Lambda}^<$, according to whether $\tilde{\lambda}(e) = c$ or $\tilde{\lambda}(e) < c$, and E_y^z is the sum of the two corresponding polynomials. The polynomial associated to $\tilde{\Lambda}^<$ is precisely E_{ys}^z. (When $c = 0$, $ys \not\leq z$, $E_{ys}^z = 0$, and there are no labelings in $\tilde{\Lambda}^<$.) Let's examine the polynomial associated to $\tilde{\Lambda}^=$. If $i < n$, this is q^c times the E polynomial associated to the tree obtained from $\widetilde{T}_y^z$ by removing the minimal (side!) edge e, and assigning the capacity of the new minimal vertex to be c. But this tree is $\widetilde{T}_{\overline{y}}^{\overline{z}}$. If $i = n$, then the above description of the polynomial associated to $\tilde{\Lambda}^=$ is still correct, except that we need to introduce an additional factor of $(-1)^c$, to account for the fact that e is now a *central* edge; *cf.* the definition of $\sigma(\tilde{\lambda})$ in Section 6.1. ∎

LEMMA 6.2E. *Let $H = K\backslash G_0$ be a Hermitian symmetric space of type I, II, or III. Fix a simple reflection s, and suppose that $z \le x$, $z < zs$, $x < xs$, and let $c = c(z,x)$ be the capacity of the minimal vertex of T_z^x corresponding to s. Then*

$$D_z^x = (-1)^r q^c (D_{\overline{z}}^{\overline{x}} - D_{\overline{z}}^{\overline{x'}})$$

where

$$r = r(z,x) = \begin{cases} c, & \text{for } s = s_n \text{ in type II} \\ 0, & \text{otherwise,} \end{cases}$$

x' is a parameter for $K\backslash G_0$ such that $z \leq x'$ (when $c > 0$; if $c = 0$ the term $D_{\bar{z}}^{\bar{x'}}$ is to be interpreted as 0), and $^-$ is defined in Lemma 6.2D.

PROOF. Let e be the minimal edge of T_z^x corresponding to s, and let v be the *upper* vertex of e. Note that e must be ordinary (in the language of Lemma 4.2C): if e is a minimal central edge in type III, then $s = s_n$ and $z = z_1\alpha\alpha$, which implies that the central edge immediately above e has no children other than e. According to the cited lemma, in computing D_z^x we may restrict our attention to the labelings $\lambda \in \Lambda_0$; thus every such labeling λ has $\lambda(e) = 1$ and exactly c edges e' above e with $\lambda(e') = -1$.

Assume first that e is not a central edge, and let T' be the tree obtained from T_z^x by removing e (and leaving all capacities unchanged). When $c > 0$, define T'' similarly except with the capacity of v reduced to $c-1$ (we ignore terms involving T'' if $c = 0$). There is an obvious bijection between Λ_0 and $\Lambda(T') - \Lambda(T'')$, which implies the identity $D_z^x = q^c\{D(T') - D(T'')\}$. Next assume e is a central edge in type II, and define T' and T'' as above. For $\lambda \in \Lambda_0$, let λ' be the restriction of λ to T'. If there are, say, k odd central edges with label -1 in λ, then there are $c-k$ odd central edges with label -1 in λ'. Hence $\sigma(\lambda') = \{(-1)^{c-k}/(-1)^k\}\sigma(\lambda) = (-1)^c\sigma(\lambda)$. Therefore $D_z^x = (-1)^c q^c\{D(T') - D(T'')\}$. Finally, assume e is a central edge in type III, and let e' be the central edge immediately above e. Denote by v' the upper vertex of e'. Let T' be the tree obtained by removing both e and e' from T_z^x. When $c > 0$ (*i.e.*, $c \geq 2$), define T'' similarly except with the capacity of v' reduced to $c-2$. It follows from the definitions in Section 4.2 that $D_z^x = q^c\{D(T') - D(T'')\}$.

It is clear in each case that $T' = T_{\bar{z}}^{\bar{x}}$. Thus it remains only to show, in the cases $c > 0$, that

$$D(T'') = D_{\bar{z}}^{\bar{x'}} \quad \text{for some parameter } x' \geq z. \tag{6.2.5}$$

First, assume that $s = s_i$, $i < N$, so that $x = x_1\alpha\beta x_2$ with $|x_1| = i-1$. In type II, if x_2 contains no α's (including the case where x_2 is empty), define x' by changing the last β of x_1 to α (and leaving all other symbols of x unchanged). In type III, if x_2 is a sequence of one or more β's, define x' by changing the last β of x_1 to α and the last β of x_2 to α; whereas if x_2 is empty, change the last *two* β's of x_1 to α's. In all other cases, we define x' by changing the last β of x_1 to α and the first α of x_2 to β.

