SIMULTANEOUS AUTOMORPHISMS IN SPACES OF ANALYTIC FUNCTIONS

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1. Introduction. Two spaces of analytic functions are considered, each comprised of functions analytic on the open disk $N_R(0)$ of radius R $(0 < R < +\infty)$ centred at the origin. The first space $\mathfrak{F}$ consists of all analytic functions on $N_R(0)$ topologized according to the metric of uniform convergence on compact sets. As the second space we allow any Fréchet space $\mathfrak{U}$ of analytic functions on $N_R(0)$ for which the topology is stronger than that induced by $\mathfrak{F}$. Our objective is then to present a scheme for constructing simultaneous automorphisms on $\mathfrak{F}$ and $\mathfrak{U}$.

The term *Fréchet space* is used as in Bourbaki (**4**, pp. 59, 110) to signify a metrizable, complete, locally convex, topological linear space over the real or complex field; however, attention here will naturally be confined to the complex field. By an *automorphism* on a topological linear space $\mathfrak{X}$ we mean a linear homeomorphic mapping of $\mathfrak{X}$ onto itself. A *simultaneous automorphism* on $\mathfrak{F}$ and $\mathfrak{U}$ is then a mapping T such that T is an automorphism on $\mathfrak{F}$ and $T|\mathfrak{U}$ is an automorphism on $\mathfrak{U}$.

Although simultaneous automorphisms are of interest in their own right, we mention two connections with other areas of mathematics. In the first place, since the topology on $\mathfrak{U}$ is stronger than that induced by $\mathfrak{F}$, the underlying structure here corresponds to that for two-norm spaces (see **1**). Simultaneous automorphisms on $\mathfrak{F}$ and $\mathfrak{U}$ thus yield analogues of "two-norm automorphisms" on $\mathfrak{U}$. In the second place, simultaneous automorphisms can be used to extend classical results on series expansions of bounded analytic functions—for example, the theorems of Steffensen and Fejér (see **6**, chap. I, §§ 1 and 3). Briefly, the ideas are as follows. Certain properties are established for the partial sums in the Taylor expansions. The latter are expansions in $\mathfrak{F}$ relative to the fundamental basis $\{\delta_n\}$ defined by

$$\delta_n(z) = z^n \qquad (|z| < R;\ n = 0, 1, \ldots), \tag{1.1}$$

but the properties in question concern only the space $\mathfrak{B}$ of bounded analytic functions on $N_R(0)$ under the usual sup norm. These properties in $\mathfrak{B}$, as properties of bases in $\mathfrak{F}$, are preserved by mappings which are simultaneous automorphisms on the two spaces.

In any Fréchet space the topology is completely determined by a sequence of semi-norms, and for $\mathfrak{U}$ we shall use $||f||_\nu$ $(f \in \mathfrak{U};\ \nu = 0, 1, \ldots)$ to specify

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such a sequence. The space $\mathfrak{F}$ is itself a Fréchet space, and in this case the semi-norms are given by

$$M_r(f) = \max_{|z|\leqslant r} |f(z)| \qquad (f \in \mathfrak{F})$$

for $0 < r < R$. As is well known (see, for example, the corollary to Proposition 9, p. 101 of **4**), the condition that the topology on $\mathfrak{U}$ is stronger than that induced by $\mathfrak{F}$ can be expressed as follows: to each r $(0 < r < R)$ there correspond a constant K and a positive integer ν such that

$$M_r(f) \leqslant K\,||f||_\nu \tag{1.2}$$

for all f in $\mathfrak{U}$.

Included among the possibilities for $\mathfrak{U}$ are the Banach space $\mathfrak{B}$ and, more generally, the $\mathfrak{L}^p$ spaces $(1 \leqslant p \leqslant +\infty)$ consisting of all analytic functions f on $N_R(0)$ having finite L^p norm,

$$||f|| = \left\{ \int_{N_R(0)} |f|^p\,da \right\}^{1/p}$$

(where a denotes 2-dimensional Lebesgue measure), the topology being that determined by this norm. In general, the semi-norms for $\mathfrak{U}$ can all be taken equal to the norm whenever $\mathfrak{U}$ is a Banach space, and the subscript ν will be omitted altogether in this case.

Our main results appear as Theorem 3 and Corollary 3.1, but their scope is perhaps best illustrated by the specialization to Pincherle sequences. For this we have the following corollary.

