DEHN FUNCTIONS AND COMPLEXES OF GROUPS

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Abstract. We study the Dehn functions of the fundamental groups of complexes of groups. We study a function known as *the Howie function*, which has a natural geometric formulation. We make use of the Howie function to obtain an upper bound for the Dehn function of the complex of groups. And we show a connection between the Howie function and actions on complexes.

0. Introduction. If G is a finitely presented group then its Dehn function speaks of the underlying geometry of the group. For example, G is hyperbolic (in the sense of Gromov, see [**5**]) iff its Dehn function is linear. Thus studying the Dehn function is one of the basic problems in geometric group theory.

In [**1**], the Dehn functions of amalgamations and HNN extensions were studied. The results there can easily be applied to graphs of groups where the edge groups are finite. Modulo a technical result about subnegativity (see below for the definition), the result is that the Dehn function of the fundamental group of the graph of groups is bounded above by the maximum of the Dehn functions of the vertex groups.

Complexes of groups with finite edge groups (see [**2**]) are the next obvious case to conisder. We restrict ourselves to developable complexes, i.e., those arising from a group action.

Suppose $\mathcal{G}$ is a developable finite complex of groups with finite edge groups. Let H be the fundamental group of the graph of groups on the one-skeleton. Letting δ be the maximum of the Dehn functions, δ_v's, of the vertex groups, from [**1**] we have that $\delta_H \preccurlyeq \bar{\delta}$, where $\preccurlyeq$ stands for being of less type (definitions given below) and $\bar{f}$ denotes the subnegative closure of f.

The fundamental group of $\mathcal{G}$ is the quotient group $G = H/N$ where N is the normal closure of the labels on the two-cells. It seems natural to expect that δ_G should depend on δ (or δ_H) and the geometry of the complex. This is made precise with the Howie function, $h_{\mathcal{G}}$, of the complex (first introduced in [**3**]). Our main result is that

$$\delta_G \preccurlyeq \bar{\delta} \circ \bar{h}_{\mathcal{G}}$$

In the latter part of the paper, we relate the Howie function to group actions on complexes, identifying it with the Dehn function of a naturally arising complex.

The layout of our paper is as follows. In § 1 we recall a few preliminary facts about Dehn functions and complexes of groups. And we algebraically define the Howie function. In § 2 we prove two lemmas and give a geometric interpretation of the Howie function.

In § 3 we prove our main result about the Dehn function for a complex of groups. In § 4 we study group actions.

1. Preliminaries. To establish notation, we start by recalling the definition and a few basic facts about Dehn functions (see [**1**]). Suppose X is a finite two-complex. Let w be an edge-circuit in X^1 that is null-homotopic in X. Then there is a singular disk (D, j)

spanning w, i.e., D is a two-dimensional disk, j is a transverse map with $j(\partial D) \subset X^1$, and $j \mid \partial D$ represents w (after choosing an orientation of ∂D). We define $\Delta_X(w) = \min\{a(D) \mid D$ is a singular disk spanning $w\}$. Here $a(D)$ is the area of the transverse map j, i.e., the number of sub-disks in the picture of the map. Note that we could get an equivalent formulation of $\Delta_X(w)$ by using van Kampen diagrams, i.e., combinatorial maps of simply connected finite planar complexes, and their area. In any case, the *Dehn function* of X is the function

$$\delta_X(n) = \max\{\Delta_X(w) \mid w \text{ is a circuit in } X^1, \text{ null-homotopic in } X, \text{ with } |w| \leq n\}.$$

We could also, in an analogous fashion, define the Dehn function, $\delta_{\mathcal{P}}$, of a finite presentation $\mathcal{P}$: Inessential edge-circuits are replaced by words w that represent the identity and $\Delta_{\mathcal{P}}(w)$ is defined to be the least n such that we can write w as a product of n conjugates of relators or their inverses. It is clear that if P is a finite presentation and X is the associated two-complex, then $\delta_X = \delta_P$. Also observe that if X is not a finite complex, we can still define the Dehn function by replacing "max" with "sup". Of course in such a case, the Dehn function may take on the value of $+\infty$.

The Dehn function may change with change of presentation. However, the *type* of function does not change. By "type" we mean the following: Given $f, g: \mathbb{N}\backslash\{0\} \to \mathbb{N}$, we say that $f \preccurlyeq g$ if there are constants a, b, c so that for all n the following holds:

$$f(n) \leq a \,.\, n + b \,.\, g(c \,.\, n)$$

We say that f and g are of the same type, and we write $f \equiv g$, if both $f \preccurlyeq g$ and $g \preccurlyeq f$ are true.

If P_1 and P_2 are finite presentations of the same group, then δ_{P_1} and δ_{P_2} are of the same type. More generally, if X_1 and X_2 are finite complexes with isomorphic fundamental groups, then $\delta_{X_1} \equiv \delta_{X_2}$. Thus when we speak of the Dehn function of a group, we are speaking of a function only defined up to equivalence of type.

We say a function f is *subnegative* if $f(n) + f(m) \leq f(n+m)$. Given a function h, we write $\bar{h}$ for the subnegative closure of h, i.e., the smallest subnegative function greater than or equal to h. For convenience sake, when we speak of the subnegative closure, $\bar{f}$, of a function f, we will assume that $\bar{f}(1) \geq 1$. This does not affect the type of the resulting function so has no impact on our results, as we are only interested in functions up to type. We make use of the subnegative closure when we obtain our upper bound for the Dehn function of a complex of groups.

We now turn to complexes of groups (see **[2]** and **[3]**). For a complex of groups, the fundamental group (and hence its Dehn function) is determined by the 2-complex of groups on the 2-skeleton. Moreover, the groups assigned to the 2-cells have no effect on the fundamental group. Hence, for our purposes, it suffices to regard a *complex of groups* as a triple $(X, \mathcal{G}, \phi)$ where X is a connected 2-complex, $(X^1, \mathcal{G})$ is a graph of groups on the 1-skeleton of X, and ϕ is a *corner labeling function*: for each directed corner α of X at the vertex v (i.e., directed edge of the link $\mathrm{lk}(v)$), $\phi(\alpha) \in \mathcal{G}_v$ and $\phi(\bar{\alpha}) = \phi(\alpha)^{-1}$.

