SOLVABILITY OF ORDINARY DIFFERENTIAL EQUATIONS NEAR SINGULAR POINTS: AN ANALYTIC CASE

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1. Introduction. The question of solvability of the differential equation

$$f'(x) = G(x, f(x)), \tag{1}$$

with x ranging over an interval $(0, a]$, and with the boundary condition $f(0+) = 0$, can be investigated as an initial-value problem at 0, which may be a singular point for the equation. This has been done in **(1)**, where it was shown by application of a fixed-point theorem that the problem is solvable if G is "restricted in growth somewhere near 0," and where criteria for unsolvability of the problem also were given. In case $G(x, w)$ is analytic in w, that is

$$G(x, w) = \sum_{p=0}^{\infty} g_p(x) w^p, \tag{2}$$

somewhat different and more transparent conditions for solvability and for unsolvability seem natural. Under these conditions the proofs become more elementary, and in addition solutions can be expanded in infinite series of integrals which are improper at 0.

The scheme for finding these expansions seems to have been hit upon first by O. Perron **(2)**. First one introduces the equation

$$f'(x) = G(x, zf(x)) = \sum_{p=0}^{\infty} g_p(x) z^p f^p(x),$$

which reduces to the analytic case of (1) when $z = 1$; then one supposes that f has a series expansion of the form

$$f = \sum_{p=0}^{\infty} c_p z^p$$

and seeks to determine recurrently the coefficient functions c_p. There is then the question of the convergence of this series, consideration of which will be facilitated by a piece of notation and a lemma, as follows.

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If each term of the sequence $\{h_0, h_1, \ldots\}$ is a complex number or each term is a complex-valued function, the terms all having a common domain in the latter case, then for $n = 1, 2, \ldots$ and $p = 0, 1, \ldots$ the symbol $h_{n,p}$ stands for the coefficient of z^p in the formal expansion of $(\sum_{q=0}^{\infty} h_q z^q)^n$ into a power series in z—which is to say that

$$h_{n,p} = h_p \qquad \text{if } n = 1,$$

and

$$h_{n,p} = \sum_{q=0}^{p} h_q h_{n-1,p-q} \qquad \text{if } n > 1.$$

An easy consequence of this definition is that if z is a complex number and each of m and n is a positive integer, then

$$\sum_{p=0}^{m} |h_{n,p} z^p| \leqslant \left(\sum_{p=0}^{m} |h_p z^p| \right)^n .$$

(Throughout the paper the inequality $f \leqslant g$, where each of f and g is a real-valued function, is to be interpreted to mean that $f(x) \leqslant g(x)$ for every number x common to the domain of f and the domain of g.) It should be noted that if $q \leqslant p$, then the expression for $h_{q,p-q}$ does not involve h_p or any succeeding term of the sequence $\{h_n\}$, so that if each of $h_0, \ldots, h_{p-1}$ has been defined, $h_{q,p-q}$ is fully determined. This fact will be used as a hidden justification in some inductive definitions.

LEMMA. *Suppose that each of $K_0, K_1, \ldots$ is a complex number, $r \geqslant 0$, T is a finite non-negative number, and*

$$\sum_{p=0}^{\infty} |K_p| r^p T^p \leqslant T. \tag{3}$$

If $s_0 = K_0$, and, for $p = 1, 2, \ldots$,

$$s_p = \sum_{q=1}^{p} K_q s_{q,p-q}, \tag{4}$$

then for each complex number z such that $|z| \leqslant r$ the series $\sum_{q=0}^{\infty} s_p z^p$ converges absolutely to a complex number t such that $|t| \leqslant T$ and

$$t = \sum_{p=0}^{\infty} K_p z^p t^p, \tag{5}$$

the convergence of the latter series also being absolute.