COROLLARY 3.2. *Let*

$$\beta_n(z) = z^n[1 + \lambda_n(z)] \qquad (|z| < R;\ n = 0, 1, \ldots),$$

$\{\lambda_n\}$ being taken as any sequence of functions in $\mathfrak{L}^p(1 \leqslant p \leqslant +\infty)$ which vanish at the origin and for which the L^p norms satisfy

$$\limsup_{n\to\infty} ||\lambda_n||^{1/n} < 1. \tag{1.3}$$

If f is any function in $\mathfrak{F}$, having

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

as its Taylor expansion, then the series

$$g(z) = \sum_{n=0}^{\infty} c_n\,\beta_n(z)$$

converges uniformly for z on compact subsets of $N_R(0)$, and the transformation T defined by $g = Tf$ is a simultaneous automorphism on $\mathfrak{F}$ and $\mathfrak{L}^p$.

The techniques of proof make use of proper bases in $\mathfrak{F}$, and we summarize here the principal ideas. A *basis* in a topological linear space $\mathfrak{T}$ is a sequence

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$\{\alpha_n\}$ of elements of $\mathfrak{T}$ such that to every f in $\mathfrak{T}$ there corresponds a unique sequence $\{c_n\}$ of scalars for which

$$f = \sum_{n=0}^{\infty} c_n \alpha_n. \tag{1.4}$$

If $\{\alpha_n\}$ is a basis in $\mathfrak{F}$, we say that $\{\alpha_n\}$ is a *proper basis* provided the series (1.4) converges in $\mathfrak{F}$ when and only when $\{c_n\}$ is the Taylor coefficient sequence of some function in $\mathfrak{F}$, i.e. when and only when

$$\limsup_{n\to\infty} |c_n|^{1/n} \leqslant 1/R. \tag{1.5}$$

Proper bases can be characterized in terms of the following two conditions, applicable to any sequence $\{\phi_n\}$ of functions in $\mathfrak{F}$:

$$(\alpha) \qquad \limsup_{n\to\infty} [M_r(\phi_n)]^{1/n} < R \qquad (\text{all } r < R)$$

and

$$(\beta) \qquad \lim_{r\to R} \left\{\liminf_{n\to\infty} [M_r(\phi_n)]^{1/n}\right\} \geqslant R.$$

For a basis $\{a_n\}$ in $\mathfrak{F}$ to be proper it is necessary and sufficient that $\{\alpha_n\}$ satisfy both condition (α) and condition (β). (This result and the others which we proceed to state have been established in **(2)**.)

Let us suppose now that $\{\alpha_n\}$ is a proper basis in $\mathfrak{F}$. Condition (α), by itself, is necessary and sufficient for the series $\sum c_n\phi_n$ to converge in $\mathfrak{F}$ for all sequences $\{c_n\}$ of complex numbers satisfying (1.5). Hence, assuming $\{\phi_n\}$ to be any sequence of functions in $\mathfrak{F}$ for which (α) holds, we can define a mapping P of $\mathfrak{F}$ into itself by taking f as in (1.4) and setting

$$Pf = \sum_{n=0}^{\infty} c_n \phi_n. \tag{1.6}$$

Under these hypotheses P is a continuous linear mapping of $\mathfrak{F}$ into itself, and we shall refer to P as the *endomorphism mapping* $\{\alpha_n\}$ *onto* $\{\phi_n\}$. A salient feature of proper bases is that if $\{\alpha_n\}$ and $\{\beta_n\}$ are proper bases in $\mathfrak{F}$, then the endomorphism mapping $\{\alpha_n\}$ onto $\{\beta_n\}$ is, in fact, an automorphism on $\mathfrak{F}$.

2. A continuity condition for mappings of $\mathfrak{F}$ into $\mathfrak{U}$. Let us fix $\{\alpha_n\}$ as a proper basis in $\mathfrak{F}$ and $\{\phi_n\}$ as a sequence of functions in $\mathfrak{F}$ satisfying condition (α). The endomorphism mapping $\{\alpha_n\}$ onto $\{\phi_n\}$ will then be denoted by P, as above.

Our concern here lies in determining when P is a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$, and a criterion for this will now be derived by an argument paralleling that used in proving Lemma 4 of **(2)**.

LEMMA 1. *For P to be a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$ it is necessary and sufficient that all ϕ_n belong to $\mathfrak{U}$ and that*

$$\limsup_{n\to\infty} ||\phi_n||_\nu^{1/n} < R \qquad (\nu = 0, 1, \ldots). \tag{2.1}$$

The expansions in (1.6) *then converge in* $\mathfrak{U}$ *for all* f *in* $\mathfrak{F}$.

