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The Spectrum of an Infinite Graph

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Abstract. In this paper, we consider the (essential) spectrum of the discrete Laplacian of an infinite graph. We introduce a new quantity for an infinite graph, in terms of which we give new lower bound estimates of the (essential) spectrum and give also upper bound estimates when the infinite graph is bipartite. We give sharp estimates of the (essential) spectrum for several examples of infinite graphs.

1 Introduction

An infinite graph $G = (V, E)$ is a collection of a set V of infinite number of vertices and a set E of edges connecting two vertices. We always assume in this paper that G is connected and locally finite, that is, for each vertex x, the degree $m(x) = \#\{y \in V; y \sim x\}$ is finite. Here $y \sim x$ means two vertices x and y are connected by an edge. One of important problems in infinite graph theory is to estimate or determine the spectrum $\sigma(G)$ of the Laplacian $\Delta = I - P$ for a given infinite graph (*cf.* [7]). Here $\Delta f(x) = f(x) - \frac{1}{m(x)} \sum_{y\sim x} f(y)$, for $x \in V$ and a function f on V with finite support. In this paper, we give lower and upper estimates of the spectrum of the discrete Laplacian of an infinite graph.

The celebrated Cheeger type inequality says that the spectrum $\sigma(G)$ of the Laplacian Δ of an infinite graph is estimated as follows (*cf.* [1], [2], [4], [6], [7], [8]):

$$\sigma(G) \subset \left[1 - \sqrt{1 - h(G)^2}, 1 + \sqrt{1 - h(G)^2}\right] \subset [0, 2].$$

Here $h(G)$ is the Cheeger constant of an infinite graph G, that is,

$$h(G) = \inf\left\{\frac{\#(\partial A)}{m_V(A)}; A \subset V, \#A < \infty\right\},$$

where $\partial A = \{e = (x, y) \in E; x \in A \text{ and } y \notin A\}$ and $m_V(A) = \sum_{x\in A} m(x)$. Even though the above estimate is general and sharp, it is not so efficient. Because it is almost impossible in general, to know the Cheeger constant explicitly for a given infinite graph. It should be noticed that another sharp estimation of the spectrum in terms of the upper bound of degrees for an infinite tree has been obtained (*cf.* [10]).

The aim of this paper is to give new general and sharp estimations of the spectra and essential ones for an arbitrarily given infinite graph. It should be noticed that these estimates are computable for many infinite regular graphs (*cf.* Section 7). Our approach is quite different from the above Cheeger type estimations.

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To explain our results, we fix any vertex $x_0 \in V$, and consider $r(x)$, $x \in V$, the distance of x from x_0, that is, the length of the shortest path from x_0 to x. We define $m_+(x)$, $m_-(x)$ and $m_0(x)$ for $x \neq x_0 \in V$ by

$$m_+(x) = \#\{z \in V; z \sim x, \text{ and } r(z) = r(x) + 1\},$$
$$m_-(x) = \#\{z \in V; z \sim x, \text{ and } r(z) = r(x) - 1\},$$
$$m_0(x) = \#\{z \in V; z \sim x, \text{ and } r(z) = r(x)\},$$

respectively (*cf.* [9]). The main ingredient of our approach is to observe the following two subsets in the real line **R**:

$$\mathcal{M}_\pm(G) = \left\{ \frac{m_\pm(x)}{m(x)}; x \in V - \{x_0\} \right\},$$

respectively. We put

$$\mathbf{m}_\pm(G) = \inf \mathcal{M}_\pm(G), \quad \mathbf{M}_\pm(G) = \sup \mathcal{M}_\pm(G),$$

respectively. Then we obtain the following theorem.

Theorem A *Let $G = (V, E)$ be an infinite graph. Assume that two closed intervals $[\mathbf{m}_-(G), \mathbf{M}_-(G)]$ and $[\mathbf{m}_+(G), \mathbf{M}_+(G)]$ have no intersection. Then we have*

$$\inf \sigma(G) \geq \mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_-(G)}.$$

Thus, we have $\inf \sigma(G) > 0$.

