Birational transformations with isolated fundamental points

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1. It is well known that the canonical system of curves on an algebraic surface is only *relatively* invariant under birational transformations of the surface. That is, if we have a birational transformation **T** between two surfaces F and F', and if K and K' denote curves of the unreduced canonical systems on F and F', then

$$\mathbf{T}(K - E) \equiv K' - E'$$

where E and E' denote the sets of curves, on F and F' respectively, which are transformed into the neighbourhoods of simple points on the other surface.

More generally, on a variety V_d of d dimensions, the canonical system of V_{d-1}'s, and the canonical systems of lower dimensions which I have recently described[1], are only relative invariants under birational transformation of V_d, and it becomes an important problem to determine the way in which the presence of fundamental elements of various kinds affects the transformation of the canonical varieties. For a threefold this question has been solved by B. Segre[2] in a recent memoir, but the general problem is of considerable complexity, and is not yet solved.

There is, however, one case in which the problem is tractable, and it is with this that the present paper is concerned. This is the case in which the transformation **T** between V_d and V'_d, and its inverse, possess fundamental points, but no fundamental varieties of higher dimension. Thus, assuming V_d and V'_d to be free from singularities, a simple linear system of V_{d-1}'s on V_d, free from base points, will be transformed by **T** into a linear system on V'_d with at most a finite number k' of base points A'_ν $(\nu = 1, \ldots, k')$, and a linear system on V'_d which is simple and free from base-points will

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[1] J. A. Todd, "The geometrical invariants of algebraic loci," *Proc. London Math. Soc.* (2), 43 (1937), 127; "The arithmetical invariants of algebraic loci," *ibid.* (2), 43 (1937), 190. We refer to these papers as I and II.

[2] B. Segre, "Quelques resultats nouveaux dans la géométric sur une V_3 algébrique," *Mem. Acad. royale de Belgique* (2), 14 (1936).

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be transformed by $\mathbf{T}^{-1}$ into a linear system on V_d with a finite number k of base-points A_μ ($\mu = 1, \ldots, k$). We suppose that the points A_μ, A'_ν are all distinct.

We shall obtain below the relations which hold between the canonical systems on V'_d and the transforms of the canonical systems on V_d under a transformation $\mathbf{T}$ of this character, and, as an application will show, for $d \leqq 6$, the arithmetic genus of V_d is unaltered by a birational transformation of this kind.

2. If P is a point of V_d, and if Q' is a point of V'_d, then the aggregate of all pairs of points (P, Q') can be represented by an algebraic variety Φ_{2d} of $2d$ dimensions (non-singular if V_d and V'_d are free from singularities). The points of Φ_{2d} which represent pairs (P, Q') such that $Q' = \mathbf{T}(P)$ form an algebraic variety V''_d, of d dimensions. Since $\mathbf{T}$ is a birational transformation it clearly induces a birational transformation $\mathbf{T}_1$ between V_d and V''_d, and a second birational transformation $\mathbf{T}_2$ between V''_d and V_d, and $\mathbf{T}$ itself is the product $\mathbf{T}_2\,\mathbf{T}_1$ of $\mathbf{T}_1$ and $\mathbf{T}_2$ (the product symbol being read from right to left, as in the calculus of matrices). If P is a point of V_d fundamental for the transformation $\mathbf{T}$ then it is evidently fundamental also for $\mathbf{T}_1$. On the other hand, $\mathbf{T}_1^{-1}$ has no fundamental points on V''_d, since each point of Φ_{2d} represents a unique pair of points (P, Q') and hence determines a unique point on V_d. In the same way, the transformation $\mathbf{T}_2$ from V''_d to V'_d has no fundamental points on V''_d, and the fundamental points of $\mathbf{T}_2^{-1}$ on V'_d are precisely those of $\mathbf{T}^{-1}$.

We can therefore express any transformation $\mathbf{T}$ with only isolated fundamental points as the product of two transformations each of which has the property that it only possesses fundamental points on one of the two varieties concerned. We shall now suppose (until § 7) that $\mathbf{T}$ itself is of this type, *i.e.* that $\mathbf{T}^{-1}$ has no fundamental points on V'_d.

3. Suppose then that $\mathbf{T}^{-1}$ is free from fundamental points, while $\mathbf{T}$ has k fundamental points A_μ ($\mu = 1, \ldots, k$) on V_d. The neighbourhood of a fundamental point A_μ is mapped by $\mathbf{T}$ on a V_{d-1}, $\mathfrak{A}_\mu$ lying on V'_d. Since the points A_μ and A_ν are distinct if μ and ν are different, it follows that the varieties $\mathfrak{A}_\mu$, $\mathfrak{A}_\nu$ have no common points, which we express symbolically in the form

$$(\mathfrak{A}_\mu\,\mathfrak{A}_\nu) \equiv 0. \tag{1}$$

On the other hand, the virtual intersection of $\mathfrak{A}_\mu$ with itself will not in general be zero.