For the rest of this paper, we fix a complex of groups with X finite, each vertex group $\mathcal{G}_v$ finitely presented, each edge group $\mathcal{G}_e$ a finite group, and $\mathcal{G}$ developable (i.e., it arises in a natural way from a group action on a 1-connected complex). And we will abuse notation and use just $\mathcal{G}$ to refer to the complex of groups.

We need to recall some notation about words in a graph of groups. As in **[7]**, to define the group of a graph of groups, we could either choose a basepoint and work with loops based at that point, or work modulo a maximal tree. We will take the latter approach. Let y be a directed edge of X^1. We identify $\mathcal{G}_y$ with its image in $\mathcal{G}_{o(y)}$. Write $a \mapsto a^y$ for the monomorphism $\mathcal{G}_y \to G_{t(y)}$, and $\mathcal{G}_y^y$ for its image. A *word* W is a pair (c, μ) where $c = y_1 y_2 \dots y_n$ is an edge-path in X^1 and $\mu = (g_0, \dots, g_n)$ is a sequence of elements $g_i \in \mathcal{G}_{v_i}$ with $v_i = o(y_{i+1}) = t(y_i)$. We will refer to c as the X-path of the word W. Then W represents the element $g_0 y_1 g_1 \dots y_n g_n$.

By the *length* of a word we mean the length of the X-path. Note that length zero words are those where the edge-path c is a degenerate edge-path, i.e., a single vertex. If the X-path is a circuit then allowing for any cyclic permutation of it, yields the notion of a *cyclic word*.

A cyclic word W of positive length can be represented by a pair (c, μ) where $\mu = (1, g_1, g_2, \dots, g_n)$. If W has length zero, say with c the degenerate path at the vertex v, then $\mu = g_0 \in \mathcal{G}_v$. We say that two cyclic words W and W' of length ≥ 1 are *equivalent* if, for some closed edge-path c, W and W' are represented by words (c, μ) and (c, μ'), respectively, where $\mu = (1, g_1, \dots, g_n)$, $\mu' = (1, g_1', \dots, g_n')$, and there is a sequence $(a_1, \dots, a_n)$, $a_i \in \mathcal{G}_{y_i}$, such that $g_i' = a_i^{y_i} g_i a_{i+1}^{-1}$ with indices mod n.

The elements of the fundamental group H of the graph of groups $(X^1, \mathcal{G})$ are represented by words (after choosing a maximal tree as in [**7**]). Equivalent cyclic words represent conjugate elements. Note that by [**1**], the Dehn function of H is bounded above by the maximum of the Dehn functions of the vertex groups.

To define the fundamental group of the complex of groups $\mathcal{G}$, we need to use the corner labels on the two-cells of X. For each two-cell σ of X, choose a boundary cycle of σ. Let r_σ denote the cyclic word obtained by reading the edges and corner labels in order around the chosen boundary cycle. Then the fundamental group of the complex of groups $\mathcal{G}$ is the quotient group $G = H/N$ where N is the normal closure of the r_σ's.

A cyclic word W is *inessential* if it represents an element of N. Given an inessential word W, the element it represents in H can be expressed as a product of conjugates of the r_σ's or their inverses. We write $\Delta_{\mathcal{G}}(W)$ for the minimal number of conjugates required in such a factorization. The function defined by

$$h_{\mathcal{G}}(n) = \sup\{\Delta_{\mathcal{G}}(W) \mid W \text{ is an inessential word of length at most } n\}$$

is called the *Howie function* of the complex of groups. Note that even though the underlying complex X is finite, we need, a priori, to take a supremum instead of a maximum as we are working with the length of the X-path instead of a length function arising from a finite set of generators, and there may be some n with infinitely many words W of length $\leq n$.

To bound the Dehn function of G we need to construct a finite 2-complex K with fundamental group isomorphic to G. We proceed as follows.

For each directed edge x of X, let A_x be the set of non-trivial elements of the finite group $\mathcal{G}_x$. For each vertex v of X, choose a finite presentation $\mathcal{P}_v = \langle A_v \mid R_v \rangle$ for $\mathcal{G}_v$ with the property: for each directed edge x with $o(x) = v$, we have $A_x \subset A_v$. Let K_v denote the canonical 2-complex (with a single vertex) associated to the presentation $\mathcal{P}_v$.

To construct K, start with the disjoint union of the complexes K_v. Attach a 1-cell e^* corresponding to each 1-cell e of X; if v and v' are the endpoints of e, then the endpoints of e^* are attached to the unique vertices of K_v and $K_{v'}$. (Of course v and v' could be the same point.) Finally, we attach 2-cells corresponding to the 1- and 2-cells of X as follows. Let e be a 1-cell of X and let x be the directed edge obtained by choosing an orientation of e. For each $a \in A_x$, attach a 2-cell along the circuit $x^* a^x \bar{x}^* a^{-1}$. Now let σ be a 2-cell of X. Choose an orientation of σ and let (c, μ) be the corresponding cyclic word around its boundary. Suppose $c = x_1 \dots x_n$ and $\mu = (1, g_1, \dots, g_n)$ and choose edge-paths w_i in $K_{t(x_i)} = K_{o(x_{i+1})}$ representing the elements g_i. Then we attach a 2-cell along the circuit $w_\sigma = x_1^* w_1 x_2^* \dots w_{n-1} x_n^* w_n$.

Let $p: K \to X$ be the obvious projection; thus $p^{-1}(v) = K_v$ for each vertex v of X, and $p^{-1}(Q)$ is a wedge of circles (resp. a point) for each interior point Q of a 1-cell (resp. 2-cell) of X. Observe that K is the two-skeleton of a complex of spaces associated to the given complex of groups. In particular, $\pi_1(K)$ is isomorphic to the fundamental group of the complex of groups. (Note that the complex of groups being developable is equivalent to the inclusion $K_v \hookrightarrow K$ inducing a monomorphism on fundamental groups for each vertex v of X.)