Corollary B *Let $G = (V, E)$ be an infinite bipartite graph. Assume that two closed intervals $[\mathbf{m}_-(G), \mathbf{M}_-(G)]$ and $[\mathbf{m}_+(G), \mathbf{M}_+(G)]$ have no intersection. Then the supremum of spectrum is estimated as follows:*

$$\sigma(G) \subset \left[\left(\sqrt{\mathbf{m}_+(G)} - \sqrt{\mathbf{M}_-(G)} \right)^2, \left(\sqrt{\mathbf{m}_+(G)} + \sqrt{\mathbf{M}_-(G)} \right)^2 \right].$$

Remark Y. Higuchi pointed out (*cf.* Proposition 4.1) that for any infinite graph $G = (V, E)$, $\mathbf{M}_+(G) \geq \mathbf{m}_-(G)$.

The main ingredients to obtain our main results are Theorem 3.8 and Corollary 3.13. The similar estimations to the essential spectrum $\sigma_{\text{ess}}(G)$ of an infinite graph can be also done in Section 5 (*cf.* Theorem 5.3 and Corollary 5.4). We give also a discrete analogue of the estimation of the essential spectrum to Riemannian manifolds (*cf.* [3], [5]) in Section 6. We give many examples in Section 7 which show how our estimations are effective.

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2 Preliminaries

In this section, we give the notation that will be needed later.

Let $G = (V, E)$ be a locally finite, infinite graph. Let $C(V)$ be the space of all real valued functions on V, and $C_c(V)$ the space of f in $C(V)$ with finite support. We consider the inner product on $C_c(V)$ defined by

$$(f_1, f_2) = \sum_{x \in V} m(x) f_1(x) f_2(x), \tag{2.1}$$

and the norm $\|f\| = \sqrt{(f, f)}$, for $f \in C_c(V)$, where $m(x)$, $x \in V$, is the degree of x which is by definition $m(x) = \#\{z \in V; z \sim x\}$. Hereafter we denote by $z \sim x$, that two vertices z and x are connected by an edge. The transition operator P acting $C_c(V)$ is defined by

$$(Pf)(x) = \frac{1}{m(x)} \sum_{z \sim x} f(z), \quad x \in V, \quad f \in C_c(V),$$

and the *Laplacian* Δ acting on $C_c(V)$ is defined by

$$(\Delta f)(x) = f(x) - Pf(x) = \frac{1}{m(x)} \sum_{z \sim x} \bigl(f(x) - f(z)\bigr), \tag{2.2}$$

for all $x \in V$ and $f \in C_c(V)$. The closure of the Laplacian to the L^2-space $L^2(V) = \{f \in C(V); \|f\| < \infty\}$ is also denoted by the same symbol Δ.

We fix an orientation on $G = (V, E)$, once and for all, and let $\mathbf{E}$ be the set of all oriented edges. For $e = [x, y] \in \mathbf{E}$, $x = o(e)$ and $y = t(e)$, the origin and terminal vertices of e, respectively, and $\bar{e} = [y, x] \in \mathbf{E}$, the reverse oriented edge of e.

Let $C(\mathbf{E})$ be the space of all real valued function φ on $\mathbf{E}$ satisfying

$$\varphi(\bar{e}) = -\varphi(e),$$

for all $e \in \mathbf{E}$. Let $C_c(\mathbf{E})$ be the space of $\varphi \in C(\mathbf{E})$ with finite support.

For $f \in C(V)$, let $df \in C(\mathbf{E})$ be the *co-boundary operator* which is defined by

$$df(e) = df([x, y]) = f(y) - f(x), \tag{2.3}$$

for all $e = [x, y] \in \mathbf{E}$. The inner product on $C_c(\mathbf{E})$ is defined by

$$(\varphi_1, \varphi_2) = \sum_{e \in E} \varphi_1(e)\varphi_2(e) = \frac{1}{2} \sum_{e \in \mathbf{E}} \varphi_1(e)\varphi_2(e), \tag{2.4}$$

and the norm is given by $\|\varphi\| = \sqrt{(\varphi, \varphi)}$.

The *co-differential operator* $\delta \colon C_c(\mathbf{E}) \to C_c(V)$ is defined by

$$(\delta\varphi)(x) = -\frac{1}{m(x)} \sum_{e \in \mathbf{E}, o(e) = x} \varphi(e), \tag{2.5}$$

for all $x \in V$ and $\varphi \in C_c(\mathbf{E})$. Then it is known (see for instance [11]) that

Proposition 2.6

(a) For $f \in C_c(V)$ and $\varphi \in C_c(\mathbf{E})$, we have

$$(df, \varphi) = (f, \delta\varphi).$$

(b) For $f \in C_c(V)$, we have

$$\Delta f = \delta df.$$

The following proposition is also useful.