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4. Let $|S|$ be a simple linear system of V_{d-1}'s on V_d, of freedom $d+1+k$ at least, free from base-points. A general linear sub-system $|S^{(h+1)}|$ of $|S|$ of freedom $h+1$ will contain ∞^h members with double points, and the locus of these double points, if $h < d$, will be a variety M_h of h dimensions, the *Jacobian* of $|S^{(h+1)}|$. The *canonical variety of h dimensions* X_h on V_d is expressible in terms of the Jacobian varieties M_h by the equivalence[1],

$$X_h \equiv M_h + \sum_{i=1}^{d-h} (-)^i \binom{h+1+i}{h+1} (M_{h+i} S^i). \tag{2}$$

Under the transformation $\mathbf{T}$, $|S|$ becomes transformed into a linear system $|S'|$ on V'_d, free from base-points since $\mathbf{T}^{-1}$ has no fundamental points on V'_d, and M_h is transformed into a locus M'_h. However, we cannot assert that M'_h is the Jacobian of the transform of $|S^{(h+1)}|$, for if one of the varieties S passes through a fundamental point A_μ the corresponding variety S' splits up into $\mathfrak{A}_\mu$ and a residual V_{d-1} whose intersection with $\mathfrak{A}_\mu$ belongs to the locus of double points on varieties of the linear system containing S'.

5. Consider now the sub-system $|S_1|$ of $|S|$ which has simple base-points at each of the points A_μ. By hypothesis, $|S_1|$ has freedom at least equal to $d+1$, and we assume that $|S|$ was chosen so generally that the points A_μ constitute the only base-points for $|S_1|$. The transform of a member of $|S_1|$ by $\mathbf{T}$ breaks up into the set of varieties $\mathfrak{A}_\mu$, each counted simply, and a residual locus S'_1. By considering the variation of $\mathbf{T}(S)$ as S approaches a variety of $|S_1|$ it is easily seen that

$$S' \equiv S'_1 + \Sigma\, \mathfrak{A}_\mu. \tag{3}$$

Further, since $|S|$ has no base-points, $(S' \mathfrak{A}_\mu) \equiv 0$, and hence

$$(S'_1 \mathfrak{A}_\mu) \equiv -(\mathfrak{A}_\mu^2). \tag{4}$$

Now let $M_{h,1}$ be the Jacobian of a linear sub-system $|S_1{}^{(h+1)}|$ of $|S_1|$ of freedom $h+1$. It is easily seen that the imposition of simple base-points on the members of $|S|$ is without effect on the varieties M_h except that $M_{d-1,1}$ has a double point at each simple base-point of $|S_1^{(d)}|$. Thus, if $M'_{h,1}$ denotes the transform of $M_{h,1}$ by $\mathbf{T}$, we have

$$M'_h \equiv M'_{h,1} \quad (h = 0, 1, \ldots, d-2). \tag{5}$$

$$M'_{d-1} \equiv M'_{d-1,1} + 2\Sigma\, \mathfrak{A}_\mu. \tag{6}$$

[1] II, formula (4).

Also, since the varieties M_h have no assigned base-points, $(M'_h \mathfrak{A}_\mu) \equiv 0$, so that

$$(M'_{h,1} \mathfrak{A}_\mu) \equiv 0 \quad (h = 0, 1, \ldots, d-2). \tag{7}$$

$$(M'_{d-1,1} \mathfrak{A}_\mu) \equiv -2 (\mathfrak{A}_\mu^2). \tag{8}$$

Now it is easily seen that $M'_{h,1}$ is the Jacobian of the linear system $|S_1^{(h+1)'}|$, the *variable* part of the transform of $|S_1^{(h+1)}|$ by $\mathbf{T}$. For if P' is a point of V'_d, double for a locus S'_1 of $|S_1^{(h+1)'}|$, and if P' does not lie on any of the varieties $\mathfrak{A}_\mu$, then it is the image of a unique point P on V_d, which is a double point of the variety S_1 which transforms into S'_1; thus P lies on $M_{h,1}$ and P' lies on $M'_{h,1}$. If on the other hand P', a double point of S'_1, lies on $\mathfrak{A}_\mu$, then the corresponding S_1 has a double point at A_μ. This cannot happen if $h < d-1$, since then M_h does not pass through A_μ; and if $h = d-1$ then S_1 must be the unique variety of $|S_1^{(d)}|$ which has a double point at A_μ. The complete transform of S_1 by $\mathbf{T}$ then contains $\mathfrak{A}_\mu$ doubly, and thus the locus S'_1 consists of $\mathfrak{A}_\mu$ and a residual variety S'_2. The point P' is thus on the intersection of S'_2 and $\mathfrak{A}_\mu$. But