Next, assume $s = s_N$. (Recall that this only occurs in types II and III, where $N = n$.) In type II we have $x = x_1\alpha$; define x' by changing the last β of x_1 to α. In type III, $x = x_1\alpha\alpha$; define x' by changing the last *two* β's of x_1 to α's. (Notice that in type III, the parities of the number of α's and of the number of β's are preserved. Also, the existence of x' relies in each case heavily on the fact that $c > 0$; *i.e.*, x is not minimal in the ordering.)

We leave to the reader the details of checking that (6.2.5) holds in each case. In so doing it should be kept in mind that there may be some vertex $v_1 \neq v$ with capacity $c + \ell$ in T'' and capacity $c + \ell - 1$ in $D_{\bar{z}}^{\bar{x'}}$. But in this case there will also be a vertex v_2 above both v and v_1, separated from v_1 by at most ℓ edges. Since at most ℓ labels -1 can be put on edges between v_1 and v, and at most $c-1$ labels -1 above v_2, decreasing the capacity of v_1 from $c + \ell$ to $c + \ell - 1$ does not change the set of allowed labelings. ∎

COMPLETION OF PROOF OF THEOREM 6.2A. Recall that we had reduced to showing (6.2.4), in the case where $x < xs$ and $y < ys \le x$. By Lemmas 6.2D and 6.2E, the left hand side of (6.2.4) is equal to

$$\begin{aligned}
\sum_{\substack{y \le z \le x \\ z < zs}} &(-1)^{r(y,z)} q^{c(y,z)} E_{\bar{y}}^{\bar{z}} \cdot (-1)^{r(z,x)} q^{c(z,x)} (D_{\bar{z}}^{\bar{x}} - D_{\bar{z}}^{\bar{x'}}) \\
&= (-1)^{r(y,x)} q^{c(y,x)} \Big\{ \sum_{\bar{y} \le \bar{z} \le \bar{x}} E_{\bar{y}}^{\bar{z}} D_{\bar{z}}^{\bar{x}} - \sum_{\bar{y} \le \bar{z} \le \bar{x'}} E_{\bar{y}}^{\bar{z}} D_{\bar{z}}^{\bar{x'}} \Big\} \\
&= (-1)^{r(y,x)} q^{c(y,x)} \{ \mathbf{S}_{\bar{y}}^{\bar{x}} - \mathbf{S}_{\bar{y}}^{\bar{x'}} \} \\
&= (-1)^{r(y,x)} q^{c(y,x)} \{ 1 - 1 \} \quad \text{by induction on rank} \\
&= 0,
\end{aligned}$$

as claimed. ∎

6.3. *Types IV and V.* Finally, we attend to the "easy" cases $\big(\mathrm{SO}(n) \times \mathrm{SO}(2)\big) \setminus \mathrm{SO}(n+2)$.

We first compute the MacPherson coefficients. We adopt the notation of Section 3.4; the cases of odd and even n admit a parallel treatment. From the given descriptions of the Schubert varieties it is easy to compute that each variety Q_i, $i = \ell, \dots, n$ is isomorphic to the complex join of P_{n-j-1} and R_{2j-n}. In particular the normal slice $(Q_j)_{P_{n-j-1}}$ is equal to the cone $X \subset \mathbb{C}^{2j-n+2}$ over R_{2j-n}. Putting $k := 2j - n$ we may take the function

$$g(z_0, \dots, z_{k+1}) := \Big| \sum_{i=1}^{k+1} z_i z_{k+1-i} \Big|^2$$

as an aura for X, and compute

$$\nabla g(z_0, \dots, z_{k+1}) = 4 \Big(\sum_{i=1}^{k+1} z_i z_{k+1-i} \Big) (\bar{z}_{k+1}, \bar{z}_k, \dots, \bar{z}_0).$$