Proposition 2.7

(a) For $f, g \in C_c(V)$, it holds that

$$\Delta(fg)(x) = (\Delta f)(x)g(x) - 2\langle df, dg\rangle(x) + f(x)(\Delta g)(x), \quad x \in V, \tag{2.8}$$

where $\langle df, dg\rangle(x)$ is the pointwise norm which is defined by

$$\langle df, dg\rangle(x) = \frac{1}{2m(x)} \sum_{z\sim x} \big(f(z) - f(x)\big)\big(g(z) - g(x)\big), \quad x \in V, \tag{2.9}$$

and satisfies

$$(df, dg) = \sum_{x \in V} m(x)\langle df, dg\rangle(x). \tag{2.10}$$

(b) For $f \in C(V)$, define a function, $\widetilde{f}$, on $\mathbf{E}$ by

$$\widetilde{f}(e) = \frac{1}{2}\Big(f\big(o(e)\big) + f\big(t(e)\big)\Big), \quad e \in \mathbf{E}. \tag{2.11}$$

Then, for all $g \in C(V)$, $\widetilde{f}dg \in C(\mathbf{E})$, and we have

$$\delta(\widetilde{f}dg)(x) = -\langle df, dg\rangle(x) + f(x)\Delta g(x), \quad x \in V.$$

Proof The proof is a straightforward computation, so we omit the proof of (a). For (b), we show the following two equations:

$$-2\langle df, dg\rangle(x) + f(x)(\Delta g)(x) = -\frac{1}{m(x)} \sum_{e \in \mathbf{E}, o(e)=x} f\big(t(e)\big)dg(e),$$

$$f(x)(\Delta g)(x) = -\frac{1}{m(x)} \sum_{e \in \mathbf{E}, o(e)=x} f\big(o(e)\big)dg(e),$$

which yield (b). The left hand side of the first equation above equals

$$
\begin{aligned}
-\frac{1}{m(x)}\sum_{z\sim x}\big(f(z)-f(x)\big)\big(g(z)-g(x)\big)+f(x)\left(g(x)-\frac{1}{m(x)}\sum_{z\sim x}g(z)\right)\\
=-\frac{1}{m(x)}\sum_{z\sim x}f(z)\big(g(z)-g(x)\big)\\
=-\frac{1}{m(x)}\sum_{e\in\mathbf{E},o(e)=x}f\big(t(e)\big)dg(e),
\end{aligned}
$$

which shows the first equation. For the second one, it suffices only to recall

$$(\Delta g)(x)=-\frac{1}{m(x)}\sum_{e\in\mathbf{E},o(e)=x}dg(e).$$

■

3 Fundamental Formulas

In this section, we preserve the situation in Section 2. For every function $\gamma\in C(V)$, we define a new operator $\mathcal{L}$ and a new inner product $(\ ,\ )_\gamma$ on $C_c(V)$ as follows:

Definition 3.1 For $f\in C_c(V)$, define $\mathcal{L}f\in C_c(V)$ by

$$\mathcal{L}f=\Delta f-e^{\gamma}\langle d(e^{-\gamma}),df\rangle,$$

that is,

$$\mathcal{L}f(x)=(\Delta f)(x)-e^{\gamma(x)}\langle d(e^{-\gamma}),df\rangle(x),\quad x\in V.$$

Definition 3.2 For $f_1,f_2\in C_c(V)$, we define the inner product $(f_1,f_2)_\gamma$ by

$$(f_1,f_2)_\gamma=\sum_{x\in V}m(x)e^{-\gamma(x)}f_1(x)f_2(x).$$

Theorem 3.3 *The operator $\mathcal{L}$ is symmetric with respect to the inner product $(\ ,\ )_\gamma$, that is, it holds that for $f_1,f_2\in C_c(V)$,*

(3.4) $$(\mathcal{L}f_1,f_2)_\gamma=(f_1,\mathcal{L}f_2)_\gamma=(\widetilde{e^{-\gamma}}df_1,df_2).$$

Moreover, the operator $\mathcal{L}$ is positive, that is, for all $f\in C_c(V)$,

$$(\mathcal{L}f,f)_\gamma=\sum_{e\in E}\widetilde{e^{-\gamma}}(e)df(e)^2\geq 0.$$