Proof. Let

$$t_n = \sum_{p=0}^{n} |s_p z^p| \qquad \text{for } n = 0, 1, \ldots .$$

Then

$$t_n = |K_0| + \sum_{p=1}^{n} \left| \sum_{q=1}^{p} K_q \, s_{q,p-q} \right| \cdot |z^p|$$

$$\leqslant |K_0| + \sum_{p=1}^{n} \sum_{q=1}^{p} |(K_q z^q)(s_{q,p-q} z^{p-q})|.$$

Now $t_0 = |s_0| = |K_0| \leqslant T$, and if $t_p \leqslant T$ for $p = 0, 1, \ldots, n-1$, then

$$|K_0| + \sum_{p=1}^{n} \sum_{q=1}^{p} |(K_q z^q)(s_{q,p-q} z^{p-q})|$$

$$= |K_0| + \sum_{q=1}^{n} \sum_{p=q}^{n} |(K_q z^q)(s_{q,p-q} z^{p-q})|$$

$$= |K_0| + \sum_{q=1}^{n} |K_q z^q| \left(\sum_{p=0}^{n-q} |s_{q,p} z^p| \right)$$

$$\leqslant |K_0| + \sum_{q=1}^{n} |K_q z^q| \left(\sum_{p=0}^{n-q} |s_p z^p| \right)^q$$

$$= |K_0| + \sum_{q=1}^{n} |K_q z^q| \, t_{n-q}^q$$

$$\leqslant |K_0| + \sum_{q=1}^{n} |K_q| \, r^q T^q$$

$$\leqslant T.$$

Hence the sequence $\{t_n\}$ is bounded above by T, so that the series $\sum_{p=0}^{\infty} s_p z^p$ converges absolutely to a complex number t such that $|t| \leqslant T$. It is not difficult to show that

$$\sum_{p=0}^{\infty} s_{q,p} z^p = t^q \qquad \text{for } q = 0, 1, \ldots,$$

and that, by an interchange of order of summation in an absolutely convergent iterated sum,

$$t = K_0 + \sum_{q=1}^{\infty} (K_q z^q) \left(\sum_{p=0}^{\infty} s_{q,p} z^p \right)$$

$$= \sum_{q=0}^{\infty} K_q z^q t^q;$$

absolute convergence of the latter series is implied by the inequalities

$$\sum_{p=0}^{\infty} |K_p z^p t^p| \leqslant \sum_{p=0}^{\infty} |K_p| r^p T^p \leqslant T.$$

This ends the proof.

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This lemma is slightly more general than subsequent applications require.

2. A criterion for solvability. If $a > 0$ and h is a function whose domain includes the interval $(0, a]$, then the statement that h is integrable from $0+$ to x, x being any number in $(0, a]$, will mean that if $0 < \delta < x$, then h is Lebesgue-integrable on the interval $[\delta, x]$ and $\int_\delta^x h$ approaches a finite limit as $\delta \to 0$; the symbol $\int_{0+}^x h$ will stand for this limit.

Let P denote the non-compact complex plane and P_∞ the compact complex plane. Suppose that, for each non-negative integer p, $g_p : (0, \infty) \to P_\infty$ and g_p is measurable.

For the analytic case of the differential equation under consideration the following condition upon the coefficients g_p will take the place of Hypothesis A of **(1)**.

Hypothesis A′. There exist a finite positive number a, a finite-valued non-negative function u on $(0, a]$, and, for each non-negative integer p, a finite non-negative number K_p such that

(i) g_0 is integrable from $0+$ to a and

$$|\textstyle\int_{0+} g_0| \leqslant K_0\, u, \tag{6}$$

and

(ii) if p is a positive integer, then $|g_p|u^p$ is integrable from $0+$ to a and

$$\textstyle\int_{0+} |g_p|u^p \leqslant K_p\, u. \tag{7}$$

The following theorem is almost, but not quite, an existence theorem for the initial-value problem. The property of the G's being "restricted in growth somewhere near 0," defined in **(1)**, finds its analogue for the analytic case in the conjunction of the hypothesis of this theorem with the hypothesis of (ii) of the conclusion of this theorem, with $r = 1$.