Given an inessential edge-circuit w in K, we need to be able to construct a singular disk (D, j) for w and measure its area. To do this, we use the projection p in order to make use of the geometry of the complex of groups. Our construction works for both essential and inessential edge-circuits. So suppose w is an edge-circuit in K. We will associate a cyclic word, Word(w), in $(X^1, \mathcal{G})$ to w as follows: The circuit w projects to the circuit $p(w)$ in X^1. Write w in the form $w = x_1^* w_1 \dots x_n^* w_n$ where each x_i^* is a directed edge in K^1 projecting to an edge x_i in X^1 and each w_i is a closed edge-path in K^1 projecting to a vertex v_i of X^1. Let g_i be the element of the vertex group $\mathcal{G}_{v_i}$ represented by w_i. Then take Word(w) to be the cyclic word represented by (c, μ), where $c = x_1 \dots x_n$ and $\mu = (1, g_1, \dots, g_n)$. Observe that it is immediate that if w is inessential in K then $W = \text{Word}(w)$ is an inessential cyclic word in $\mathcal{G}$.

2. Howie diagrams and the Howie function. If we take an inessential edge-circuit w in K and project and perform the above construction, then we get an inessential cyclic word $W = \text{Word}(w)$ in $\mathcal{G}$. The Howie function tells us that we can write W as a product of no more than $h_{\mathcal{G}}(|W|)$ conjugates of the two-cell labels r_σ's or their inverses. We need to use this to construct a singular disk for w mapping into K and be able to bound its area. First we will construct a labelled van Kampen diagram over X, also known as a *Howie diagram.*

A Howie diagram (see [**3**]) in $\mathcal{G}$ is a triple (Ω, f, λ) where $f: \Omega \to X$ is a combinatorial map, Ω is a planar 1-connected finite 2-complex, and λ is a corner labeling function assigning to each directed corner α of Ω, both interior and exterior corners, an element of the vertex group $\mathcal{G}_{f(v)}$ (where α is incident to v) with $\lambda(\bar{\alpha}) = \lambda(\alpha)^{-1}$, and such that the following properties are satisfied:

(H1) If v is a vertex of Ω and $\alpha_1, \dots, \alpha_t$ are the directed corners of Ω at v in order according to some orientation of the plane (thus, forming a loop around v), then $\lambda(\alpha_1) \dots \lambda(\alpha_t) = 1$ where the product is taken in $\mathcal{G}_{f(v)}$.

(H2) If σ is an oriented 2-cell of Ω, then the cyclic word obtained by reading the labels on the directed edges (via f) and the corners of σ (via λ) in order around $\partial\sigma$ (in the direction of the orientation of σ) is equivalent to the cylic word on the oriented boundary of $f(\sigma)$.

We will often write $f:\Omega\to X$ for the Howie diagram, i.e., the corner labelling function λ is understood. Also, we write $a(\Omega)$ to mean the area of the combinatorial map, i.e., the number of two-cells of Ω.

By the *cyclic boundary word* of a Howie diagram we shall mean the cyclic word obtained by reading the images (in X^1) of the directed edges and the exterior corner labels in order around a boundary cycle tracing the boundary of the planar complex once in the direction of the preferred orientation of the plane.

Recall that the Howie function is defined in terms of $\Delta_{\mathcal{G}}$. The following proposition relates Howie diagrams and the Howie function. See [**3**, Theorem 2.6] for a related result.

Proposition 2.1. *Let W be a non-trivial inessential cyclic word in $\mathcal{G}$. There exists a Howie diagram $f:\Omega\to X$ with corner labelling function λ whose cyclic boundary word U is equivalent to W and with $a(\Omega)=\Delta_{\mathcal{G}}(W)$.*

Proof. The essence of the proof is a "bunch of lollipops" construction as in [**6**]. However, as there are some subtle technical considerations involved with Howie diagrams (for example, the resulting diagram has boundary word *only equivalent* to the original word), we provide a detailed proof.

Choose a word (c,μ) representing W, say where $c=x_1x_2\ldots x_n$ and $\mu=(1,g_1,\ldots,g_n)$. It should be noted that c is non-trivial, by the developability assumption. Let w_i be an edge-path in $K_{t(x_i)}$ representing g_i for each i. Then the closed edge-path $x_1^*w_1x_2^*w_2\ldots x_n^*w_n$ determines a circuit w in K^1 such that $\mathrm{Word}(w)=W$.

Since W is inessential, we can choose a sequence $r_1,r_2,\ldots,r_m$ of r_σ's or their inverses, such that the element of H represented by (c,μ) is a product of conjugates of $r_1,r_2,\ldots,r_m$. Corresponding to r_i is an attaching circuit u_i of a 2-cell of K such that $\mathrm{Word}(u_i)$ is a cyclic word representing r_i.

A standard construction yields a transverse singular disk (D,j) in K spanning w having subdisks $D_1,D_2,\ldots,D_m$ where D_i maps to the two-cell of K with boundary u_i. Now the composition $p\circ j:D\to X$ is a singular disk in X. Let ξ be the preimage of the set of midpoints of 1-cells of X; thus ξ is a compact 1-manifold properly embedded in D. Put $\Lambda=\partial D\cup(\bigcup D_i)\cup\xi$, the "picture" of $(D,p\circ j)$. Since $\mathcal{G}$ is developable, each vertex group $\mathcal{G}_v$ embeds in G. Now if Λ were not connected, we could find a loop ω in $D\setminus\Lambda$ separating Λ. And then we could use the injectivity of $\pi_1(p^{-1}(v))$ in $\pi_1(K)$ to redefine the map $p\circ j$ on the interior of ω so that the new map has picture missing this interior. Hence we may assume that Λ is connected. Then taking the complex dual to the picture we get a van Kampen diagram $f:\Omega\to X$. Observe that we can take Ω to be embedded in D with one vertex in each component of $D\setminus\Lambda$ and with a 1-cell dual to each component of ξ.

The proof is completed by defining a corner labelling function λ satisfying the conditions (H1) and (H2) and showing that the boundary word U is equivalent to W.