Proof By Proposition 2.6, we obtain that, for $f_1,f_2\in C_c(V)$,

$$\big(\delta(\widetilde{e^{-\gamma}}df_1),f_2\big)=(\widetilde{e^{-\gamma}}df_1,df_2).$$

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By Proposition 2.7 (b),

$$\begin{aligned}\big(\delta(\widetilde{e^{-\gamma}} df_1), f_2\big) &= (e^{-\gamma}\Delta f_1 - \langle d(e^{-\gamma}, df_1\rangle, f_2) \\ &= (\mathcal{L} f_1, f_2)_\gamma.\end{aligned}$$

Since

$$(\widetilde{e^{-\gamma}} df_1, df_2) = (\widetilde{e^{-\gamma}} df_2, df_1),$$

we obtain Theorem 3.3. ■

Now we take as $\gamma \in C(V)$, $\gamma(x) = cr(x)$, $x \in V$, for every real number $c \in \mathbf{R}$, where $r(x)$, $x \in V$, is the graph distance between x and a fixed vertex p, *i.e.*, the length of a shortest path from p to x. From now on, we denote by $(\ ,\ )_c$, the above inner product $(\ ,\ )_\gamma$ for $\gamma = cr$ with $c \in \mathbf{R}$. We define the transformation T defined by

$$(Tf)(x) = e^{\frac{c}{2}r(x)} f(x), \quad x \in V,$$

for $f \in C(V)$. Then we have

$$(Tf_1, Tf_2)_c = (f_1, f_2), \quad f_1, f_2 \in C_c(V).$$

Moreover, we introduce the following operators.

Definition 3.5

(a) The operator Δ_0 and Δ_1 are defined by

$$\Delta_0 f(x) = -\frac{1}{m(x)} \sum_{\substack{y\sim x,\\ r(y)=r(x)}} \big(f(y) - f(x)\big),$$

$$\Delta_1 f(x) = -\frac{1}{m(x)} \sum_{\substack{y\sim x,\\ r(y)\neq r(x)}} \big(f(y) - f(x)\big),$$

for $x \in V$ with $x \neq x_0$, and $\Delta_0 f(x_0) = 0$ and $\Delta_1 f(x_0) = \Delta f(x_0)$, respectively.

(b) And define also $d_0 f$ and $d_1 f$ in $C(\mathbf{E})$ by

$$d_0 f(e) = d_0 f([x, y]) = \begin{cases} f(y) - f(x), & \text{if } r(y) = r(x) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_1 f(e) = d_1 f([x, y]) = \begin{cases} f(y) - f(x), & \text{if } r(y) \neq r(x) \\ 0, & \text{otherwise.} \end{cases}$$

(c) For $\varphi \in C_c(\mathbf{E})$, define $\delta_0\varphi$ and $\delta_1\varphi$ in $C_c(V)$ by

$$\delta_0\varphi(x) = -\frac{1}{m(x)} \sum_{\substack{e\in\mathbf{E}, o(e)=x,\\ r(t(e))=r(x)}} \varphi(e),$$

$$\delta_1\varphi(x) = -\frac{1}{m(x)} \sum_{\substack{e\in\mathbf{E}, o(e)=x,\\ r(t(e))\neq r(x)}} \varphi(e),$$

respectively.

Then, we have immediately

Proposition 3.6 *For $f \in C_c(V)$ and $\varphi \in C_c(\mathbf{E})$, we have the following.*

(1) $\Delta_0 f(x) = \delta_0 d_0 f(x)$, $\Delta_1 f(x) = \delta_1 d_1 f(x)$, $x \in V$.
(2) $(\delta_0\varphi, f) = (\varphi, d_0 f)$, $(\delta_1\varphi, f) = (\varphi, d_1 f)$.
(3) $(\Delta_0 f, f) \geq 0$, $(\Delta_1 f, f) \geq 0$.
(4) $df = d_0 f + d_1 f$, $\delta\varphi = \delta_0\varphi + \delta_1\varphi$.
(5) $\delta_1 d_0 f = 0$, $\delta_0 d_1 f = 0$.
(6) $\Delta f = \Delta_0 f + \Delta_1 f$.

Moreover, we have

Proposition 3.7 *For a finite or infinite graph G, the following conditions are equivalent.*

(1) G is bipartite,
(2) $m_0(x) = 0$ for each $x \in V$ with $x \neq x_0$,
(3) $r(y) \neq r(x)$ for each $x, y \in V$ with $y \sim x$,
(4) $\Delta_0 = 0$.