THEOREM 1. *Suppose that Hypothesis* A′ *is true and that* u *and* $\{K_0, K_1, \ldots\}$ *denote a function and a number sequence which jointly verify it. Then each of the following statements is true.*

(i) *The recurrence formula*

$$c_p = \begin{cases} \displaystyle\int_{0+} g_0 & \text{if } p = 0, \\ \displaystyle\sum_{q=1}^{p} \int_{0+} g_q\, c_{q,p-q} & \text{if } p = 1, 2, \ldots, \end{cases} \tag{8}$$

defines a function sequence $\{c_0, c_1, \ldots\}$ *each term of which includes the interval* $(0, a]$ *in its domain.*

(ii) *If* $r \geqslant 0$ *and there exists a finite non-negative number* T *such that*

$$\sum_{p=0}^{\infty} K_p\, r^p T^p \leqslant T,$$

then for each complex number z such that $|z| \leqslant r$ *the series* $\sum_{p=0}^{\infty} c_p z^p$ *converges absolutely on the interval* $(0, a]$, *also uniformly on every set on which u is bounded, to a function f which has the property that*

(a) $|g_p f^p|$ *is integrable from* $0+$ *to a for* $p = 1, 2, \ldots$,

(b) $f = \sum_{p=0}^{\infty} \int_{0+} g_p z^p f^p$, *the convergence of this series being absolute, also uniform on every set on which u is bounded, and*

(c) $|f| \leqslant tu$, *where t satisfies*

$$t = \sum_{p=0}^{\infty} K_p |z|^p t^p$$

and is the smallest non-negative number which does so.

Proof. As in the lemma let $s_0 = K_0$ and

$$s_p = \sum_{q=1}^{p} K_q s_{q,p-q} \qquad \text{for } p = 1, 2, \ldots .$$

The fact that the K_p's are all non-negative implies by induction that each $s_p \geqslant 0$ and each $s_{p,q} \geqslant 0$.

Let $c_0 = \int_{0+} g_0$. Since g_0 is integrable from $0+$ to a, the domain of c_0 includes the interval $(0, a]$. Now, by (6), $|c_0| \leqslant K_0 u = s_0 u$, and therefore

$$|c_{1,0}| = |c_0| \leqslant s_0 u = s_{1,0} u^1,$$

while if $n > 0$ and $|c_{m,0}| \leqslant s_{m,0} u^m$ for $m = 1, \ldots, n-1$, then

$$|c_{n,0}| = |c_0 c_{n-1,0}| \leqslant (s_0 u)(s_{n-1,0} u^{n-1}) = s_{n,0} u^n.$$

Hence if $p = 0$,

$$|c_p| \leqslant s_p u \tag{9}$$

and, for each positive integer n,

$$|c_{n,p}| \leqslant s_{n,p} u^n. \tag{10}$$

An inductive argument will now establish that if d is a positive integer and $c_0, \ldots, c_{d-1}$ is a sequence which satisfies the recurrence formula (8), and in addition the inequalities (9) and (10), for $0 \leqslant p \leqslant d-1$, and each of $c_0, \ldots, c_{d-1}$ includes the interval $(0, a]$ in its domain, then the recurrence formula generates a function c_d, with $(0, a]$ as a subset of its domain, for which (9) and (10) hold when $p = d$. The argument is as follows.

Suppose that $1 \leqslant q \leqslant d$. From (10) it follows that $|g_q c_{q,d-q}| \leqslant s_{q,d-q} |g_q| u^q$. Because each of $c_0, \ldots, c_{d-1}$ is continuous, $c_{q,d-q}$ is continuous, and since g_q is measurable, $g_q c_{q,d-q}$ is measurable. But this function, being measurable and dominated by a function which is integrable from $0+$ to a, namely $s_{q,d-q} |g_q| u^q$, must itself be integrable from $0+$ to a, along with its absolute value, and what is more,

$$\left|\int_{0+} g_q c_{q,d-q}\right| \leqslant \int_{0+} |g_q c_{q,d-q}| \leqslant s_{q,d-q} \int_{0+} |g_q| u^q. \tag{11}$$