Suppose v is a vertex of Ω and α is a corner incident at v. Then v is contained in some component V of $D\setminus\Lambda$ and α corresponds to an arc γ in $\partial V\setminus\Omega^1$ joining two points P_1 and P_2 of $\Omega^1\cap\xi$. Now $f(v)$ is a vertex in X^1. Consider the closed neighborhood $S=\mathrm{star}_{X^1}(f(v))$ of $f(v)$ in X^1. Observe that the inclusion map of $K_{f(v)}\hookrightarrow p^{-1}(S)$ induces an isomorphism of $\mathcal{G}_{f(v)}$ onto $\pi_1(p^{-1}(S))$. Recall that the basepoint of $K_{f(v)}$ is $f(v)^*$. We

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also take $f(v)^*$ to be the basepoint of $p^{-1}(S)$. For each point P of $\Omega^1 \cap \xi$, choose a path β_P joining $f(v)^*$ to P. Then the loop $\beta_{P_1} . \gamma . \beta_{P_2}^{-1}$ is a loop in S based at $f(v)^*$, and so represents an element of $\mathcal{G}_{f(v)}$. Define the label of the corner α to be this group element.

By construction, the product of the corner labels around the vertex v is homotopic to $j(\partial V)$, a null-homotopic loop in $p^{-1}(S)$. And if τ is a two-cell of Ω, then the cyclic boundary word of τ is equivalent to $\mathrm{Word}(u_i)$, the cyclic word around a 2-cell of X. And finally, the cyclic boundary word U of Ω is equivalent to $\mathrm{Word}(w) = W$, as required. ☐

An immediate consequence of this proposition is the characterization of the minimal spanning area of an inessential cyclic word W as

$$\Delta_{\mathcal{G}}(W) = \min\{a(\Omega) \mid \text{where } \Omega \text{ is a Howie diagram with boundary word equivalent to } W\}$$

And the Howie function of $\mathcal{G}$ is given by

$$h_{\mathcal{G}}(n) = \sup\{\Delta_{\mathcal{G}}(W) \mid W \text{ is an inessential cyclic word of length at most } n\}.$$

We now turn to a lemma that helps us handle equivalent cyclic words. Recall that we are writing δ for the maximum of the Dehn functions, δ_v's, of the vertex groups.

LEMMA 2.2. *Let w and u be circuits in K^1 such that* Word(w) and Word(u) *are equivalent cyclic words. Let l be the maximum of the lengths of w and u. Then there exists a singular annulus (A, j) in K with one boundary component mapped to each of w and u such that*

$$a(A) \le l + \bar{\delta}(4 . l) \le \bar{\delta}(5 . l).$$

Proof. The hypothesis implies that w and u may be expressed as edge-paths:

$$x_1^* w_1 x_2^* w_2 \dots x_n^* w_n \quad \text{and} \quad x_1^* u_1 x_2^* u_2 \dots x_n^* u_n$$

where $p(x_1^*) = x_1, \dots, p(x_n^*) = x_n$ are directed edges. Let g_i and h_i be the vertex group elements determined by the w_i and u_i, respectively. By the definition of equivalent cyclic words, there exist elements $a_i \in \mathcal{G}_{x_i}$ satisfying

$$h_i = a_i^{x_i} g_i a_{i+1}^{-1} \text{ (indices modulo } n).$$

It follows that for each i, the circuit $u_i^{-1} a_i^{x_i} w_i a_{i+1}^{-1}$ is null-homotopic in the appropriate vertex complex K_v. Hence, we can form a singular annulus A mapping into K as depicted in Figure 1 (the case $n = 4$ is shown) such that

$$\begin{aligned} a(A) &\le n + \sum_{i=1}^{n} \delta(|w_i| + |u_i| + 2) \\ &\le n + \bar{\delta}(|w| + |u| + 2n) \\ &\le l + \bar{\delta}(4 . l) \end{aligned}$$

as required.

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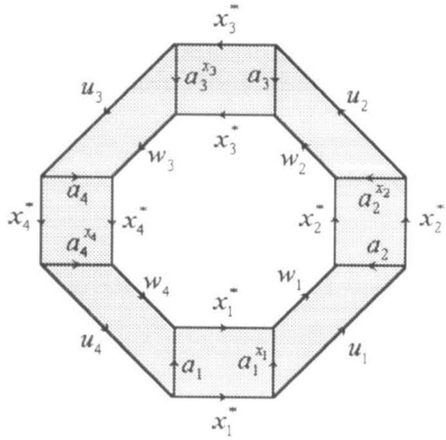

Figure 1

3. Bounding the Dehn function. We now come to our result about the Dehn function of $\mathcal{G}$. We recall some of our notation in the statement of the theorem.

THEOREM 3.1. *Let $(X, \mathcal{G}, \phi)$ be, as above, a developable complex of groups, with underlying complex X a finite complex, vertex groups finitely presented, and edge groups finite groups. Write G for the fundamental group of the complex of groups. Let δ be the maximum of the Dehn functions of the vertex groups. Then*

$$\delta_G \preccurlyeq \bar{\delta} \circ \bar{h}_{\mathcal{G}}.$$

Proof. Since K is a finite complex with fundamental group G, it is enough to show that there are positive integers C_1, C_2, C_3, such that

$$\delta_K(l) \leq C_1 \, . \, h_{\mathcal{G}}(l) + \bar{\delta}(C_2 \, . \, l + C_3 \, . \, h_{\mathcal{G}}(l)).$$

For this implies that

$$\delta_K \preccurlyeq h_{\mathcal{G}} + \bar{\delta} \circ \bar{h}_{\mathcal{G}} \preccurlyeq (\mathrm{id} + \bar{\delta}) \circ \bar{h}_{\mathcal{G}} \preccurlyeq \bar{\delta} \circ \bar{h}_{\mathcal{G}}.$$

Recall that for each 2-cell σ of X, there is a corresponding 2-cell of K attached, say, along the circuit w_σ. Let $C = \max\{|w_\sigma| : \sigma \text{ is a 2-cell of } X\}$. Then we shall verify that the above inequality holds with the constants:

$$C_1 = 1 + \bar{\delta}(15 \, . \, C), \qquad C_2 = 18, \qquad C_3 = 3 \, . \, C.$$

To this end, let w be a null-homotopic edge-loop in K such that $|w| \leq l$. Then Word(w) is an inessential cyclic word.