Proof Recall that G is bipartite if and only if G contains no odd cycle. Assume that $m_0(x) > 0$ for some $x \in V$. Then there exists $y \in V$ with $y \sim x$ such that $y(y) = r(x)$. Let $z \in V$ be the last branch point in two shortest paths from x_0 to x and y. Then we have an odd cycle in G. Conversely, if we have an odd cycle, $m_0(x) > 0$ for some $x \in V$, clearly. The other equivalence are also clear. ■

We obtain

Theorem 3.8 *For all $f \in C_c(V)$, we have*

$$T^{-1}\big(\mathcal{L}(Tf)\big)(x)$$
$$= \cosh\frac{c}{2}\left\{\Delta_1 f(x) + \left(\frac{m_+(x)}{m(x)}(e^{-\frac{c}{2}} - 1) + \frac{m_-(x)}{m(x)}(e^{\frac{c}{2}} - 1)\right) f(x)\right\} + \Delta_0 f(x).$$

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Proof By definition and (2.8),

$$
\begin{aligned}
T^{-1}\big(\mathcal{L}(Tf)\big)(x) &= e^{-\frac{c}{2}r(x)}\Delta(e^{\frac{c}{2}r}f)(x) - e^{\frac{c}{2}r(x)}\langle de^{-cr}, d(e^{\frac{c}{2}r}f)\rangle(x)\\
&= e^{-\frac{c}{2}r(x)}\{(\Delta e^{\frac{c}{2}r})(x)f(x) - 2\langle de^{\frac{c}{2}r}, df\rangle(x) + e^{\frac{c}{2}r(x)}\Delta f(x)\}\\
&\qquad - e^{\frac{c}{2}r(x)}\langle de^{-cr}, d(e^{\frac{c}{2}r}f)\rangle(x)\\
&= \Delta f(x) + e^{\frac{c}{2}r(x)}\Delta e^{\frac{c}{2}r}(x)f(x) - 2e^{-\frac{c}{2}r(x)}\langle de^{\frac{c}{2}r}, df\rangle(x)\\
&\qquad - e^{\frac{c}{2}r(x)}\langle de^{-cr}, d(e^{\frac{c}{2}r}f)\rangle(x)\\
&= \Delta f(x) + e^{\frac{c}{2}r(x)}\Delta e^{\frac{c}{2}r}(x)f(x)\\
&\qquad - 2e^{\frac{c}{2}r(x)}\langle (e^{-\frac{c}{2}r})^{\sim}de^{-\frac{c}{2}r}, de^{\frac{c}{2}r}\rangle(x)f(x)\\
&\qquad - e^{-\frac{c}{2}r(x)}\langle de^{\frac{c}{2}r}, df\rangle(x) - e^{\frac{c}{2}r(x)}\langle de^{-\frac{c}{2}}, df\rangle(x)\\
&= Hf(x) - Df(x),
\end{aligned}
\tag{3.9}
$$

where we used the equation below, for $de^{-c\gamma} = d(e^{-\frac{c}{2}\gamma}e^{-\frac{c}{2}\gamma})$,

$$
d(fg)(e) = \widetilde{f}(e)dg(e) + \widetilde{g}(e)df(e),
\tag{3.10}
$$

for $f, g \in C(V)$ and $e \in E$, and we put

$$
\begin{cases}
Df(x) = e^{-\frac{c}{2}r(x)}\langle de^{\frac{c}{2}r}, df\rangle(x) + e^{\frac{c}{2}r(x)}\langle de^{-\frac{c}{2}r}, df\rangle(x),\\
Hf(x) = \Delta f(x) + e^{-\frac{c}{2}r(x)}\Delta e^{\frac{c}{2}r}(x)f(x)\\
\qquad -2e^{\frac{c}{2}r(x)}\langle (e^{-\frac{c}{2}r})^{\sim}de^{-\frac{c}{2}r}, de^{-\frac{c}{2}r}\rangle(x)f(x).
\end{cases}
\tag{3.11}
$$

Theorem 3.8 can be obtained immediately by the following lemma.