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Proof. In view of the fact that there exists an automorphism on $\mathfrak{F}$ mapping $\{\alpha_n\}$ onto $\{\delta_n\}$, there is no loss of generality in supposing, as we do, that $\alpha_n = \delta_n$ $(n = 0, 1, \ldots)$. Also, for notational brevity we shall fix ν arbitrarily and omit this subscript in the semi-norm notation.

Then, assuming (2.1) to hold, we can choose ρ such that

$$\limsup_{n\to\infty} ||\phi_n||^{1/n} < \rho < R. \tag{2.2}$$

Since this implies the existence of a real number C for which $||\phi_n|| \leqslant C\rho^n$ $(n = 0, 1, \ldots)$, it follows that the series in (1.6) converges in $\mathfrak{U}$ (the convergence is, in fact, absolute). Hence, P maps $\mathfrak{F}$ into $\mathfrak{U}$. To show that this mapping is continuous, we choose r as any number such that $\rho < r < R$ and use the Cauchy inequalities on the Taylor coefficients c_n of f to write

$$|c_n| \leqslant M_r(f)/r^n \qquad (n = 0, 1, \ldots).$$

Continuity then results from the inequality

$$||Pf|| \leqslant \frac{Cr}{r-\rho} M_r(f).$$

Conversely, let us assume that P is a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$. If $\{c_n\}$ is any sequence of complex numbers for which the series in (1.4) converges, then $c_n\delta_n \to 0$ in $\mathfrak{F}$ and the continuity of P forces

$$|c_n| \cdot ||\phi_n|| \to 0. \tag{2.3}$$

It is easily seen from this that condition (2.1) must hold. Indeed, if (2.1) were not to hold, then for any sequence of positive numbers r_k increasing strictly to R we could find a sequence of positive integers n_k such that

$$||\phi_{n_k}||^{1/n_k} > r_k \qquad (k = 0, 1, \ldots),$$

the semi-norm appearing here being taken as one for which (2.1) fails. Setting

$$c_{n_k} = 1/||\phi_{n_k}|| \qquad \textit{and} \qquad c_n = 0 \qquad (n \neq n_k)$$

would then serve to specify $\{c_n\}$ as a sequence for which (1.5) holds (so that the series in (1.4) converges) but for which condition (2.3) is violated. This contradiction completes the proof.

A particularly simple condition for P to map $\mathfrak{F}$ continuously into $\mathfrak{U}$ is available whenever $\mathfrak{U}$ is comprised of bounded functions and the topology on $\mathfrak{U}$ is weaker than that given by the sup norm.

THEOREM 1. *Suppose that $\mathfrak{U}$ consists of bounded functions and that the topology on $\mathfrak{U}$ is weaker than that determined by the sup norm. If the functions ϕ_n $(n = 0, 1, \ldots)$ belong to $\mathfrak{U}$ and are uniformly continuous on $N_R(0)$, then P is a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$.*

Proof. It suffices to assume that $\mathfrak{U}$ is a subspace of $\mathfrak{B}$, and we then introduce the mappings P_t $(0 < t < 1)$ defined by

$$(P_t f)(z) = (Pf)(tz) \qquad (|z| < R)$$

for all f in $\mathfrak{F}$. Thus,

$$(P_t f)(z) = \sum_{n=0}^{\infty} c_n \, \phi_n^t(z) \qquad (|z| < R),$$

where

$$\phi_n^t(z) = \phi_n(tz) \qquad (|z| < R; n = 0, 1, \ldots).$$

From condition (α), with $r = tR$, we obtain

$$\limsup_{n\to\infty} ||\phi_n^t||^{1/n} < R,$$

and Lemma 1 assures us that P_t maps $\mathfrak{F}$ continuously into $\mathfrak{U}$.

The proof is completed by means of the Banach-Steinhaus theorem (**5**, p. 55, Theorem 18). In the first place, the family of mappings P_t $(0 < t < 1)$ is pointwise bounded, since obviously $||P_t f|| \leqslant ||Pf||$ for all t and each f in $\mathfrak{F}$. In the second place, the uniform continuity of the functions ϕ_n results in

$$\lim_{t\to 1} ||P_t \delta_n - P\delta_n|| = 0 \qquad (n = 0, 1, \ldots),$$

so that $\{P_t\}$ converges to P on a total subset of $\mathfrak{F}$. Consequently, P is a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$.