Let H be the fundamental group of the graph of groups on X^1. First of all, assume that Word(w) is trivial. Then w represents the identity in H. Since the edge groups are finite, it follows from [**1**] that there exists a singular disk for w with area at most $\delta(l)$.

So assume that Word(w) is non-trivial. Then, by virtue of Proposition 2.1, there exists a Howie diagram (Ω, f, λ) whose cyclic boundary word U is equivalent to

Word(w), and $a(\Omega) = \Delta_{\mathscr{G}}(\text{Word}(w)) \leq h_{\mathscr{G}}(l)$. We construct a singular disk (D, j) in K spanning w by replacing the vertices of the Howie diagram with singular disks and attaching an annulus to the outer boundary component.

Start with the graph Ω^1; remember that it is embedded in the plane. Choose small disjoint closed balls $V_1, \ldots, V_k$ in the plane, one centered about each vertex of Ω^1. The boundaries of these balls are unions of arcs that correspond to the corners of Ω; define j to map each arc (in a piecewise linear fashion) to an edge-path in the appropriate K_v representing the label assigned by λ to the corresponding corner of Ω. Then, by the definition of a Howie diagram, it follows that the boundary of each ball, V_i, is mapped to a null-homotopic edge-loop, u_i, in some K_v; extend the map j over the interior of V_i to a least area singular disk in K_v; thus $a(V_i) \leq \delta_v(|u_i|)$. Let $D_0 = \Omega^1 \cup (\bigcup V_i)$, and extend the map j over D_0 in the essentially unique way so that the remnants of the edges of Ω^1 are mapped (in a PL fashion) onto edges of K.

Let $\Sigma_1, \ldots, \Sigma_m$ be the bounded components of $\mathbb{R}^2 \setminus D_0$; notice that $m \leq h_{\mathscr{G}}(l)$. Then the boundary of each Σ_i is mapped to a circuit w_i in K^1 such that, by the definition of a Howie diagram, Word(w_i) is equivalent to Word(w_σ) for some 2-cell σ of X (recall that w_σ is the boundary of a two-cell in K). We may assume that w_i is gotten from w_σ by interleaving elements from the various edge groups. Each element of each edge group was chosen as a generator (recall that the edge groups were all finite groups). Hence if the X-path of w_σ was of length n, then the length of w_i is no more than $2n$ larger than that of w_σ. It follows that $|w_i| \leq 3 \,.\, |w_\sigma| \leq 3 \,.\, C$.

So, by Lemma 2.2, there is a singular annulus in K of area at most $\bar{\delta}(15 \,.\, C)$ with one boundary component mapped to each of w_i and w_σ. Furthermore, the boundary component mapped to w_σ can be capped off with a disk mapped to a 2-cell in K, thus giving a singular disk in K with area bounded above by $1 + \bar{\delta}(15 \,.\, C) = C_1$.

Similarly, the boundary of the unbounded component of $\mathbb{R}^2 \setminus D_0$ is mapped to a circuit, w_∞, in K^1 such that Word(w_∞) is equivalent to Word(w), and $|w_\infty| \leq 3 \,.\, |w| \leq 3 \,.\, l$. Again, invoking Lemma 2.2, we see that there is a singular annulus in K of area at most $\bar{\delta}(15 \,.\, l)$ with one boundary component mapped to each of w_∞ and w.

The singular disk D spanning w is now obtained from D_0 by attaching a singular disk of area at most C_1 to the boundary of each Σ_i and by attaching a collar (i.e., an annulus) of area at most $\bar{\delta}(15 \,.\, l)$ along the boundary of the unbounded component of $\mathbb{R}^2 \setminus D_0$. Hence,

$$a(D) \leq C_1 \,.\, a(\Omega) + \bar{\delta}(15 \,.\, l) + a(D_0).$$

But since $\bar{\delta}$ is subnegative and increasing,

$$\begin{aligned} a(D_0) &\leq \sum_{i=1}^{k} \delta(|u_i|) \\ &\leq \bar{\delta}(|u_1| + \ldots + |u_k|) \\ &\leq \bar{\delta}(|w_\infty| + |w_1| + \ldots + |w_m|) \\ &\leq \bar{\delta}(3 \,.\, l + 3C \,.\, a(\Omega)). \end{aligned}$$

We conclude that

$$a(D) \leq C_1 \,.\, a(\Omega) + \bar{\delta}(18 \,.\, l + C_3 \,.\, a(\Omega)) \leq C_1 \,.\, h_{\mathscr{G}}(l) + \bar{\delta}(C_2 \,.\, l + C_3 \,.\, h_{\mathscr{G}}(l)).$$

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Taking the supremum over all null-homotopic edge-loops w in K with $|w| \leq l$ yields the desired inequality.

4. Group actions and Dehn functions. Turning now to group actions, we make the following restriction: Let G be a (discrete) group. Then a G-complex shall mean a CW-complex Y, whose 2-cells are attached along non-trivial circuits in Y^1, upon which G acts cellularly such that the cells of Y are permuted without inversions. Hence, given a G-complex Y, the quotient Y/G has a natural cell structure such that the orbit map is combinatorial. We say that a G-complex Y is *cocompact* if Y/G is compact.

Our considerations only depend on the 2-skeleton of a G-complex. Thus, we shall henceforth assume that all G-complexes are 2-dimensional. Indeed the following situation is our primary concern: let Y be a (2-dimensional) connected G-complex such that:

- the stabilizer of every vertex is finitely presented,
- the stabilizer of every 1-cell is finite
- Y is cocompact.

These are the circumstances that correspond to the type of complexes of groups we have been considering.