Lemma 3.12 *The operators D and H are calculated as follows:*

$$
Df(x) = \left(1 - \cosh\frac{c}{2}\right)\Delta_1 f(x),
$$

$$
Hf(x) = \Delta f(x) + \cosh\frac{c}{2}\left(\frac{m_+(x)}{m(x)}(e^{-\frac{c}{2}} - 1) + \frac{m_-(x)}{m(x)}(e^{\frac{c}{2}} - 1)\right)f(x),
$$

for $f \in C_c(V)$ and $x \in V$.

Proof For Df, we have

$$
\begin{aligned}
Df(x) &= e^{-\frac{c}{2}r(x)}\langle de^{\frac{c}{2}r}, df\rangle(x) + e^{\frac{c}{2}r(x)}\langle de^{-\frac{c}{2}r}, df\rangle(x)\\
&= e^{-\frac{c}{2}r(x)}\frac{1}{2m(x)}\sum_{y\sim x}(e^{\frac{c}{2}r(y)} - e^{\frac{c}{2}r(x)})\big(f(y) - f(x)\big)\\
&\qquad + e^{\frac{c}{2}r(x)}\frac{1}{2m(x)}\sum_{y\sim x}(e^{-\frac{c}{2}r(y)} - e^{-\frac{c}{2}r(x)})\big(f(y) - f(x)\big),
\end{aligned}
$$

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where

$$r(y) = \begin{cases} r(x) - 1, \\ r(x), & \text{or} \\ r(x) + 1 \end{cases}$$

if $y \sim x$. Therefore, we obtain

$$\begin{aligned}
Df(x) &= e^{-\frac{c}{2}r(x)} \frac{1}{2m(x)} \sum_{\substack{y\sim x,\\ r(y)=y(x)+1}} (e^{\frac{c}{2}(r(x)+1)} - e^{-\frac{c}{2}r(x)})\big(f(y) - f(x)\big) \\
&\quad + e^{-\frac{c}{2}r(x)} \frac{1}{2m(x)} \sum_{\substack{y\sim x,\\ r(y)=r(x)-1}} (e^{\frac{c}{2}(r(x)-1)} - e^{\frac{c}{2}r(x)})\big(f(y) - f(x)\big) \\
&\quad + e^{\frac{c}{2}r(x)} \frac{1}{2m(x)} \sum_{y\sim x,\ r(y)=r(x)+1} (e^{-\frac{c}{2}(r(x)+1)} - e^{-\frac{c}{2}r(x)})\big(f(y) - f(x)\big) \\
&\quad + e^{\frac{c}{2}r(x)} \frac{1}{2m(x)} \sum_{\substack{y\sim x,\\ r(y)=r(x)-1}} (e^{-\frac{c}{2}(r(x)-1)} - e^{-\frac{c}{2}r(x)})\big(f(y) - f(x)\big) \\
&= \frac{1}{2m(x)}(e^{\frac{c}{2}} - 1) \sum_{\substack{y\sim x,\\ r(y)=r(x)+1}} \big(f(y) - f(x)\big) \\
&\quad + \frac{1}{2m(x)}(e^{-\frac{c}{2}} - 1) \sum_{\substack{y\sim x,\\ r(y)=r(x)-1}} \big(f(y) - f(x)\big) \\
&\quad + \frac{1}{2m(x)}(e^{-\frac{c}{2}} - 1) \sum_{\substack{y\sim x,\\ r(y)=r(x)+1}} \big(f(y) - f(x)\big) \\
&\quad + \frac{1}{2m(x)}(e^{\frac{c}{2}} - 1) \sum_{\substack{y\sim x,\\ r(y)=r(x)-1}} \big(f(y) - f(x)\big) \\
&= (e^{\frac{c}{2}} - 1)\frac{1}{2m(x)} \sum_{\substack{y\sim x,\\ r(y)\neq r(x)}} \big(f(y) - f(x)\big) \\
&\quad + (e^{-\frac{c}{2}} - 1)\frac{1}{2m(x)} \sum_{\substack{y\sim x,\\ r(y)\neq r(x)}} \big(f(y) - f(x)\big) \\
&= -\frac{e^{\frac{c}{2}} - 1}{2}\Delta_1 f(x) - \frac{e^{-\frac{c}{2}} - 1}{2}\Delta_1 f(x) \\
&= \left(1 - \cosh\frac{c}{2}\right)\Delta_1 f(x),
\end{aligned}$$

by definition of Δ_1. Thus, we have the equation for Df in Lemma 3.12.