$$(S'_2 . \mathfrak{A}_\mu) \equiv (S'_1 - \mathfrak{A}_\mu . \mathfrak{A}_\mu) \equiv -2 (\mathfrak{A}_\mu^2) \equiv (M'_{d-1,1} \mathfrak{A}_\mu)$$

by (4) and (8). Thus $M'_{d-1,1}$ and S'_2 meet $\mathfrak{A}_\mu$ in equivalent loci. But *any* point of $M'_{d-1,1}$ obviously lies on the Jacobian of $|S_1^{(d)'}|$, and hence these intersections coincide. It follows that, for all values of $h\,(< d)$, $M'_{h,1}$ is the Jacobian of the system $|S_1^{(h+1)'}|$.

6. The canonical system X'_h on V'_d is, by (2), given by

$$X'_h \equiv M'_{h,1} + \sum_{i=1}^{d-h} (-)^i \binom{h+1+i}{h+1} (M'_{h+i,1}\, S'^{\,i}_1). \tag{9}$$

When $h = d-1$ this becomes

$$X'_{d-1} \equiv M'_{d-1,1} - (d+1) S'_1.$$

Using (3) and (6) we obtain

$$X'_{d-1} \equiv M'_{d-1} - (d+1) S' + (d-1) \Sigma \mathfrak{A}_\mu .$$

But $\qquad \mathbf{T}(X_{d-1}) \equiv M'_{d-1} - (d+1) S', \qquad$ and so

$$\mathbf{T}(X_{d-1}) \equiv X'_{d-1} - (d-1) \Sigma \mathfrak{A}_\mu . \tag{10}$$

When $h < d - 1$, on substituting in (9) from (3), (5) and (6), we get

$$X'_h \equiv M'_h + \sum_{i=1}^{d-h-2} (-)^i \binom{h+1+i}{h+1} (M'_{h+i} \, . \, (S' - \Sigma \mathfrak{A}_\mu)^i)$$

$$+ (-)^{d-h-1} \binom{d}{h+1} ((M'_{d-1} - 2\Sigma\mathfrak{A}_\mu) \, . \, (S' - \Sigma\mathfrak{A}_\mu)^{d-h-1}) + (-)^{d-h} \binom{d+1}{h+1} ((S' - \Sigma\mathfrak{A}_\mu)^{d-h}).$$

But $(M'_k . \mathfrak{A}_\mu) \equiv 0$, $(S' . \mathfrak{A}_\mu) \equiv 0$, and so, using (1),

$$X'_h \equiv M'_h + \sum_{i=1}^{d-h} (-)^i \binom{h+1+i}{h+1} (M'_{h+i} S'^i) - 2\binom{d}{h+1} \Sigma (\mathfrak{A}_\mu^{d-h}) + \binom{d+1}{h+1} \Sigma (\mathfrak{A}_\mu^{d-h}),$$

i.e.

$$X'_h \equiv \mathbf{T}(X_h) - \left(2\binom{d}{h+1} - \binom{d+1}{h+1}\right) \Sigma (\mathfrak{A}_\mu^{d-h}). \tag{11}$$

If we write $d - h$ for h in (11) we see that it can be written in the form

$$\mathbf{T}(X_{d-h}) \equiv X'_{d-h} - \lambda_h \Sigma (\mathfrak{A}_\mu^h), \tag{12}$$

where λ_h is found (after a little reduction) to be given by

$$\lambda_h = \binom{d}{h} - \binom{d}{h-1}. \tag{13}$$

Since (11) holds for $0 \leqq h < d - 1$, (12) holds for $1 < h \leqq d$; but from (10) we see also that (12) holds when $h = 1$, if we make the usual convention that $\binom{d}{0} = 1$. Thus (12) may be regarded as giving the rule by which *all* the canonical systems on V_d are transformed.