3. Application of proper bases to the construction of simultaneous automorphisms. We confine our attention henceforth to sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathfrak{F}$ for which the difference functions

$$\phi_n = \beta_n - \alpha_n \qquad (n = 0, 1, \ldots) \tag{3.1}$$

belong to $\mathfrak{U}$ and satisfy (2.1). Corresponding to any prescribed semi-norm for $\mathfrak{U}$ there then exists a number ρ such that (2.2) holds. In conjunction with (1.2) this shows that for any r $(0 < r < R)$ there are positive constants K and ρ $(<R)$ such that

$$M_r(\phi_n) \leqslant K\rho^n \qquad (n = 0, 1, \ldots), \tag{3.2}$$

These inequalities lead at once to the following lemma.

LEMMA 2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathfrak{F}$ for which the functions ϕ_n of* (3.1) *belong to $\mathfrak{U}$ and satisfy* (2.1). *Then the sequence $\{\beta_n\}$ satisfies condition* (α) *if and only if $\{\alpha_n\}$ does.*

Proof. By symmetry of the given data it suffices to establish that condition (α) holds for $\{\beta_n\}$ whenever it holds for $\{\alpha_n\}$. We thus assume that $\{\alpha_n\}$ satisfies condition (α). Then, fixing r $(0 < r < R)$ arbitrarily, we can choose the constant ρ in (3.2) large enough so that

$$\limsup_{n\to\infty} [M_r(\alpha_n)]^{1/n} < \rho.$$

Hence, for large n,

$$M_r(\beta_n) \leqslant M_r(\alpha_n) + M_r(\phi_n) \leqslant (1+K)\rho^n,$$

and it follows that $\{\beta_n\}$ satisfies condition (α).

Lemma 2 can be paraphrased for condition (β), but this direct counterpart turns out to be false; it fails, for example, when $\mathfrak{U} = \mathfrak{F}$ and

$$-\alpha_n(z) = \phi_n(z) = z^n \qquad (|z| < R;\ n = 0, 1, \ldots).$$

On the other hand, the resulting assertion is valid for $\mathfrak{L}^p$ spaces $(1 \leqslant p \leqslant +\infty)$ and, more generally, for all $\mathfrak{U}$ which are Banach spaces. We have, in fact, the following lemma.

LEMMA 3. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathfrak{F}$ for which the functions ϕ_n of* (3.1) *belong to $\mathfrak{U}$ and satisfy*

$$\sup_{\nu \geqslant 0} \Big[\limsup_{n\to\infty} ||\phi_n||_\nu^{1/n} \Big] < R. \tag{3.3}$$

Then the sequence $\{\beta_n\}$ satisfies condition (β) if and only if $\{\alpha_n\}$ does.

Proof. The hypothesis (3.3) is clearly equivalent to the existence of a number ρ $(<R)$ such that

$$\limsup_{n\to\infty} ||\phi_n||_\nu^{1/n} < \rho$$

for all ν $(\geqslant 0)$. This, in turn, implies that for each r $(0 < r < R)$

$$\limsup_{n\to\infty} [M_r(\phi_n)]^{1/n} < \rho, \tag{3.4}$$

in view of (1.2).

Assuming now that $\{\alpha_n\}$ satisfies condition (β), let us take σ as any number such that $\rho < \sigma < R$. For r sufficiently near R we then have

$$\liminf_{n\to\infty} [M_r(\alpha_n)]^{1/n} > \sigma,$$

and together with (3.4) this yields

$$M_r(\beta_n) \geqslant M_r(\alpha_n) - M_r(\phi_n) > \sigma^n - \rho^n$$

for large n. There results

$$\liminf_{n\to\infty} [M_r(\beta_n)]^{1/n} \geqslant \sigma \lim_{n\to\infty} [1 - (\rho/\sigma)^n]^{1/n} = \sigma.$$

Hence, $\{\beta_n\}$ satisfies condition (β), and the lemma follows by symmetry.

Inasmuch as conditions (α) and (β) are necessary and sufficient for a basis in $\mathfrak{F}$ to be proper, Lemmas 2 and 3 give rise to the following theorem.

THEOREM 2. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be bases in $\mathfrak{F}$ for which the functions ϕ_n of* (3.1) *belong to $\mathfrak{U}$ and satisfy* (3.3). *Then for $\{\beta_n\}$ to be proper it is necessary and sufficient that $\{\alpha_n\}$ be proper.*

Our method for generating simultaneous automorphisms hinges on the following simple result.