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For Hf, we calculate

$$
\begin{aligned}
&e^{-\frac{c}{2}r(x)}\Delta e^{\frac{c}{2}r}(x)\\
&= e^{-\frac{c}{2}r(x)}\frac{1}{m(x)}\sum_{y\sim x}(e^{\frac{c}{2}r(x)}-e^{\frac{c}{2}r(y)})\\
&= e^{-\frac{c}{2}r(x)}\frac{1}{m(x)}\Big\{\sum_{\substack{y\sim x,\\ r(y)=r(x)+1}}(e^{\frac{c}{2}r(x)}-e^{\frac{c}{2}(r(x)+1)})+\sum_{\substack{y\sim x,\\ r(y)=r(x)-1}}(e^{\frac{c}{2}r(x)}-e^{\frac{c}{2}(r(x)-1)})\Big\}\\
&= \frac{1}{m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)+1}}(1-e^{\frac{c}{2}})+\frac{1}{m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)-1}}(1-e^{-\frac{c}{2}})\\
&= (1-e^{\frac{c}{2}})\frac{m_+(x)}{m(x)}+(1-e^{-\frac{c}{2}})\frac{m_-(x)}{m(x)},
\end{aligned}
$$

and we have

$$
\begin{aligned}
&2e^{\frac{c}{2}r(x)}\langle (e^{-\frac{c}{2}r})^{\sim} de^{-\frac{c}{2}r}, de^{\frac{c}{2}r}\rangle(x)\\
&= 2e^{\frac{c}{2}r(x)}\frac{1}{2m(x)}\sum_{y\sim x}\frac{e^{-\frac{c}{2}r(x)}+e^{-\frac{c}{2}r(y)}}{2}(e^{-\frac{c}{2}r(y)}-e^{-\frac{c}{2}r(x)})(e^{\frac{c}{2}r(y)}-e^{\frac{c}{2}r(x)})\\
&= 2e^{\frac{c}{2}r(x)}\frac{1}{2m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)+1}}\frac{e^{-\frac{c}{2}r(x)}+e^{-\frac{c}{2}(r(x)+1)}}{2}\\
&\qquad\times(e^{-\frac{c}{2}(r(x)+1)}-e^{-\frac{c}{2}r(x)})(e^{\frac{c}{2}(r(x)+1)}-e^{\frac{c}{2}r(x)})\\
&\qquad+2e^{\frac{c}{2}r(x)}\frac{1}{2m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)-1}}\frac{e^{-\frac{c}{2}r(x)}+e^{-\frac{c}{2}(r(x)-1)}}{2}\\
&\qquad\times(e^{-\frac{c}{2}(r(x)-1)}-e^{-\frac{c}{2}r(x)})(e^{\frac{c}{2}(r(x)-1)}-e^{\frac{c}{2}r(x)})\\
&= \frac{1}{m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)+1}}\frac{e^{-\frac{c}{2}r(x)}+e^{-\frac{c}{2}(r(x)-1)}}{2}(e^{-\frac{c}{2}}-1)(e^{\frac{c}{2}}-1)\\
&\qquad+\frac{1}{m(x)}\sum_{\substack{y\sim x,\\ r(y)=r(x)-1}}\frac{e^{-\frac{c}{2}r(x)}+e^{-\frac{c}{2}(r(x)-1)}}{2}(e^{\frac{c}{2}}-1)(e^{-\frac{c}{2}}-1)\\
&= \frac{m_+(x)}{m(x)}\frac{1+e^{-\frac{c}{2}}}{2}(e^{-\frac{c}{2}}-1)(e^{\frac{c}{2}}-1)+\frac{m_-(x)}{m(x)}\frac{1+e^{\frac{c}{2}}}{2}(e^{\frac{c}{2}}-1)(e^{-\frac{c}{2}}-1)\\
&= \frac{m_+(x)}{m(x)}\frac{1-e^{-c}}{2}(1-e^{\frac{c}{2}})+\frac{m_-(x)}{m(x)}\frac{1-e^{c}}{2}(1-e^{-\frac{c}{2}}).
\end{aligned}
$$