7. It is now easy to deduce the formulae appropriate to the more general case in which $\mathbf{T}$ and $\mathbf{T}^{-1}$ both have fundamental points. To do this we introduce the variety V''_d of § 2, and write $\mathbf{T} = \mathbf{T}_2 \mathbf{T}_1$, where $\mathbf{T}_1$ is the transformation between V_d and V''_d, and $\mathbf{T}_2$ is the transformation between V''_d and V'_d; $\mathbf{T}_1$ and $\mathbf{T}_2^{-1}$ are transformations of the type considered in §§ 3-6. Thus, if $\mathfrak{B}_\mu$ are the fundamental varieties on V''_d corresponding to the fundamental points A_μ on V_d, and if X''_h is the canonical system of dimension h on V''_d, then, by (12),

$$\mathbf{T}_1 (X_{d-h}) \equiv X''_{d-h} - \lambda_h \Sigma (\mathfrak{B}_\mu^h).$$

Similarly, if $\mathfrak{B}'_\nu$ are the fundamental varieties on V''_d corresponding to the fundamental points A'_ν which $\mathbf{T}_2^{-1}$ has on V'_d,

$$\mathbf{T}_2^{-1} (X'_{d-h}) \equiv X''_{d-h} - \lambda_h \Sigma (\mathfrak{B}'^h_\nu).$$

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The transformation $\mathbf{T}_2$ changes the varieties $\mathfrak{B}_\mu$ into the varieties $\mathfrak{A}_\mu$ which correspond to the fundamental points A_μ on V_d, while $\mathbf{T}_1^{-1}$ transforms $\mathfrak{B}'_\nu$ into the fundamental variety $\mathfrak{A}'_\nu$ corresponding to the fundamental point A'_ν on V'_d. That is,

$$\mathbf{T}_2(\mathfrak{B}_\mu) \equiv \mathfrak{A}_\mu ; \quad \mathbf{T}_1^{-1}(\mathfrak{B}'_\nu) \equiv \mathfrak{A}'_\nu.$$

Hence

$$\begin{aligned} \mathbf{T}(X_{d-h} - \lambda_h \Sigma(\mathfrak{A}'^h_\nu)) &\equiv \mathbf{T}_2 \mathbf{T}_1 (X_{d-h} - \lambda_h \Sigma(\mathfrak{A}'^h_\nu)) \\ &\equiv \mathbf{T}_2(X''_{d-h} - \lambda_h [\Sigma(\mathfrak{B}^h_\mu) + \Sigma(\mathfrak{B}'^h_\nu)]) \\ &\equiv X'_{d-h} - \lambda_h \Sigma(\mathfrak{A}^h_\mu), \end{aligned}$$

and so the general law of transformation is

$$\mathbf{T}(X_{d-h} - \lambda_h \Sigma(\mathfrak{A}'^h_\nu)) \equiv X'_{d-h} - \lambda_h \Sigma(\mathfrak{A}^h_\mu). \tag{14}$$

8. If $(h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s})$ is a partition of the integer d then the intersection of the canonical varieties $(X^{a_1}_{d-h_1} X^{a_2}_{d-h_2} \ldots . X^{a_s}_{d-h_s})$ is in general a finite set of points. The number of points in this set is a relative invariant of V_d, and is denoted by the symbol $\{h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s}\}$. I have shown[1] that the arithmetic genus of V_d can be expressed as a linear function of these invariants for all values of d, and have determined the explicit form of the expression for $d \leqq 6$. By way of application of the foregoing work we shall now verify that the arithmetic genus of V_d is unaltered by a transformation of the kind considered above, if $d \leqq 6$. This means that P_a, which is *a priori* given as a function of *relative* invariants of V_d is an *absolute* invariant for birational transformations of this particular type.

It is evidently sufficient to consider the case in which $\mathbf{T}$ has fundamental points on V_d only, and not on V'_d. We first examine the effect of $\mathbf{T}$ on the intersections of the various canonical systems.

In the first place, since the points A_μ are not assigned base-points for any of the systems $\{X_h\}$, it follows that $(\mathfrak{A}_\mu . \mathbf{T}(X_{d-h})) \equiv 0$. Thus, from (1) and (12),

$$(\mathfrak{A}_\mu . X'_{d-h}) \equiv \lambda_h (\mathfrak{A}^{h+1}_\mu). \tag{15}$$

Hence

$$\begin{aligned} \mathbf{T}(X_{d-h}.X_{d-l}) &\equiv (X'_{d-h} - \lambda_h \Sigma(\mathfrak{A}^h_\mu))(X'_{d-l} - \lambda_l \Sigma(\mathfrak{A}^l_\mu)) \\ &\equiv (X'_{d-h} . X'_{d-l}) - \lambda_h (\Sigma(\mathfrak{A}^h_\mu) . X'_{d-l}) - \lambda_l (\Sigma(\mathfrak{A}^l_\mu) . X'_{d-h}) \\ &\qquad + \lambda_h \lambda_l \Sigma(\mathfrak{A}^{h+l}_\mu) \\ &\equiv (X'_{d-h} . X'_{d-l}) - \lambda_h \lambda_l \Sigma(\mathfrak{A}^{h+l}_\mu). \end{aligned}$$

[1] II, § 7.