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This being true for $q = 1, \ldots, d$, the recurrence formula is thereby able to generate the function c_d determined by

$$c_d = \sum_{q=1}^{d} \int_{0+} g_q \, c_{q,d-q};$$

c_d is certainly defined at least on the interval $(0, a]$, and

$$|c_d| \leqslant \sum_{q=1}^{d} s_{q,d-q} \int_{0+} |g_q| u^q \leqslant \left(\sum_{q=1}^{d} K_q \, s_{q,d-q} \right) u = s_d \, u.$$

It follows as before by induction on n that $|c_{n,d}| \leqslant s_{n,d} \, u^n$. This completes the induction on d and establishes (i).

Suppose now that $|z| \leqslant r$. According to the lemma the series $\sum_{k=0}^{\infty} s_p |z|^p$ converges to a number t such that

$$t = \sum_{p=0}^{\infty} K_p \, |z|^p t^p.$$

Since in the present instance each $s_p \geqslant 0$, so is t. Also, it follows immediately from the lemma that t is the smallest non-negative solution of the last equation.

If n is a non-negative integer, then

$$\sum_{p=0}^{n} |c_p \, z^p| \leqslant \left(\sum_{p=0}^{n} s_p |z|^p \right) u \leqslant tu, \tag{12}$$

and therefore the series $\sum_{p=0}^{\infty} c_p \, z^p$ converges absolutely on the interval $(0, a]$ to a function whose domain is that interval. Call this function f. Since, for $n = 0, 1, \ldots$,

$$\left| f - \sum_{p=0}^{n} c_p \, z^p \right| = \left| \sum_{p=n+1}^{\infty} c_p \, z^p \right| \leqslant \left(\sum_{p=n+1}^{\infty} s_p |z|^p \right) u = \left(t - \sum_{p=0}^{n} s_p |z|^p \right) u,$$

it follows that the convergence is uniform on every set on which u is bounded.

Since

$$f = \sum_{p=0}^{\infty} c_p \, z^p,$$

(12) implies that $|f| \leqslant tu$, so that if p is a positive integer, then $|g_p f^p| \leqslant t^p |g_p| u^p$. Because f is a pointwise limit of continuous functions, it is measurable, and so, therefore, is $g_p f^p$. Furthermore, in view of the last inequality and (ii) of Hypothesis A′, $g_p f^p$ is dominated by a function which is integrable from $0+$ to a, and, consequently, is itself integrable from $0+$ to a.

Suppose that $x \in (0, a]$. Then

$$f(x) = c_0(x) + \sum_{p=1}^{\infty} \sum_{q=1}^{p} \left(\int_{0+}^{x} g_q \, c_{q,p-q} \right) z^p.$$

The double sum converges absolutely, for if n is a positive integer, then, in view of (11), (7), and (4),

$$\sum_{p=1}^{n}\sum_{q=1}^{p}\left|\left(\int_{0+}^{x} g_q\, c_{q,p-q}\right) z^p\right| \leqslant \sum_{p=1}^{n}\sum_{q=1}^{p}\left(\int_{0+}^{x}|g_q|u^q\right) s_{q,p-q}|z^p|$$
$$\leqslant \sum_{p=1}^{n}\left(\sum_{q=1}^{p} K_q\, s_{q,p-q}\right)|z^p|u(x)$$
$$= \left(\sum_{p=1}^{n} s_p|z|^p\right)u(x)$$
$$\leqslant (t-K_0)u(x).$$

Therefore the order of summation can be changed to yield

$$f(x) = c_0(x) + \sum_{q=1}^{\infty}\sum_{p=q}^{\infty}\left(\int_{0+}^{x} g_q\, c_{q,p-q}\, z^{p-q}\right) z^q \tag{13}$$
$$= c_0(x) + \sum_{q=1}^{\infty}\left(\sum_{p=0}^{\infty}\int_{0+}^{x} g_q\, c_{q,p}\, z^p\right) z^q.$$