Therefore the restriction ϕ to $\mathbb{S}^{2k+3}$ of the normalized gradient map $\frac{\nabla g}{|\nabla g|}$ covers the map $[z_0, \dots, z_{k+1}] \longmapsto [\bar{z}_{k+1}, \dots, \bar{z}_0]$ of $\mathbb{CP}^{k+1}$ to itself, and ϕ clearly preserves the orientation of the circles of the Hopf fibration. Thus the degree of ϕ is $(-1)^{k+1} = (-1)^{n+1}$, so by Theorem 2.2A

$$d_{P_{n-j-1}}^{Q_j} = (-1)^{n+1}. \tag{6.3.1}$$

Next we define polynomial versions of the MacPherson coefficients and "invert" via (6.0.1) to obtain "Euler obstruction polynomials." Begin with type IV (n odd). Set $\ell = (n+1)/2$, and label the Schubert varieties $1, 2, \dots, 2\ell$ from smallest to largest (so that $\dim(i) = i - 1$; recall (3.4.2)). Define

$$D_i^j = \begin{cases} q^{j-\ell}, & j > i = 2\ell - j \\ 1, & i = j \\ 0, & \text{otherwise.} \end{cases} \tag{6.3.2}$$

(The Hermitian symmetric space of type IV with $n = 3$ is isomorphic to the Hermitian symmetric space of type II with $n = 2$. This definition of D_i^j is chosen so that these two

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isomorphic cases agree.) Comparing (6.3.2) with (6.3.1), and taking the index shift into account, we have $D_i^j(1) = d_i^j$. Next we set

$$E_i^j = \begin{cases} 1 - q^{j-\ell}, & j > 2\ell - j \geq i \\ 1, & j \geq i > 2\ell - j \text{ or } 2\ell - j \geq j \geq i \\ 0, & i > j. \end{cases} \tag{6.3.3}$$

It is routine to check that, for $i \leq j$, $\sum_{i \leq k \leq j} E_i^k D_k^j = 1$. Hence $e_i^j = E_i^j(1)$. In particular,

$$e_i^j = 0 \neq 1 = p_i^j \qquad \text{for } j > 2\ell - j \geq i,$$

(see, *e.g.*, [Boe, (5.1a)]).

Now consider type V (n even). Set $\ell = (n+2)/2$, and label the Schubert varieties from 1 to 2ℓ, with $1, \ldots, \ell$ corresponding to $P_0, \ldots, P_{\ell-1}$, and $\ell + 1, \ldots, 2\ell$ corresponding to $P'_{\ell-1}, Q_\ell, \ldots, Q_n$, respectively (recall (3.4.3)). Define

$$D_i^j = \begin{cases} -q^{j-\ell-1}, & \ell + 2 \leq j \leq 2\ell - 1,\ i = 2\ell - j \\ 1, & i = j \\ 0, & \text{otherwise,} \end{cases} \tag{6.3.4}$$

$$E_i^j = \begin{cases} 1 + q^{j-\ell-1}, & \ell + 2 \leq j \leq 2\ell - 1,\ i \leq 2\ell - j \\ 0, & i > j \text{ or } i = \ell,\ j = \ell + 1 \\ 1, & \text{otherwise.} \end{cases} \tag{6.3.5}$$

Again, comparing (6.3.4) with (6.3.1) gives $D_i^j(1) = d_i^j$. Also the fundamental relation (6.0.1) holds; therefore $E_i^j(1) = e_i^j$. However, in this case $E_i^j = P_i^j$ for all i and j, by [Boe, (5.1b)].

7. **Multiplicities in Characteristic Cycles of Intersection Homology Sheaves.**

In this section we use the results of Section 6 to decompose the characteristic cycles of intersection homology sheaves on Schubert varieties in the compact classical Hermitian symmetric spaces:

$$\mathrm{CC}(\mathrm{IH}_X) = \sum_{Y \subset X} m_Y^X \vec{\mathbf{N}}(Y^\circ). \tag{7.0.1}$$

Recall from (2.1.7) the relation

$$p = e \cdot m, \tag{7.0.2}$$

where $p = (p_Y^X)$ is the matrix of Kazhdan-Lusztig numbers, and e is the matrix of Euler obstruction numbers. Knowing e and p, we can now solve for m.

7.1. *Multiplicity formulas.* Recall that a Dynkin diagram is said to be *simply laced* if it contains no multiple edges.