By repetition of this argument we see that

$$\mathbf{T}\,(X^{a_1}_{d-h_1}\,X^{a_2}_{d-h_2}\ldots X^{a_s}_{d-h_s}) \equiv (X'^{a_1}_{d-h_1}\,X'^{a_2}_{d-h_2}\ldots X'^{a_s}_{d-h_s}) - \lambda^{a_1}_{h_1}\lambda^{a_2}_{h_2}\ldots\lambda^{a_s}_{h_s}\,\Sigma\,(\mathfrak{A}^a_\mu), \qquad (16)$$

where $a = h_1 a_1 + h_2 a_2 + \ldots. + h_s a_s$.

Thus, if $\{h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s}\}'$ is the invariant of V'_d corresponding to the invariant $\{h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s}\}$ of V_d, it follows from (16) that

$$\{h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s}\} = \{h_1^{a_1} h_2^{a_2} \ldots h_s^{a_s}\}' - \lambda^{a_1}_{h_1}\lambda^{a_2}_{h_2}\ldots\lambda^{a_s}_{h_s}\,.\,\theta \qquad (17)$$

where

$$\theta = \sum_{\mu=1}^{k}[(\mathfrak{A}^d_\mu)]$$

is independent of the h's and a's.

Since P_a is a *linear* function of the invariants $\{h_1^{a_1}\ldots. h_s^{a_s}\}$, in order to prove that P_a is the same for V_d and V'_d it is enough, by (17), to show that the same linear function of the quantities $\lambda^{a_1}_{h_1}\ldots.\lambda^{a_s}_{h_s}$ is equal to zero. The λ's are determined by (13), so that the verification is reduced to simple substitution in known formulae.

For a V_3 it is known[1] that

$$24\,(P_a - 1) = \{21\}$$

and to prove the invariance of P_a it is thus only necessary to verify that $\lambda_2 \lambda_1 = 0$; as is in fact the case, since $\lambda_2 = 0$.

For[2] a V_4,

$$720\,(P_a + 1) = -\{4\} + \{31\} + 3\,\{2^2\} + 4\,\{21^2\} - \{1^4\}.$$

Now when $d = 4$, (13) gives $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = -2$, $\lambda_4 = -3$, and

$$-\lambda_4 + \lambda_3\lambda_1 + 3\,\lambda_2^2 + 4\,\lambda_2\,\lambda_1^2 - \lambda_1^4 = 3 - 6 + 12 + 72 - 81 = 0,$$

so that P_a is invariant for these transformations of a V_4.

For[3] a V_5,

$$1440\,(P_a - 1) = -\{41\} + \{31^2\} + 3\,\{2^2\,1\} - \{21^3\},$$

while

$$\lambda_1 = 4,\ \lambda_2 = 5,\ \lambda_3 = 0,\ \lambda_4 = -5.$$

[1] B. Segre, Mem. R. Acc. d'Italia, 5 (1934), 505.

[2] II, formula (37).

[3] II, formula (56).

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Thus, since

$$-\lambda_4\lambda_1+\lambda_3\lambda_1^2+3\lambda_2^2\lambda_1-\lambda_2\lambda_1^3=20+0+300-320=0,$$

P_a is invariant.

Finally, for[1] a V_6,

$$60480(P_a-1)=2\{6\}-2\{51\}-9\{42\}-5\{41^2\}-\{3^2\}+11\{321\}+5\{31^3\}+10\{2^3\}$$
$$+11\{2^2\,1^2\}-12\{21^4\}+2\{1^6\},$$

and $\lambda_1=5,\ \lambda_2=9,\ \lambda_3=5,\ \lambda_4=-5,\ \lambda_5=-9,\ \lambda_6=-5.$

Hence

$$2\lambda_6-2\lambda_5\lambda_1-9\lambda_4\lambda_2-5\lambda_4\lambda_1^2-\lambda_3^2+11\lambda_3\lambda_2\lambda_1+5\lambda_3\lambda_1^3+10\lambda_2^3+11\lambda_2^2\lambda_1^2-12\lambda_2\lambda_1^4+2\lambda_1^6$$
$$=-10+90+405+625-25+2475+3125+7290+22275-67500+31250=0$$

and P_a is invariant for V_6.

[1] II, formula (61).

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