Now from the absolute convergence of $\sum_{q=0}^{\infty} c_p\, z^p$ to f it follows by induction that if q is a positive integer, then

$$\sum_{p=0}^{\infty} c_{q,p}\, z^p = f^{\,q},$$

with absolute convergence here, also. But from the last equation it follows that if n is a non-negative integer, then

$$\left|f^{\,q} - \sum_{p=0}^{n} c_{q,p}\, z^p\right| = \left|\sum_{p=n+1}^{\infty} c_{q,p}\, z^p\right| \leqslant \left(t^q - \sum_{p=0}^{n} s_{q,p}\, |z|^p\right)u^q,$$

where use has been made of inequality (10) and the fact, mentioned in the proof of the lemma, that

$$\sum_{p=0}^{\infty} s_{q,p}\, |z|^p = t^q.$$

(Here $t = \sum_{p=0}^{\infty} s_p|z|^p$, whereas in the lemma $t = \sum_{p=0}^{\infty} s_p\, z^p$.)

Therefore

$$\left|\int_{0+}^{x} g_q f^{\,q} - \sum_{p=0}^{n}\int_{0+}^{x} g_q\, c_{q,p}\, z^p\right| \leqslant \int_{0+}^{x}|g_q|\cdot\left|f^{\,q} - \sum_{p=0}^{n} c_{q,p}\, z^p\right|$$
$$\leqslant \left(t^q - \sum_{p=0}^{n} s_{q,p}\, |z|^p\right)\int_{0+}^{x}|g_q|u^q,$$

so that

$$\sum_{p=0}^{\infty}\int_{0+}^{x} g_q\, c_{q,p}\, z^p = \int_{0+}^{x} g_q f^{\,q};$$

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hence in view of (13)

$$f = \sum_{p=0}^{\infty} \int_{0+} g_p z^p f^p.$$

Now since, according to the lemma,

$$t = \sum_{p=0}^{\infty} K_p |z|^p t^p,$$

it follows that for each positive integer n,

$$\left| \sum_{p=0}^{\infty} \int_{0+} g_p z^p f^p - \sum_{p=0}^{n} \int_{0+} g_p z^p f^p \right| \leqslant \sum_{p=n+1}^{\infty} \left(\int_{0+} |g_p f^p| \right) |z^p|$$

$$\leqslant \left(t - \sum_{p=0}^{n} K_p |z|^p t^p \right) u.$$

This inequality implies that the convergence of $\sum_{p=0}^{\infty} \int_{0+} g_p z^p f^p$ is absolute and is uniform on every set on which u is bounded. This completes the proof.

The preceding theorem will become an existence theorem for the differential equation initial-value problem if its hypothesis is strengthened enough to allow the conclusion that

$$f = \sum_{p=0}^{\infty} \int_{0+} g_p z^p f^p = \int_{0+} \sum_{p=0}^{\infty} g_p z^p f^p,$$

and hence the conclusion that

$$f'(x) = \sum_{p=0}^{\infty} g_p(x) z^p f^p(x)$$

almost everywhere. The next theorem details such a procedure.

THEOREM 2. *If to the hypothesis of Theorem* 1 *there were added the assumptions that* $u(0+) = 0$ *and that there exists a positive number* R *such that* $\sum_{p=0}^{\infty} |g_p| R^p$ *converges uniformly on every interval* $[d, a)$ *for which* $0 < d < a$, *and the number* b *were then defined by*

$$b = \sup\{a' \in (0, a] \mid rTu(x) \leqslant R \text{ if } 0 < x \leqslant a'\},$$

where r *and* T *are the numbers mentioned in statement* (ii) *of the conclusion of Theorem* 1, *then to statement* (ii) *of the conclusion could be added the following:*

(d) *the series* $\sum_{p=0}^{\infty} g_p z^p f^p$ *converges absolutely on the open interval* $(0, b)$ *to a function which, if* $0 < x < b$, *is integrable from* $0+$ *to* x, *the convergence being uniform on every interval* $[d, b)$ *for which* $0 < d < b$, *and*

(e) *if* $0 < x < b$, *then*

$$f(x) = \int_{0+}^{x} \sum_{p=0}^{\infty} g_p z^p f^p,$$

and

$$f'(x) = \sum_{p=0}^{\infty} g_p(x) z^p f^p(x)$$

almost everywhere (*everywhere, if each* g_p *is continuous*).

