THE K-PRODUCT OF ARITHMETIC FUNCTIONS

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1. Introduction. In this note we introduce a natural generalization of the ordinary convolution of arithmetic functions: If f and g are arithmetic functions,

$$(f \times g)(n) = \sum_{ab=n} f(a)g(b)K((a, b))$$

defines the K-*product* of f and g. If the kernel $K(n) \equiv E(n) = 1$, the K-product is the ordinary convolution $\sum_{d|n} f(d)g(n/d)$; if $K(n) \equiv \epsilon(n) = [1/n]$, then the K-product is the unitary product $\sum f(d)g(n/d)$, summed over $d|n$, $(d, n/d) = 1$ **(1, 2)**. We give in Theorem 1 a characterization of all associative kernels, i.e., kernels for which the corresponding K-product is associative.

In the latter half of this paper we study multiplicative functions under the K-product. It is shown that under certain conditions the function $\epsilon(n)$ (defined above) is the identity for the K-product, a multiplicative function has a multiplicative inverse, and the K-product of multiplicative functions is multiplicative. Finally, we derive some general identities involving multiplicative functions defined in terms of the K-product.

2. The associative kernels.

THEOREM 1. *The K-product is associative if and only if either $K(n) \equiv 0$ or $K(n)$ is of the form*

$$K(n) = \begin{cases} O, & \text{if } m \nmid n, \\ K(m) \prod_{\substack{q^b \| n \\ q \nmid m}} K^*(q^b), & \text{if } m|n, \end{cases}$$

where m is the smallest integer such that $K(m) \neq 0$, and $K^*(n) \equiv K(mn)/K(m)$ is a multiplicative function having values $K^*(p^a) = 1$ for all a if $p|m$, and if $q \nmid m$, $K^*(q^a) = 0$ or $K^*(q^{B(q)})$ according as $a < B(q)$ or $a \geqslant B(q)$, for $B(q)$ a positive integer or ∞.

Proof. Since

$$((f \times g) \times h)(n) = \sum_{abc=n} f(a)g(b)h(c)K((a, b))K((ab, c))$$

and

$$(f \times (g \times h))(n) = \sum_{abc=n} f(a)g(b)h(c)K((a, bc))K((b, c)),$$

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it is clear that the K-product is associative if and only if

(1) $$K((a, b))K((ab, c)) = K((a, bc))K((b, c)) \qquad \text{for all } a, b, c.$$

First, suppose K is a given function satisfying (1), and K is not identically zero. Let m be the smallest integer such that $K(m) \neq 0$. If n is any integer and $K(n) \neq 0$, take $a = m$, $b = n$, $c = mn$ in (1). We have

$$K((m, n))K(mn) = K(m)K(n).$$

The right side is not zero, so $K((m, n)) \neq 0$. By definition of m, then, $m \leqslant (m, n)$. But $(m, n)|m$, so n is a multiple of m. Thus, if $K(n) \neq 0$, then $m|n$, or as stated in the theorem $K(n) = 0$ if $m \nmid n$.

We consider now the function $K^*(n) \equiv K(mn)/K(m)$. Clearly $K^*(1) = 1$. Let $(r, s) = 1$, and replace a by ms, b by mr, and c by mrs in (1):

$$K(m)K(mrs) = K(mr)K(ms).$$

Dividing both sides by $(K(m))^2$, we have $K^*(rs) = K^*(r)K^*(s)$, so K^* is multiplicative.

NOTE. If $K(1) = 1$, then $K(n) = K^*(n)$ is multiplicative, so $K(1) = 1$ *is necessary and sufficient for* $K(n)$ *to be multiplicative.*

In view of the multiplicativity of K^*, it suffices to find K^* at the prime powers. Let N be a positive integer, q any prime, and x, y, z integers such that $0 \leqslant x \leqslant y \leqslant z$. Take $a = Nq^x$, $b = Nq^y$, $c = Nq^z$. Then (1) yields

(2) $$K(Nq^x)K(Nq^{\min(x+y,z)}) = K(Nq^x)K(Nq^y), \qquad 0 \leqslant x \leqslant y \leqslant z.$$

Suppose q is any prime divisor of m. In (2), take $N = m/q$, $x = y = 1$, and $z = 2$. We have $K(m)K(mq) = K(m)K(m)$, and since $K(m) \neq 0$, $K(mq) = K(m)$ or $K^*(q) = 1$. Assume that $K^*(q^t) = 1$, $t > 0$. In (2), take $N = m$, $x = y = t$, $z = t + 1$ to get

$$K(mq^t)K(mq^{t+1}) = K(mq^t)K(mq^t).$$

Dividing both sides by $(K(m))^2$ and applying the inductive assumption, we obtain $K^*(q^{t+1}) = K^*(q^t) = 1$. This proves that $K^*(q^a) = 1$ for all a, if $q|m$.