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THEOREM 7.1A. *Let H_G be a Hermitian symmetric space associated to a classical Lie group G. The characteristic cycles of the intersection homology sheaves associated to the Schubert varieties $X \subset H_G$ are all irreducible iff the Dynkin diagram of G is simply laced.*

PROOF. The Dynkin diagram of G is simply laced in types I, III, and V. We have seen that in each of these types, $E_Y^X = P_Y^X$ for all Schubert varieties Y, X; on the other hand, in types II and IV we have given examples where $e_Y^X \neq p_Y^X$ (*cf.* Remark 6.1B and Section 6.3). But by (7.0.2), the condition $e = p$ is equivalent to the irreducibility of all the characteristic cycles associated to the Schubert variety intersection homology sheaves. ∎

It remains to find formulas for the multiplicities m in the non-simply-laced types. In type IV, it is well-known that $p = \zeta$; *i.e.*, $p_Y^X = 1$ iff $Y \subset X$ (*cf.* [Boe, (5.1a)]). Comparing the fundamental relation (2.1.4) $e \cdot d = \zeta$ to (7.0.2), it is clear that $m = d$. Thus (using the classification of Section 6.3),

$$m_i^j = \begin{cases} 1, & i = j \\ 1, & i = 2\ell - j < j \\ 0, & \text{otherwise.} \end{cases} \tag{7.1.1}$$

In particular, each $\mathrm{CC}(\mathrm{IH}_X)$ has at most two summands $\vec{\mathbf{N}}(Y^\circ)$.

Finally, we treat type II. Recall from Section 6.2 the parametrization of Schubert varieties by Weyl group elements: sequences of n symbols chosen from $\{\alpha, \beta\}$; recall also from (6.2.1) the action of the simple reflections $s_1, \ldots, s_n$. The following theorem gives quick recursions for the multiplicities m_y^x.

THEOREM 7.1B. *Let $Y \subset X$ be Schubert varieties in a compact Hermitian symmetric space of type II. Let $y \leq x$ be the corresponding Weyl group words in α and β. The multiplicity m_y^x of $\vec{\mathbf{N}}(Y^\circ)$ in $\mathrm{CC}(\mathrm{IH}_X)$ is given recursively as follows.*

(a) If, for some simple reflection s, $y < ys$ but $x \not< xs$, then $m_y^x = 0$.
(b) If $y = y_1\alpha\beta y_2$, $x = x_1\alpha\beta x_2$ with $|y_1| = |x_1|$, then $m_y^x = m_{y_1y_2}^{x_1x_2}$.
(c) If $y = y_1\alpha\alpha$, $x = x_1\sigma\alpha$, $\sigma \in \{\alpha, \beta\}$, then $m_y^x = m_{y_1}^{x_1}$.

REMARKS 7.1C. (1) Of course the recursion ends with the rules $m_x^x = 1$, $m_y^x = 0$ if $y \not\leq x$.

(2) The effectiveness of the recursion also depends on the following facts. Given any Y which is not the whole manifold Λ_n, there exists a simple reflection s such that $y < ys$. If the *only* such s is $s = s_n$ and none of (a), (b), or (c) applies, then $y = x = \beta\beta \cdots \beta\alpha$, and hence $m_y^x = 1$.

PROOF OF THEOREM 7.1B. We prove each of (a), (b), and (c) in turn by downward induction on y; the case $y = x$ is trivial. So assume $y < x$.

(a) Assume $y < ys$, $x \not< xs$. Then

$$\begin{aligned} \sum_z e_{ys}^z m_z^x &= p_{ys}^x && \text{by (7.0.2)} \\ &= p_y^x && \text{by [Boe, (3.5)]} \\ &= \sum_z e_y^z m_z^x && \text{by (7.0.2)} \end{aligned}$$

$$= \sum_{z<zs} e^z_y m^x_z + \sum_{z\not<zs} e^z_y m^x_z$$
$$= \sum_{z<zs} e^z_y m^x_z + \sum_{z\not<zs} e^z_{ys} m^x_z \qquad \text{by Lemma 6.2C}$$

and therefore

$$\sum_{z<zs} (e^z_y - e^z_{ys})\, m^x_z = 0.$$

But by induction, $m^x_z = 0$ for $y < z < zs$. Hence

$$0 = (e^y_y - e^y_{ys})\, m^x_y = (1-0)\, m^x_y = m^x_y.$$

(b) Set $i = |y_1| + 1$ and $s = s_i$ so that $y < ys = y_1\beta\alpha y_2$, and similarly for x. For any $z = z_1\alpha\beta z_2$ with $|z_1| = i-1$ put $\bar{z} = z_1 z_2$, as in Lemma 6.2D. Then