LEMMA 4. *Let $T = S + P$, where S is a simultaneous automorphism on $\mathfrak{F}$ and $\mathfrak{U}$ and P is a continuous linear mapping of $\mathfrak{F}$ into $\mathfrak{U}$. If T is an automorphism on $\mathfrak{F}$, then T is, in fact, a simultaneous automorphism on $\mathfrak{F}$ and $\mathfrak{U}$.*

Proof. Since the topology on $\mathfrak{U}$ is stronger than that on $\mathfrak{F}$, P maps $\mathfrak{U}$ continuously into itself, and the same is therefore true of T. We observe next that T actually maps $\mathfrak{U}$ *onto* itself. Indeed, if g is any point of $\mathfrak{U}$, so that $g = Tf$ for some f in $\mathfrak{F}$, then $Sf(= g - Pf)$ lies in $\mathfrak{U}$, and this forces f to lie in $\mathfrak{U}$. The open mapping theorem (**5**, p. 57, Theorem 2) then guarantees that $T|\mathfrak{U}$ is an automorphism on $\mathfrak{U}$.

Proper bases furnish a convenient tool for dealing with the conditions encountered in Lemma 4. Initially S can be taken as the identity mapping I, and subsequent choices for S can be made from among the simultaneous automorphisms T which result. However, to simplify the statements, we shall treat explicitly only the case of $S = I$.

THEOREM 3. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be proper bases in $\mathfrak{F}$, and let T be the endomorphism mapping $\{\alpha_n\}$ onto $\{\beta_n\}$. If the functions $\phi_n = \beta_n - \alpha_n$ $(n = 0, 1, \ldots)$ belong to $\mathfrak{U}$ and satisfy the condition*

$$\limsup_{n\to\infty} ||\phi_n||_\nu^{1/n} < R \qquad (\nu = 0, 1, \ldots),$$

then T is a simultaneous automorphism on $\mathfrak{F}$ and $\mathfrak{U}$.

Proof. For f any function in $\mathfrak{F}$, having

$$f = \sum_{n=0}^{\infty} c_n \alpha_n$$

as its expansion in the basis $\{\alpha_n\}$, Tf is given by

$$Tf = \sum_{n=0}^{\infty} c_n \beta_n = \sum_{n=0}^{\infty} c_n \alpha_n + \sum_{n=0}^{\infty} c_n \phi_n.$$

Thus, $T = I + P$, where P is defined as in (1.6). By Lemma 1, P maps $\mathfrak{F}$ continuously into $\mathfrak{U}$, and the theorem follows from Lemma 4.

A further result of the same sort is now immediate from Theorem 2.

COROLLARY 3.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be bases in $\mathfrak{F}$ for which the functions $\phi_n = \beta_n - \alpha_n$ $(n = 0, 1, \ldots)$ belong to $\mathfrak{U}$ and satisfy the condition*

$$\sup_{\nu\geqslant 0}\left[\limsup_{n\to\infty} ||\phi_n||_\nu^{1/n}\right] < R.$$

If one of the given bases is proper, then both are, and the endomorphism T mapping $\{\alpha_n\}$ onto $\{\beta_n\}$ is a simultaneous automorphism on $\mathfrak{F}$ and $\mathfrak{U}$.

We turn finally to the Pincherle case, presented in § 1 as Corollary 3.2. Here the functions ϕ_n have the form

$$\phi_n(z) = z^n \lambda_n(z) \qquad (|z| < R;\ n = 0, 1, \ldots),$$

where $\{\lambda_n\}$ is a sequence of functions in $\mathfrak{L}^p$ which vanish at the origin and satisfy (1.3). A simple application of the Cauchy inequalities shows that the coefficients in the Taylor expansions

$$\lambda_n(z) = \sum_{k=1}^{\infty} h_{nk}\, z^k \qquad (n = 0, 1, \ldots)$$

have the property that

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |h_{nk}|\, r^k = 0 \qquad (r < R).$$

Hence, by the theorem of Boas (**3**, Theorem 2), $\{\beta_n\}$ is a basis (in fact, a proper basis) in $\mathfrak{F}$. The evident inequalities

$$||\phi_n|| \leqslant R^n\, ||\lambda_n|| \qquad (n = 0, 1, \ldots)$$

ensure, moreover, that

$$\limsup_{n\to\infty} ||\phi_n||^{1/n} < R,$$

and Corollary 3.2 now appears as a consequence of either Theorem 3 or Corollary 3.1.

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