First of all, we construct an associated complex of groups. Let $X = Y/G$ with cell structure inherited from Y. Form a graph $\bar{\Gamma}$, called the *principal face graph* of X, by taking as vertex set the cells of X and attach an edge between a pair of vertices, corresponding to cells of codimension one, for each occurrence of the lower dimensional cell in the attaching region of the other cell. We view $\bar{\Gamma}$ as being embedded in X in the obvious piecewise linear fashion with the vertices of $\bar{\Gamma}$ at the "centers" of the cells of X. Likewise, let Γ be the principal face graph of Y, embedded G-equivariantly in Y so that $\Gamma/G = \bar{\Gamma}$.

Choose a maximal tree T of $\bar{\Gamma}$, and orient each 1-cell γ of $\bar{\Gamma}$ so that its origin corresponds to a cell of X of greater dimension than that of its terminus. We write $o(\gamma)$ and $t(\gamma)$ for the origin and terminus, respectively, of this orientation of γ. Then, by **[7]**, there is a section s: cells $\bar{\Gamma} \to$ cells Γ of the orbit map, and an element $g(\gamma) \in G$ for each 1-cell γ of Γ satisfying:

(1) $o(s\gamma) = so(\gamma)$;
(2) $t(s\gamma) = g(\gamma) \,.\, st(\gamma)$;
(3) $g(\gamma) = 1$ for all γ in T.

(Hence the restriction of s to T determines an embedding into Γ.)

We use the section s on $\bar{\Gamma}$ to define a complex of groups on X. Recall that a vertex of $\bar{\Gamma}$ corresponds to a cell of X, and an edge of $\bar{\Gamma}$ corresponds to a pair of cells (a cell and a principal face). We assign to the vertex b of $\bar{\Gamma}$, i.e., b a cell of X, the stabilizer subgroup $\mathcal{G}_b = G_{s(b)}$. We assign to the 1-cell γ of $\bar{\Gamma}$, i.e., the pair of cells $b \subset c$ where $b = t(\gamma)$ is a principal face of $c = o(\gamma)$, the monomorphism $\mathcal{G}_c \to \mathcal{G}_b$ given by $a \mapsto a^{g(\gamma)} = g(\gamma)^{-1} a g(\gamma)$. Note that γ corresponds to an occurrence of b in the attaching map of c—there may be other occurrences. Note that $\mathcal{G}_c^{g(\gamma)} \subset \mathcal{G}_b$ by condition (2) above.

To define the corner labeling function, we associate to each directed corner α of X at a vertex v the unique edge-path $\bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_4$ in Γ (of length four) with endpoints at v going around the corner in the appropriate direction. The label on α is then defined as

$$\phi(\alpha)=g(\gamma_1)^{-1}g(\gamma_2)^{-1}g(\gamma_3)g(\gamma_4)$$

It can be readily verified that $(X, \mathcal{G}, \phi)$ is a complex of groups and is developable.

Let $p:K\to X$ be a complex of spaces, constructed as in § 1, for $(X, \mathcal{G}, \phi)$. Recall that each cell of X has a canonical "lift" to K, i.e., the preimage of each n-cell of X contains a unique n-cell of K. In this way, the principal face graph $\bar{\Gamma}\subset X$ can be "lifted" to K; identify $\bar{\Gamma}$ with the image of such an embedding into K. Let $\pi_1(K)$ be the fundamental group defined using the tree T as an "extended base point" (in an analogous fashion to that of [**7**] for graphs of groups). Then $\pi_1(K)$ is generated by the edges of $\bar{\Gamma}\setminus T$ and the edges of the vertex spaces K_v, $v\in X^1$.

Given a directed edge z in K_v, denote by $g(z)$ the image of the element it represents in $\pi_1(K)$ via the natural homomorphism $\pi_1(K_v)\to G_v\subset G$. And for each directed edge z of $\bar{\Gamma}$, let $g(z)$ be as above. Then the function $z\mapsto g(z)$ determines a homomorphism $h:\pi_1(K)\to G$; see [**2**].

Let $H=\ker h$. Incidentally, it can be shown [**2**] that $\ker h$ is isomorphic to $\pi_1(Y)$. Denote by $\pi:K_H\to K$ the (regular) covering space corresponding to H, and identify G with the group of covering transformations as follows: Since T is contractible, its preimage in K_H is a disjoint union of homeomorphic copies of T that are permuted transitively by the covering group. Let $\tilde{T}$ be a fixed choice of such a lift of T. Given any directed edge z, in the set of generators specified above, let $\tilde{z}$ denote the unique lift of z to K_H with origin in $\tilde{T}$. Then identify $g(z)$ with the unique covering transformation such that $g(z)\,.\,\tilde{T}$ contains the terminus of $\tilde{z}$. It is easily seen, by elementary covering space theory, that this identification is merely the isomorphism of G with the group of deck transformations.

To complete our set-up, we observe that there is a unique G-equivariant map $\tilde{p}:K_H\to Y$ such that $\tilde{p}(\tilde{T})=s(T)$ and the diagram:

$$\begin{array}{ccc} K_H & \xrightarrow{\tilde{p}} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle q} \\ K & \xrightarrow{p} & X \end{array}$$

is commutative (see also a similar construction in [**4**]). To see this, let e be an open cell of X and put $p^{-1}(e)=K_e$. Note that $\pi^{-1}(K_e)$ is a disjoint union of copies of a covering space of K_e that are permuted transitively by G. Denote by $\tilde{K}_e$ the component of $\pi^{-1}(K_e)$ that meets $\tilde{T}$. Then for $g\in G$, the component $g(\tilde{K}_e)$ is mapped by $\tilde{p}$ onto the cell $g\,.\,s(e)$ of Y in the unique way that induces p on the orbit spaces.

We say that a circuit β in Y^1 and a cyclic word W over $(X, \mathcal{G}, \phi)$ are *related* if there exists a circuit w in K^1_H such that $\tilde{p}(w)=\beta$ and $\mathrm{Word}(\pi(w))=W$.