$$\sum_{z<zs} e^z_y m^x_z + \sum_{z\not<zs} e^z_y m^x_z = p^x_y = p^x_{ys} + p^{\bar{x}}_{\bar{y}} \qquad \text{by [Boe, (3.14a)]}$$
$$\sum_{z<zs} e^z_y m^x_z + \sum_{z\not<zs} e^z_{ys} m^x_z = \sum_{z} e^z_{ys} m^x_z + \sum_{\bar{z}} e^{\bar{z}}_{\bar{y}} m^{\bar{x}}_{\bar{z}} \qquad \text{by Lemma 6.2C}$$
$$\sum_{z<zs} (e^z_y - e^z_{ys})\, m^x_z = \sum_{\bar{z}} e^{\bar{z}}_{\bar{y}} m^{\bar{x}}_{\bar{z}}$$
$$m^x_y + \sum_{y<z<zs} e^{\bar{z}}_{\bar{y}} m^{\bar{x}}_{\bar{z}} = m^{\bar{x}}_{\bar{y}} + \sum_{\bar{y}<\bar{z}} e^{\bar{z}}_{\bar{y}} m^{\bar{x}}_{\bar{z}} \qquad \text{by Lemma 6.2D and induction.}$$

Clearly the map $z \longmapsto \bar{z}$ is a bijection between the parameters z in the summation on the left, and the parameters $\bar{z}$ in the summation on the right. Therefore $m^x_y = m^{\bar{x}}_{\bar{y}}$ as claimed.

(c) Fix $y < x$ as in the statement of (c), and let $s = s_n$. For any z let $|z|_\beta$ denote the number of symbols β in z. Put $c^z_y = |z|_\beta - |y|_\beta$, the capacity between y and z "on the diagonal," and similarly c^x_z. Set $\delta_z = 0$ if c^x_z is odd, $= 1$ if c^x_z is even. For $z = z_1\tau\alpha$ $(\tau \in \{\alpha,\beta\})$ put $\bar{z} = z_1\tau$ as in Lemma 6.2D. Proceeding as in (a) and (b), we have

$$p^x_{ys} + \delta_y p^{x_1}_{y_1} = p^x_y = \sum_{z<zs} e^z_y m^x_z + \sum_{z\not<zs} e^z_y m^x_z \qquad \text{by [Boe, (3.14b)]}$$

$$\delta_y p^{x_1}_{y_1} = \sum_{z<zs} (e^z_y - e^z_{ys}) m^x_z \qquad \text{by Lemma 6.2C}$$

$$(7.1.2) \qquad \delta_y p^{x_1}_{y_1} = \sum_{z<zs} (-1)^{c^z_y} e^{\bar{z}}_{\bar{y}} m^x_z \qquad \text{by Lemma 6.2D}$$
$$= \sum_{z=z_1\beta\alpha} (-1)^{c^z_y} e^{z_1\beta}_{y_1\alpha} m^x_z + \sum_{z=z_1\alpha\alpha} (-1)^{c^z_y} e^{z_1\alpha}_{y_1\alpha} m^x_z$$
$$= \sum_{z=z_1\beta\alpha} (-1)^{c^z_y} e^{z_1\beta}_{y_1\alpha} m^x_z + \sum_{z=z_1\alpha\alpha} (-1)^{c^z_y} (e^{z_1\alpha}_{y_1\beta} + (-1)^{c^z_y} e^{z_1}_{y_1}) m^x_z$$
$$\text{by Lemma 6.2D, and since } c^{z_1\alpha}_{y_1\alpha} = c^{z_1\alpha\alpha}_{y_1\alpha\alpha}$$
$$= \sum_{z<zs} (-1)^{c^z_y} e^{\bar{z}}_{y_1\beta} m^x_z + \sum_{z=z_1\alpha\alpha} e^{z_1}_{y_1} m^x_z \qquad \text{by Lemma 6.2C}$$

Put $y' = y_1\beta\alpha$ and observe that $c^z_{y'} = c^z_y - 1$. Then

$$\delta_y p^{x_1}_{y_1} = -\sum_{z<zs} (-1)^{c^z_{y'}} e^{\bar{z}}_{\bar{y}'} m^x_z + \sum_{z=z_1\alpha\alpha} e^{z_1}_{y_1} m^x_z.$$