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Proof. From the lemma it follows that $t \leqslant T$, and therefore $|f| \leqslant Tu$. Hence, if $0 < x < b$ and p is a non-negative integer, then

$$|g_p(x)z^p f^p(x)| \leqslant |g_p(x)|[rTu(x)]^p \leqslant |g_p(x)|R^p.$$

The series $\sum_{p=0}^{\infty} g_p z^p f^p$, being thus majorized on the interval $(0, b)$ by the series $\sum_{p=0}^{\infty}|g_p|R^p$, converges absolutely on $(0, b)$, and, also like the latter series, it converges uniformly on every interval $[d, b)$ for which $0 < d < b$.

Let

$$h = \sum_{p=0}^{\infty} g_p z^p f^p$$

and suppose that $x \in (0, b)$ and $0 < d < x$. On the closed interval $[d, x]$, h is a uniform limit of functions which are integrable from $0+$ to a and, consequently, from d to x. Therefore h is integrable from d to x, and

$$\begin{aligned}\int_d^x h &= \lim_{n\to\infty} \int_d^x \sum_{p=0}^{n} g_p z^p f^p \\ &= \lim_{n\to\infty} \left(\int_{0+}^x \sum_{p=0}^{n} g_p z^p f^p - \int_{0+}^d \sum_{p=0}^{n} g_p z^p f^p \right) \\ &= \sum_{p=0}^{\infty} \int_{0+}^x g_p z^p f^p - \sum_{p=0}^{\infty} \int_{0+}^d g_p z^p f^p \\ &= f(x) - f(d).\end{aligned}$$

Since $u(0+) = 0$ and $|f| \leqslant tu$, $f(0+)$ is also 0, and it follows, upon letting d tend to 0, that h is integrable from $0+$ to x, and that

$$f(x) = \int_{0+}^x h = \int_{0+}^x \sum_{p=0}^{\infty} g_p z^p f^p,$$

so that

$$f'(x) = \sum_{p=0}^{\infty} g_p(x) z^p f^p(x)$$

almost everywhere in $(0, b)$. If each g_p is continuous, then so is h, so that the word "almost" can be omitted in that case.

3. A criterion for unsolvability. The final theorem to be presented is approximately a converse of Theorems 1 and 2. It is analogous to the last theorem in **(1)** and requires for its formulation an assumption about the g_p's somewhat antithetical to Hypothesis A′, i.e. the following Hypothesis B′. It is for the analytic case the counterpart of Hypothesis B in **(1)**.

Hypothesis B′. There exist a finite positive number a, a non-negative measurable function u on $(0, a]$ and, for each non-negative integer p, a non-negative number K_p such that

(i) if $b \in (0, a]$, then there is a number x such that $0 < x \leqslant b$ and $u(x) \neq 0$,

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(ii) if $x \in (0, a]$ and g_0 is integrable from $0+$ to x, then

$$\left| \int_{0+}^{x} g_0 \right| \geqslant K_0\, u(x), \tag{14}$$

and

(iii) if $x \in (0, a]$ and p is a positive integer, then either $|g_p|u^p$ is not integrable from $0+$ to x, or else it is and

$$\int_{0+}^{x} |g_p|u^p \geqslant K_p\, u(x). \tag{15}$$

THEOREM 3. *Suppose that Hypothesis* B′ *is true and that* u *and* $\{K_0, K_1, \ldots\}$ *denote a function and a number sequence which jointly verify it. If* $r > 0$ *and there does not exist a finite non-negative number* T *such that*

$$\sum_{p=0}^{\infty} K_p\, r^p T^p \leqslant T,$$

then for no complex number z *such that* $|z| \geqslant r$ *do there exist a number* b *in the interval* $(0, a]$ *and a function* $f\colon (0, b] \to P$ *such that*