Consider now $K^*(q^a)$ for prime q, $q \nmid m$. If $k^*(q^a) = 0$ for $a = 1, 2, \ldots$, define $B(q) = \infty$, and if $K^*(q^a) \neq 0$ for some positive a, define $B(q)$ to be the least such a. Obviously if $B(q) = \infty$ for every $q \nmid m$, the function K^* has been completely determined. If there is a prime q such that $B(q) < \infty$, put $N = m$, $z = x + 1$, $x = y = B(q)$ in (2). Dividing by $(K(m))^2$, we get

(3) $$K^*(q^B)K^*(q^{B+1}) = K^*(q^B)K^*(q^B), \qquad B \equiv B(q).$$

Since $K^*(q^B) \neq 0$, by definition of B, $K^*(q^{B+1}) = K^*(q^B)$. Continuing by induction, we obtain

$$K^*(q^B) = K^*(q^{B+1}) = K^*(q^{B+2}) = \ldots .$$

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Together with the previous results, this shows that

$$K^*(q^b) = \begin{cases} 1, & \text{for all } b \geqslant 0 \text{ if } q|m, \\ 0, & \text{if } q \nmid m \text{ and } b < B(q), \\ K^*(q^{B(q)}), & \text{if } q \nmid m \text{ and } b \geqslant B(q). \end{cases}$$

Suppose n is any multiple of m. Then n can be written uniquely as

$$n = m^a \prod_{\substack{q^b \| n \\ q \nmid m}} q^b, \qquad a \geqslant 1,$$

and

$$K(n) = K(m)K^*(m^{a-1} \prod q^b) = K(m)K^*(m^{a-1}) \prod K^*(q^b)$$
$$= K(m) \prod K^*(q^b).$$

This completes the first part of the proof.

It remains to show that every such function satisfies (1). Suppose a, b, c are any positive integers. If m fails to divide (a, b, c), then at least one factor on each side of (1) vanishes. If (a, b, c) has an exact divisor q^b, $q \nmid m$, with $0 < b < B(q)$, then again at least one factor on each side of (1) vanishes, so we may assume that $K((a, b, c)) \neq 0$. Then

$$\begin{aligned} K((a, b))K((ab, c)) &= (K(m))^2 \prod_{\substack{q^b \| (a,b) \\ q \nmid m}} K^*(q^b) \prod_{\substack{q^b \| (ab,c) \\ q \nmid m}} K^*(q^b) \\ &= (K(m))^2 \prod_{q | (a,b)} K^*(q^{B(q)}) \prod_{q | (ab,c)} K^*(q^{B(q)}) \\ &= (K(m))^2 \prod_{q | (a,bc)} K^*(q^{B(q)}) \prod_{q | (b,c)} K^*(q^{B(q)}) \\ &= (K(m))^2 \prod_{q^b \| (a,bc)} K^*(q^b) \prod_{q^b \| (b,c)} K^*(q^b) \\ &= K((a, bc))K((b, c)). \end{aligned}$$

The proof is complete.

In the balance of this paper we consider only K-products with associative kernels.

3. Evidently $K(1) \neq 0$ is necessary and sufficient for the K-product operation to have the identity

$$\epsilon_K(n) = \begin{cases} 1/K(1), & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

and in particular if $K(1) = 1$, the identity is $\epsilon(n)$. If $K(1) \neq 0$ and f is any arithmetic function, the inverse f^{-1} (if is exists) is defined by $f \times f^{-1} = \epsilon_K$.

THEOREM 2. *If* $K(1) \neq 0$, *the inverse* f^{-1} *exists if and only if* $f(1) \neq 0$.

Proof. If $K(1) \neq 0$ and f^{-1} exists, then $(f \times f^{-1})(1) = \epsilon_K(1)$, or

$f(1)f^{-1}(1)K(1) = 1/K(1)$, so $f(1) \neq 0$. Conversely, if $K(1) \neq 0$ and $f(1) \neq 0$, then the defining relation $f \times f^{-1} = \epsilon_K$ can be used to construct f^{-1} by induction: $f^{-1}(1) = 1/f(1)(K(1))^2$, and if $f^{-1}(n)$ has been constructed for $1 \leqslant n \leqslant c$, then

$$f^{-1}(c+1) = \frac{-1}{f(1)K(1)} \sum_{\substack{ab=c+1 \\ b<c+1}} f(a)f^{-1}(b)K((a,b)).$$

COROLLARY (Inversion Formula). *If $K(1) \neq 0$, then $\mu^* \equiv E^{-1}$ exists, and for any functions f and g, $f \times E = g$ if and only if $f = g \times \mu^*$.*

THEOREM 3. *The inverse of a multiplicative function is multiplicative if and only if $K(n)$ is multiplicative.*