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Now substitute the analog of (7.1.2) with y' in place of y:

$$\begin{aligned} \delta_y p_{y_1}^{x_1} &= -\delta_{y'} p_{y_1}^{x_1} + \sum_{z=z_1\alpha\alpha} e_{y_1}^{z_1} m_z^x \\ p_{y_1}^{x_1} &= \sum_{z=z_1\alpha\alpha} e_{y_1}^{z_1} m_z^x \qquad \text{since } \delta_y + \delta_{y'} = 1 \\ m_{y_1}^{x_1} + \sum_{y_1<z_1} e_{y_1}^{z_1} m_{z_1}^{x_1} &= m_y^x + \sum_{y<z=z_1\alpha\alpha} e_{y_1}^{z_1} m_{z_1}^{x_1} \qquad \text{by induction} \\ m_{y_1}^{x_1} &= m_y^x \end{aligned}$$

This proves (c). ∎

The theorem admits at least two useful reformulations, the first of which gives a closed form for the non-zero multiplicities, and the second being a geometric restatement of the first.

THEOREM 7.1D. *Let $Y \subset X$ be Schubert varieties in a compact Hermitian symmetric space of type II. Let $y \leq x$ be the corresponding Weyl group words in α and β. The multiplicity m_y^x of $\vec{\mathbf{N}}(\operatorname{reg} Y)$ in $\mathrm{CC}(\mathrm{IH}_X)$ is either 0 or 1.*

(1) $m_y^x = 1$ iff the following conditions hold:

- *(a) y is obtained from x by changing certain β's to α's;*
- *(b) each β in (a) occurs an even number of symbols from the end of the word x;*
- *(c) every subword of x beginning with any β in (a) contains at most one more β than α (where a subword is a sequence of consecutive symbols).*

(2) $m_y^x = 1$ iff the following conditions hold:

- *(a) there is a sequence of Schubert varieties $X = X_0 \supset X_1 \supset \cdots \supset X_r = Y$ such that*
- *(b) $\dim X_{i-1} - \dim X_i$ is even for each $1 \leq i \leq r$, and*
- *(c) the Schubert strata corresponding to X_{i-1} and X_i have conormal varieties whose closures meet in codimension one, for each $1 \leq i \leq r$.*

In particular, the number of summands in $\mathrm{CC}(\mathrm{IH}_X)$ is equal to the number of β's in x satisfying the conditions in (1)(b) and (c).

PROOF. (1) follows from Theorem 7.1B by a straightforward induction on the rank n. The details are left to the reader.

(2) is shown to be equivalent to (1) as follows. Assuming y is obtained from x by changing r β's to α's as in (1), define x_i to be the word obtained from x by reversing only the first i of these β's. Put X_i to be the associated Schubert variety. One checks easily that if $x = \sigma_1 \cdots \sigma_n$ with $\sigma_j \in \{\alpha, \beta\}$ then $\dim X = \ell(x) = \sum_{\sigma_j=\beta}(n+1-j)$. This immediately implies the equivalence of (1)(b) and (2)(b). The condition in (2)(c) (recall Remark 5.2B(1)) amounts to the following: $\Delta(X_i)\setminus\Delta(X_{i-1})$ is a (connected) ribbon which, when followed from upper left to lower right, always remains no wider than it is tall. (See Figure 5.4, where the ribbon in the left diagram and the short ribbon in the right diagram are of this type, while the long ribbon in the right diagram is not.) But it's easy to see that this is equivalent to the statement that x_i is obtained from x_{i-1} by reversing a

single β satisfying the condition in (1)(c). Thus (1) implies (2), and the converse is now also clear. ∎

COROLLARY 7.1E. *The characteristic cycles of the intersection homology sheaves on Schubert varieties for classical Hermitian symmetric spaces are multiplicity-free. Moreover, the multiplicity* $m_Y^X = 0$ *unless* $\ell(x) - \ell(y) \equiv \operatorname{codim}_X Y$ *is even.*

PROOF. This follows from Theorem 7.1A, (7.1.1), and Theorem 7.1D. ∎

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Department of Mathematics
University of Georgia
Athens, Georgia
USA 30602
e-mail: brian@math.uga.edu
fu@math.uga.edu

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