$$|f| \geqslant \left| \int_{0+} g_0 \right| + \sum_{p=1}^{\infty} \int_{0+} |g_p\, z^p f^{\,p}|; \tag{16}$$

in particular if $z \geqslant r$, $\int_{0+} g_0 \geqslant 0$, *and* $g_p \geqslant 0$ *for* $p = 1, 2, \ldots$, *then there do not exist a number* b *in* $(0, a]$ *and a non-negative function* f *on* $(0, b]$ *such that*

$$f = \int_{0+} \sum_{p=0}^{\infty} g_p\, z^p f^{\,p},$$

or even such that

$$f = \sum_{p=0}^{\infty} \int_{0+} g_p\, z^p f^{\,p}.$$

Proof. Suppose that there do exist such a number b and a function f : $(0, b] \to P$. This implies, among other things, that if $0 < x \leqslant b$, then g_0 is integrable from $0+$ to x, and that $|f| \geqslant |\int_{0+} g_0|$, so that, according to (14), $|f| \geqslant K_0\, u$.

Now let $t_0 = K_0$, and define $t_1, t_2, \ldots$ inductively by the formula

$$t_n = \sum_{p=0}^{\infty} K_p\, r^p t_{n-1}{}^p,$$

allowing the possibility that $t_n = \infty$ from some term onward. The sequence $\{t_0, t_1, \ldots\}$ is non-decreasing, for clearly $t_0 \leqslant t_1$, and if $t_{n-1} \leqslant t_n$, then

$$t_n = \sum_{p=0}^{\infty} K_p\, r^p t_{n-1}{}^p \leqslant \sum_{p=0}^{\infty} K_p\, r^p t_n{}^p = t_{n+1}.$$

Furthermore, $\lim_{n\to\infty} t_n = \infty$, for if $T = \lim_{n\to\infty} t_n$, then

$$T = \sum_{p=0}^{\infty} K_p r^p T^p$$

and by hypothesis cannot be finite.

On the other hand, $|f| \geqslant K_0 u = t_0 u$, while if n is a positive integer and $|f| \geqslant t_{n-1} u$, then $|f| \geqslant t_n u$. This latter comes from the following considerations.

If $|f| \geqslant t_{n-1} u$, and p is a positive integer, then

$$|g_p| r^p (t_{n-1} u)^p \leqslant |g_p z^p f^p|.$$

The function on the right is, by hypothesis, integrable from $0+$ to b, and since, as may easily be seen, neither r nor t_{n-1} can be 0, the function $|g_p| u^p$, being non-negative, measurable, and dominated by a function which is integrable from $0+$ to b, must itself be integrable from $0+$ to b. In addition

$$\int_{0+} |g_p z^p f^p| \geqslant \int_{0+} |g_p| r^p (t_{n-1} u)^p \geqslant K_p r^p t_{n-1}^p u,$$

in which the last step utilizes (15); therefore it follows from (14) and (16) that

$$|f| \geqslant K_0 u + \sum_{p=1}^{\infty} K_p r^p t_{n-1}^p u = t_n u.$$

The fact that $|f| \geqslant t_n u$ for $n = 0, 1, \ldots$, coupled with the fact that, for at least one number x in the interval $(0, b]$, $u(x) \neq 0$, leads to a contradiction, since $\lim t_n = \infty$, while the values of f are supposed to be finite.

The final part of the conclusion follows without difficulty from the first part.

4. Remarks. In closing, let it be remarked that the foregoing analysis can be applied to the equation

$$f' = \sum_{p=0}^{\infty} g_p z^p (f - c)^p$$

with the initial condition $f(0+) = c$, where c can be any complex number.

REFERENCES

1. H. G. Ellis, *Solvability of the initial-value problem for ordinary differential equations near singular points*, Trans. Amer. Math. Soc., *119* (1965), 1–20.

2. M. Müller, *Neure Untersuchungen über den Fundamentalsatz in der Theorie der gewöhnlichen Differentialgleichungen*, Jber. Deutsch. Math. Verein., *37* (1928), 33–48.

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