LEMMA 4.1. *Let β be a circuit in Y^1 and let W be a cyclic word related to β. Then β is null-homotopic in Y if and only if W is inessential, in which case*

$$\Delta_Y(\beta)=\Delta_{\mathcal{G}}(W).$$

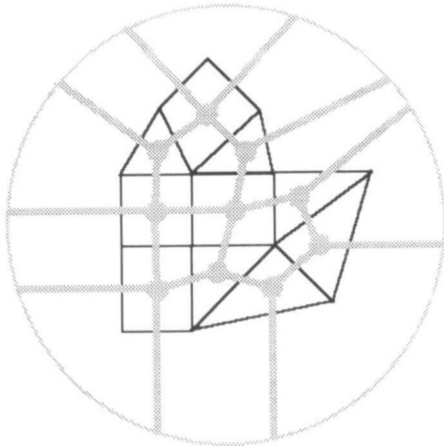

Figure 2

Proof. Fix a circuit w in K_H^1 such that $\tilde{p}(w) = \beta$ and $\mathrm{Word}(\pi(w)) = W$.

Suppose W is inessential. If W is a trivial word, then $\pi(w)$ is a (null-homotopic) circuit in some vertex space K_v, so w lies in some component of $\pi^{-1}(K_v)$. However, since each such component is mapped to a vertex of Y, $\tilde{p}(w) = \beta$ is a trivial circuit. Thus $\Delta_Y(\beta) = \Delta_{\mathscr{G}}(W) = 0$.

Having dealt with the trivial case, now assume that W is non-trivial. By Proposition 2.1, there is a Howie diagram (Ω, f, λ) whose cyclic boundary word U is equivalent to W and $a(\Omega) = \Delta_{\mathscr{G}}(W)$. Since W and U are equivalent, $\Delta_{\mathscr{G}}(W) = \Delta_{\mathscr{G}}(U)$. Our goal is to "lift" this Howie diagram to a singular disk, of the same area, spanning β.

Initially, we embed Ω in the interior of the standard closed unit ball B^2 and form a handle decomposition of B^2, dual to Ω, as follows. Let $\sigma_1, \ldots, \sigma_m$ be the open 2-cells of Ω, thus $m = a(\Omega)$, and choose an embedded disk (0-handle) D_i in σ_i for each i. Let $\Sigma^0 = \partial B^2 \cup (\bigcup D_i)$. As 1-handles, we choose a collection of disjoint "bands" (homeomorphic to $[0, 1] \times [0, 1]$) in B^2, one dual to each 1-cell of Ω, as depicted in Figure 2. Denote by Σ^1 the union of Σ^0 and all the 1-handles. Note that the closure of each component of $B^2 \backslash \Sigma^1$ is a disk (2-handle) containing a unique vertex of Ω.

Define a map $j_0 : \Sigma^0 \to K$ such that

- for each 1-handle E, say dual to the 1-cell e of Ω, the components of $E \cap \Sigma^0$ (attaching regions of E) are both mapped homeomorphically onto the unique 1-cell $f(e)^*$ of K in $p^{-1}(f(e))$;
- for each 0-handle D_i, $j_0 \mid \partial D_i$ is a piecewise linear representation of the attaching circuit of the 2-cell $p^{-1}(f(\sigma_i))$, and $j_0 \mid \mathrm{Int}\, D_i$ is a homeomorphism onto this open 2-cell of K;
- $j_0 \mid \partial B^2$ is a piecewise linear representation of the circuit $\pi(w)$.

Next we define $j_1 : \Sigma^1 \to K$, extending j_0, as follows. Let C be the closure of a component of $B^2 \backslash \Omega^1$, say with boundary cycle $e_1 \ldots e_n$. Then $\Sigma^1 \cap C$ is the union of a 0-handle D_k (or ∂B^2) and a collection $E_1, \ldots, E_n$ of 4-gons (halves of 1-handles), where E_i connects ∂D_k

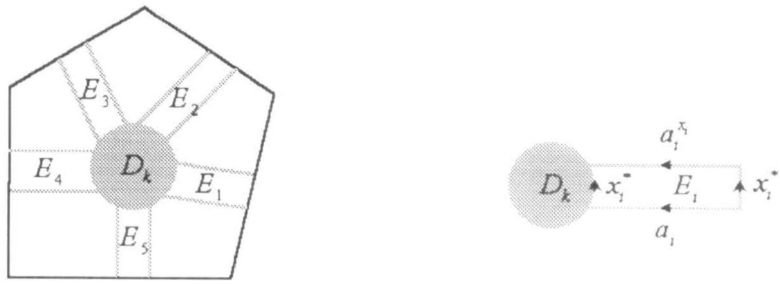

Figure 3

(or ∂B^2) to e_i; refer to Figure 3(a). By the definition of a Howie diagram (or because the boundary word is equivalent to W), there exists a sequence $(a_1, \ldots, a_n)$, $a_i \in \mathcal{G}_{x_i}$, such that for each directed corner α of Ω, say from $\bar{e}_i$ to e_{i+1}, $\lambda(\alpha) = a_i^{x_i}\phi(f(\alpha))a_{i+1}^{-1}$ where $x_i = f(e_i)$. Recall that each nontrivial element of $\mathcal{G}_{x_i}$ is a generator, thus an edge in $K_{o(x_i)}$. Then $j_1 \mid E_i$ is, defined to be, the combinatorial map into K indicated in Figure 3(b).

Finally, we define $j_2: B^2 \to K$, extending j_1, thus: Let V be a 2-handle in B^2, say dual to the vertex v of Ω. Note that $j_1 \mid \partial V$ is a null-homotopic loop in K_v representing the product of the corner labels around the vertex v. Hence we can extend to a map of V into $K_v \subset K$. Let j_2 be the extension of j_1, defined in this way.

Recall that $j_2 \mid \partial B^2$ is a piecewise linear representation of the circuit $\pi(w)$. Hence there is a unique lift $\tilde{j}_2: B^2 \to K_H$, covering j_2, such that $\tilde{j}_2 \mid \partial B^2$ is a piecewise linear representation of the circuit w. Now (B^2, j), where $j = \tilde{p} \circ \tilde{j}_2: B^2 \to Y$, is a singular disk in Y spanning $\tilde{p}(w) = \beta$. So β is null-homotopic. Moreover, note that the restriction of j to a 1- or 2-handle of B^2 contributes no area, as the image of such a handle, via j, is an edge or vertex of Y, accordingly. Hence, the area of (B^2, j) equals m, the number of 0-handles, which is the area of Ω. Consequently, $\Delta_Y(\beta) \leq \Delta_{\mathcal{G}}(W)$.

