ON THE SUMMABILITY OF A CLASS OF DERIVED FOURIER SERIES

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1. Let $f(t)$ be integrable $L(-\pi, \pi)$ and periodic with period 2π and let

$$(1.1) \qquad \frac{1}{2} a_o + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be its Fourier series. The series

$$(1.2) \qquad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} n \, B_n(t)$$

obtained by term by term differentiation of the series (1.1) is called the derived Fourier series of f.

Suppose that $(\Lambda) = (\lambda_{n,k})$ is a triangular matrix, i.e. $\lambda_{n,k} = 0$ for $k \geq n+1$, which defines a regular sequence to sequence transformation [cf. 1, page 43, theorem 2].

If $\{s_n\}$ denotes the partial sum of the series (1.2) then the (Λ) transforms $\{t_n\}$ are given by

$$t_n = \sum_{k=1}^{n} \lambda_{n,k} \, s_k .$$

and the series (1.2) is said to be summable (Λ) to the sum s, if $t_n \to s$ as $n \to \infty$.

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Summability (Λ) of the series (1.1) has been considered by Petersen [2]. In this note we consider the summability (Λ) of the series (1.2).

2. We write

$$\psi_x(t) = f(x+t) - f(x-t) \ , \ g(t) = \frac{\psi_x(t)}{\sin t/2}$$

and prove

THEOREM A. *If* (Λ) *is a regular sequence to sequence triangular matrix such that*

$$(2.1) \qquad \sum_{k=2}^{n} k(\log k) \left| \lambda_{n,k} - \lambda_{n,k+1} \right| = O \ (\log n)$$

and if

$$(2.2) \qquad \int_t^{\pi} \frac{|g(u)|}{u} du = o \ (\log 1/t) \qquad (t \to 0+) \ ,$$

then $t_n = o \ (\log n)$.

We remark (as is readily proved using integration by parts) that (2.2) implies

$$(2.3) \qquad \int_0^t |g(u)| \, du = o \ (t \log 1/t) \qquad (t \to 0+) \ ,$$

while (2.2) is implied by

$$(2.4) \qquad \int_0^t |g(u)| \, du = o(t) \qquad (t \to 0+) \ .$$

For simplicity in the proof of the theorem, we shall denote

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$$(2.5) \qquad D_k(t) = \frac{\sin t/2}{2\pi} \frac{d}{dt} \left\{ \frac{\sin (k+\frac{1}{2})t}{\sin t/2} \right\}, \quad L_k(t) = \sum_{r=1}^{k} D_r(t) .$$

Then we can show that

$$L_k(t) = \frac{\sin t/2}{2\pi} \frac{d}{dt} \left\{ \frac{\sin^2 (k+1)t/2}{\sin^2 t/2} \right\} ,$$

and making use of the elementary inequalities

$$\left|\frac{\sin pu}{\sin u}\right| \leq p , \quad \left|\frac{d}{du} \left(\frac{\sin pu}{\sin u}\right)\right| \leq \frac{2p}{|\sin u|} , \quad (p = 1, 2, 3, \ldots) ,$$

$$2u/\pi < \sin u < u \qquad (0 < u < \pi/2) ,$$

it is easy to prove that

$$(2.6) \qquad |D_k(t)| < \frac{1}{2} k , \quad |L_k(t)| < \frac{1}{2} k^2 \qquad (k = 1, 2, 3, \ldots ; t \text{ real}) ;$$

$$(2.7) \qquad |L_k(t)| < \frac{2k}{t} \qquad (k = 1, 2, 3, \ldots ; 0 < t < \pi) .$$

3. Proof of Theorem A: It is easy to see that

$$r B_r(x) = \frac{1}{\pi} \int_0^{\pi} \{f(x+t) - f(x-t)\} r \sin rt \, dt$$

so that, if $\{s_n\}$ denotes the sequence of partial sums of the series (1.2) and $D_n(t)$ is defined by (2.5), then

$$- s_n = - \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{r=1}^{n} r \sin rt \, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{d}{dt} \left\{ \frac{1}{2} + \sum_{r=1}^{n} \cos rt \right\} dt$$