Proof. As noted earlier, $K(1) = 1$ is equivalent to $K(n)$ being multiplicative. Thus, assume that $K(1) = 1$ and suppose that f is multiplicative. Then $f(1) = 1$ and $f^{-1}(1) = 1/f(1)(K(1))^2 = 1$. Suppose we have shown that $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$ whenever $(m, n) = 1$ and $1 \leqslant mn < rs$. If $(r, s) = 1$, then

$$\epsilon(rs) = \sum_{\substack{a|r \\ b|s}} f(ab)f^{-1}(rs/ab)K((ab, rs/ab)),$$

$$0 = \left\{ \sum_{a|r} f(a)f^{-1}(r/a)K((a, r/a))\right\}\left\{ \sum_{b|s} f(b)f^{-1}(s/b)K((b, s/b))\right\}$$
$$+ f(1)f^{-1}(rs)K(1) - \{f(1)f^{-1}(r)K(1)\}\{f(1)f^{-1}(s)K(1)\},$$

$$0 = \epsilon(r)\epsilon(s) + f^{-1}(rs) - f^{-1}(r)f^{-1}(s).$$

Since $rs > 1$, at least one of $\epsilon(r)$, $\epsilon(s)$ is zero, and we have $f^{-1}(rs) = f^{-1}(r)f^{-1}(s)$.

Conversely, suppose the multiplicativity of f implies that of f^{-1}, so that $f(1) = f^{-1}(1) = 1$. Then $(f \times f^{-1})(1) = \epsilon_K(1)$, or $f(1)f^{-1}(1)K(1) = 1/K(1)$, and $K(1) = 1$.

THEOREM 4. *The K-product of two multiplicative functions is multiplicative if and only if $K(n)$ is multiplicative.*

Proof. The necessity is immediate if we consider the K-product at $n = 1$. To prove that the condition is sufficient, assume that $K(1) = 1$, f and g are multiplicative, and $(m, n) = 1$. Then

$$\begin{aligned}(f \times g)(mn) &= \sum_{\substack{a|m \\ b|n}} f(ab)g(mn/ab)K((ab, mn/ab)) \\ &= \sum f(ab)g(mn/ab)K((a, m/a)(b, n/b)) \\ &= \sum f(a)f(b)g(m/a)g(n/b)K((a, m/a))K((b, n/b)) \\ &= \left\{ \sum_{a|m} f(a)g(m/a)K((a, m/a))\right\}\left\{ \sum_{b|n} f(b)g(n/b)K((b, n/b))\right\} \\ &= (f \times g)(m) \cdot (f \times g)(n).\end{aligned}$$

We now confine our attention to operations with multiplicative kernels. Suppose $I(n)$ is a multiplicative function *which is never zero.* Recall that $\mu^* \equiv E^{-1}$.

DEFINITION. *For positive integers m and n,*

$$A(n, m) \equiv \begin{cases} I(m) & \text{if } m|n, \\ 0 & \text{if } m \nmid n, \end{cases}$$

and

$$B(n, m) \equiv A(n, m) \times \mu^*(m).$$

By applying the inversion formula on the latter definition, we have

$$A(n, m) = \sum_{ab=m} B(n, a)K((a, b)).$$

The function $A(n, m)$ is multiplicative in m; that is, if $(r, s) = 1$, then $A(n, r)A(n, s) = A(n, rs)$. If $r|n$ and $s|n$, this result follows by the multiplicativity of $I(n)$, and if at least one of r, s does not divide n, then rs does not divide n, and both $A(n, r)A(n, s)$ and $A(n, rs)$ vanish. By Theorem 3, μ^* is multiplicative, and it follows by Theorem 4 that $B(n, m)$ is multiplicative in m.

Notice that $B(1, m) = \sum_{d|n} A(1, d)\mu^*(m/d)K((d, m/d)) = A(1, 1)\mu^*(m)K(1) = \mu^*(m)$.

4. In this section we develop some identities using the functions introduced above. For this purpose we require the following lemma.

LEMMA. *If $K(m) = O(1)$ and $\mu^*(m) = O(1)$, then for fixed n $B(n, m) = O(1)$.*

Proof. Suppose $|K(m)| < M_1$ and $|\mu^*(m)| < M_2$. Then

$$\begin{aligned} |B(n, m)| &= \left| \sum_{d|m} A(n, d)\mu^*(m/d)K((d, m/d)) \right| \\ &= \left| \sum_{\substack{d|m \\ d|n}} I(d)\mu^*(m/d)K((d, m/d)) \right| \\ &\leqslant M_1 M_2 M(n)\tau(n), \end{aligned}$$

where $M(n) = \max_{d|n}|I(d)|$ is independent of m.

DEFINITION. *If $K(n) = O(1)$, i is a positive integer, and s is real $(s > 1)$, then*

$$\zeta(i, s) \equiv \sum_{n=1}^{\infty} \frac{K((i, n))}{n^s}.$$

Remark. For any $s > 1$, $\zeta(i, s) = O(1)$ uniformly in i if $K(n) = O(1)$.