Conversely, suppose β is null-homotopic in Y and let (B^2, j) be a minimal area, singular disk in Y spanning β. Denote by $D_1, \ldots, D_m$ the collection of disjoint disks whose interiors form $j^{-1}(Y \setminus Y^1)$, and let ξ be the preimage of the set of midpoints of edges of Y; so ξ is a 1-manifold. Put $\Lambda = \partial B^2 \cup (\bigcup D_i) \cup \xi$ and form a 2-complex Ω, embedded in B^2, dual to Λ.

First we construct a singular disk (B^2, j_H) in K_H for w in the following manner. Initially, for each i, we define $j_H \mid D_i$ so that Int D_i is mapped homeomorphically onto an open 2-cell in $\tilde{p}^{-1}(j(D_i))$ such that on the finite set of points $D_i \cap \xi$, $\tilde{p} \circ j_H \mid (D_i \cap \xi) = j \mid (D_i \cap \xi)$. Likewise, $j_H \mid \partial B^2$ is chosen to be a piecewise linear map representing the circuit w such that $\tilde{p} \circ j_H \mid (\partial B^2 \cap \xi) = j \mid (\partial B^2 \cap \xi)$.

Next we define $j_H \mid \xi$. Let E be a component of ξ, and let P be the midpoint of a 1-cell of Y such that $E \subset j^{-1}(P)$. Then we choose a path in $\tilde{p}^{-1}(P)$ joining the points $j_H(\partial E)$; it should be remembered that $\tilde{p}^{-1}(P)$ is path connected.

Having defined j_H on Λ, let V be the closure of a component of $B^2 \setminus \Lambda$. Then $j(V) \subset N_v$ where N_v is the closed neighbourhood in Y^1, of a vertex v, consisting of the initial half of each directed edge y such that $o(y) = v$. Thus $j_H(\partial V) \subset \tilde{p}^{-1}(N_v)$, which we claim is 1-connected. To see this, note that $q(N_v) = N_{q(v)}$ is such that $K_{q(v)}$ is a strong deformation retract of $p^{-1}(N_{q(v)})$. It follows that the homomorphism induced on fundamental groups by the inclusion $p^{-1}(N_{q(v)}) \to K$ is injective. Hence every component of the preimage of $p^{-1}(N_{q(v)})$ in K_H, for example $\tilde{p}^{-1}(N_v)$, is a copy of its universal cover. Therefore, by the claim, $j_H \mid \partial V$ extends to a map of V into $\tilde{p}^{-1}(N_v)$. It is in this way that we extent the definition of j_H to all of B^2.

Projecting, via π, gives a singular disk $(B^2, \bar{j})$ in K spanning $\pi(w)$ where $\bar{j} = \pi \circ j_H$. Now we construct a Howie diagram Ω with cyclic boundary word equivalent to $\mathrm{Word}(\pi(w)) = W$ as in the proof of Proposition 2.1. Note that $a(\Omega) = m$, the number of subdisks in Λ, as $\bar{j}(B^2 \setminus \Lambda) \subset X^1$. It follows that W is inessential and that $\Delta_{\mathcal{G}}(W) \leq \Delta_Y(\beta)$, thus completing the proof. □

THEOREM 4.2. *If Y is a connected G-complex and $(X, \mathcal{G}, \phi)$ is an associated complex of groups, then $\delta_Y = \delta_{\mathcal{G}}$.*

Proof. Since a circuit, $\beta \subset Y^1$ and a related cyclic word W have the same length, the result follows directly from Lemma 4.1. □

Using Theorem 4.2 and Theorem 3.1, we immediately get the following:

COROLLARY 4.3. *Suppose G is a group. Let Y be a cocompact, 1-connected G-complex where each vertex stabilizer is finitely presented and each edge stabilizer is finite. For each vertex v of Y, let δ_v be a Dehn function for the stabilizer subgroup G_v, and let* $\delta = \max\{\delta_v \mid v \in Y^{(0)}\}$. Then

$$\delta_G \preccurlyeq \bar{\delta} \circ \bar{\delta}_Y.$$

A short argument gives the following result:

COROLLARY 4.4. *Let Y be a 1-connected cell complex, acted upon properly discontinuously and cocompactly by a group G. Then $\delta_G \equiv \delta_Y$.*

Proof. The hypothesis implies that each cell stabilizer—in particular for the vertices as well as the edges—is a finite group, and that there are only finitely many such groups, up to isomorphism. As we showed in the proof of Theorem 3.1 (here we take δ_G to be the Dehn function of the constructed complex K)

$$\delta_G(l) \leq C_1 \,.\, h_{\mathcal{G}}(l) + \bar{\delta}(C_2 \,.\, l + C_3 \,.\, h_{\mathcal{G}}(l))$$

for some constants C_1, C_2, C_3. The function δ is linear here. Hence it follows that $\delta_G \preccurlyeq h_{\mathcal{G}}$. Combining this with Theorem 4.1 yields $\delta_G \preccurlyeq \delta_Y$.

For the converse, we note that in our construction of K, we can take the generating set of each vertex stabilizer $\mathcal{G}_v$ to be the finite set $\mathcal{G}_v \setminus \{1\}$. It follows that given a cyclic word W of length n, there is a circuit w in K of length at most $2n$ such that $\mathrm{Word}(w) = W$. Moreover, if W is inessential then $\Delta_{\mathcal{G}}(W) \leq \delta_K(|w|)$; see the construction in the proof of Proposition 2.1 of a Howie diagram from a singular disk spanning w. Thus, $h_{\mathcal{G}}(n) \leq \delta_K(2n)$ for all $n \in \mathbb{N} \setminus \{0\}$, and hence $h_{\mathcal{G}} \preccurlyeq \delta_K$. But $\delta_G = \delta_K$, by construction, and $\delta_Y = h_{\mathcal{G}}$, by Theorem 4.2, thus giving the desired inequality.

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