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$$= \int_0^{\pi} g(t)\, D_n(t)\, dt$$

(3.1)

$$= \left(\int_0^{1/n^2} + \int_{1/n^2}^{\pi} \right) g(t)\, D_n(t)\, dt$$

$$\equiv P_n + Q_n, \text{ say.}$$

Therefore, $|t_n| = \left| \sum_{k=1}^{n} \lambda_{n,k}\, s_k \right|$

(3.2) $$\leq \left| \sum_{k=1}^{n} \lambda_{n,k}\, P_k \right| + \left| \sum_{k=1}^{n} \lambda_{n,k}\, Q_k \right|$$

$$\equiv |J_1| + |J_2|, \text{ say .}$$

Now $|P_n| \leq \frac{1}{2} n \cdot o\left(\frac{1}{n^2}\, 2 \log n \right)$, by (2.5) and (2.4)

$$= o(1) \text{ as } n \to \infty ;$$

thus $\{P_n\}$ is a null sequence and hence, since $(\lambda_{n,k})$ is regular,

(3.3) $$|J_1| = o(1) \text{ as } n \to \infty .$$

Since $(\lambda_{n,k})$ is regular, we may assume without loss of generality that $\lambda_{n,k} = 0$ for $k = 1, 2$. By definitions (2.5) and (3.1), and applying partial summation,

$$J_2 \equiv \sum_{k=3}^{n} \lambda_{n,k}\, Q_k$$

$$= \sum_{k=3}^{n} \lambda_{n,k} \int_{1/k^2}^{\pi} g(t) \{ L_k(t) - L_{k-1}(t) \}\, dt$$

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$$= \sum_{k=2}^{n} (\lambda_{n,k} - \lambda_{n,k+1}) \int_{1/k^2}^{\pi} g(t) \; L_k(t)dt$$

$$- \sum_{k=2}^{n} \lambda_{n,k+1} \int_{1/(k+1)^2}^{1/k^2} g(t) \; L_k(t)dt$$

$$\equiv I_1 + I_2 \text{, say .}$$

Now

$$|I_1| \leq \sum_{k=2}^{n} |\lambda_{n,k} - \lambda_{n,k+1}| \int_{1/k^2}^{\pi} |g(t)| \frac{2k}{t} \, dt \text{, by (2.7)}$$

(3.4) $$= o\{ \sum_{k=2}^{n} | \lambda_{n,k} - \lambda_{n,k+1} | \, k \, . \, 2 \log k \} \text{, by (2.2)}$$

$$= o(\log n) \text{, by (2.1).}$$

Also $$|I_2| \leq \sum_{k=2}^{n} | \lambda_{n,k+1} | \int_{0}^{1/k^2} |g(t)| \; |L_k(t)| \, dt$$

(3.5) $$\leq \sum_{k=2}^{n} |\lambda_{n,k+1}| \frac{1}{2} k^2 \; o(\frac{1}{k^2} \; 2 \log k) \text{, by (2.6) and (2.3)}$$

$$= o(\log n) \text{,}$$

since the matrix $(\lambda_{n,k})$ is regular.

It now follows, on substituting (3.3), (3.4), (3.5) into (3.2), that $t_n = o(\log n)$, and the proof of the theorem is complete.

4. In particular if we choose $\lambda_{n,k} = \frac{1}{n+1}$ for $k \leq n$ and zero for $k > n$, the (Λ) method of summability reduces to the $(C, 1)$ method of summability. Also this choice of $(\lambda_{n,k})$ satisfies all the conditions imposed on the matrix in our

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theorem, so that the theorem A reduces to

THEOREM B. <u>If</u> $\{t_n\}$ <u>denotes the</u> (C, 1) <u>mean of the series</u> (1.2) <u>and if</u>

$$\int_t^{\pi} \frac{|g(u)|}{u} du = o(\log 1/t) \text{ as } t \to 0+ ,$$

<u>then</u> $t_n = o(\log n)$ <u>as</u> $n \to \infty$.

The theorem B generalizes a theorem due to Mohanty and Nanda [3].

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