Let $F(x, y)$ denote any function of two *real* variables. If n is a positive integer and x is real $(x \geqslant n)$, then

$$\text{(4)}\qquad \sum_{ab=n} F(a,b) = \sum_{t\leqslant x} \frac{A(n,t)F(t,n/t)}{I(t)}$$
$$= \sum_{t\leqslant x} \frac{F(t,n/t)}{I(t)} \sum_{t|d} B(n,d)K((d,t/d))$$
$$= \sum_{d\leqslant x} B(n,d) \sum_{c\leqslant x/d} \frac{K((c,d))F(cd,n/cd)}{I(cd)}.$$

THEOREM 5. *If* $K(n) = O(1)$, $\mu^*(n) = O(1)$, *then*

$$n^{-s} \sum_{ab=n} I(a)b^s = \sum_{d=1}^{\infty} B(n,d)\zeta(d,s)d^{-s}, \qquad s > 1.$$

Proof. In (4), take $F(x,y) = I(x)y^s$, $s > 1$. Then

$$\text{(5)}\qquad \sum_{ab=n} I(a)b^s = n^s \sum_{d\leqslant x} B(n,d) \sum_{c\leqslant x/d} K((c,d))(cd)^{-s}, \qquad x \geqslant n.$$

But the inner sum on the right is equal to

$$\sum_{c\leqslant x/d} K((c,d))(cd)^{-s} = \sum_{c=1}^{\infty} K((c,d))(cd)^{-s} - \sum_{c=[x/d]+1}^{\infty} K((c,d))(cd)^{-s}$$
$$= \zeta(d,s)d^{-s} + O\left(\int_{x/d}^{\infty} d^{-s}y^{-s}dy\right)$$
$$= \zeta(d,s)d^{-s} + O(1/dx^{s-1}).$$

Substituting this into (5), we obtain:

$$n^{-s} \sum_{ab=n} I(a)b^s = \sum_{d\leqslant x} B(n,d)\{\zeta(d,s)d^{-s} + O(1/dx^{s-1})\}$$
$$= \sum_{d\leqslant x} B(n,d)\zeta(d,s)d^{-s} + O((\log x)/x^{s-1}),$$

by the lemma. Now let $x \to \infty$ and the proof is complete.

Among the special cases of Theorem 5 is the following well-known result (**4**, p. 184).

COROLLARY (Ramanujan). *If* $s > 1$, *then*

$$n^{1-s}\sigma_{s-1}(n) = \zeta(s) \sum_{d=1}^{\infty} c_d(n)d^{-s},$$

where $\zeta(s)$ *is the Riemann zeta function and* $c_d(n)$ *is Ramanujan's trigonometric sum.*

Proof. Take $K = E$ and $I(n) = n$. Then the K-product is the ordinary convolution and μ^* is the Möbius function, so the boundedness hypotheses are satisfied. Moreover, $\zeta(n,s)$ is the Riemann zeta function when $K = E$.

Finally, by (3, p. 237)

$$B(n, d) = \sum_{a|d} A(n, a)\mu(d/a) = \sum_{\substack{a|d \\ a|n}} a\mu(d/a) = c_d(n).$$

Since $B(1, m) = \mu^*(m)$, taking $n = 1$ in the above theorem, we obtain:

COROLLARY. *If* $K(n) = O(1)$ *and* $\mu^*(n) = O(1)$, *then*

$$1 = \sum_{d=1}^{\infty} \mu^*(d)\zeta(d, s)d^{-s}, \qquad s > 1.$$

THEOREM 6. *If* $K(n) = O(1)$, $\mu^*(n) = O(1)$, $s > 1$, *and* p *is prime, then*

$$\sum_{ab=n} I(a)a^{-s}K((a, p)) = \sum_{d=1}^{\infty} B(n, d)d^{-s}K((d, p))\zeta(dp, s).$$

Proof. In (4), take $F(x, y) = I(x)x^{-s}K((x, p))$. Then

$$\sum_{ab=n} I(a)a^{-s}K((a, p)) = \sum_{d\leqslant x} B(n, d) \sum_{c\leqslant x/d} K((c, d))K((cd, p))(cd)^{-s}.$$

But $K((c, d))K((cd, p)) = K((c, dp))K((d, p))$ by (1). After this substitution the arguments are similar to those in the proof of Theorem 5.

An interesting special case arises if $K(p) = 0$. Then, since $K(1) = 1$, the right side is the series $\sum B(n, d)\zeta(dp, s)d^{-s}$, summed over d, $(d, p) = 1$. And if $I = E$, the left side is $n^{-s} \sum d^s$, summed over $d|n$ and $(d, p) = 1$.

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