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Therefore, we have

$$
\begin{aligned}
Hf(x) &= \Delta f(x) + \frac{m_+(x)}{m(x)}\left\{1 - e^{\frac{c}{2}} - \frac{1-e^{-c}}{2}(1-e^{\frac{c}{2}})\right\} f(x) \\
&\quad + \frac{m_-(x)}{m(x)}\left\{1 - e^{-\frac{c}{2}} - \frac{1-e^{c}}{2}(1-e^{-\frac{c}{2}})\right\} f(x) \\
&= \Delta f(x) + \frac{m_+(x)}{m(x)}(1-e^{\frac{c}{2}})\left(1 - \frac{1-e^{-c}}{2}\right) f(x) \\
&\quad + \frac{m_-(x)}{m(x)}(1-e^{-\frac{c}{2}})\left(1 - \frac{1-e^{c}}{2}\right) f(x) \\
&= \Delta f(x) + \frac{m_+(x)}{m(x)}(1-e^{\frac{c}{2}})\left(1 + \frac{1-e^{-c}}{2}\right) f(x) \\
&\quad + \frac{m_-(x)}{m(x)}(1-e^{-\frac{c}{2}})\left(1 + \frac{1-e^{c}}{2}\right) f(x) \\
&= \Delta f(x) + \cosh\frac{c}{2}\left(\frac{m_+(x)}{m(x)}(e^{-\frac{c}{2}}-1) + \frac{m_-(x)}{m(x)}(e^{\frac{c}{2}}-1)\right) f(x).
\end{aligned}
$$

We have Lemma 3.12. ■

Thus, we obtain Theorem 3.8. ■

We obtain immediately

Corollary 3.13 *For $g_i = Tf_i$, $(i = 1, 2)$, we have*

$$
\begin{aligned}
&\sum_{e\in E}(e^{-cr})^{\sim}(e)dg_1(e)dg_2(e) \\
&\quad = \sum_{x\in V} m(x)T^{-1}\big(\mathcal{L}(Tf_1)\big)(x)f_2(x) \\
&\quad = \cosh\frac{c}{2}\sum_{x\in V} m(x)\left\{\Delta_1 f_1(x) + \left(\frac{m_+(x)}{m(x)}(e^{-\frac{c}{2}}-1) + \frac{m_-(x)}{m(x)}(e^{\frac{c}{2}}-1)\right) f_1(x)\right\} f_2(x) \\
&\qquad + \sum_{x\in V} m(x)\Delta_0 f_1(x)f_2(x).
\end{aligned}
$$

4 Lower Estimate of the Spectrum

4.1

In this subsection, we show

Proposition 4.1 **(Y. Higuchi)** *For an infinite graph $G = (V, E)$ let $\mathbf{m}_{\pm}(G)$ and $\mathbf{M}_{\pm}(G)$ be the positive numbers as in the introduction. Then $\mathbf{M}_{+}(G) \geq \mathbf{m}_{-}(G)$.*

Proof Assume that $\mathbf{M}_+(G) < \mathbf{m}_-(G)$. Putting $\alpha = \mathbf{M}_+(G)$ and $\beta = \mathbf{m}_-(G)$, we have

$$m_-(x) \geq \beta m(x), \quad \text{and} \quad m_+(x) \leq \alpha m(x), \quad (\forall x \neq x_0).$$

By the assumption, we have

$$m_+(x) - m_-(x) \leq (\alpha - \beta) m(x) < 0 \quad (\forall x \neq x_0).$$

For any $n = 1, 2, \ldots$, let

$$B_n(x_0) = \{x \in V; d(x, x_0) < n\},$$

and

$$\partial B_n(x_0) = \{e \in E; e = (x, y), x \in B_n(x_0), y \notin B_n(x_0)\}.$$

Since for $r = 0, 1, \ldots,$

$$\sum_{r(x)=r} m_+(x) = \sum_{r(x)=r+1} m_-(x),$$

and $m_-(x_0) = 0$, we have

$$\begin{aligned} \#\partial B_n(x_0) &= \sum_{r(x)=n-1} m_+(x) \\ &= \sum_{r=0}^{n-1} \sum_{r(x)=r} \bigl(m_+(x) - m_-(x)\bigr) \\ &= \sum_{r=1}^{n-1} \sum_{r(x)=r} \bigl(m_+(x) - m_-(x)\bigr) + m(x_0) \\ &\leq (\alpha - \beta) \sum_{r=1}^{n-1} \sum_{r(x)=r} m(x) + m(x_0) \\ &= (\alpha - \beta) m_V\bigl(B_n(x_0)\bigr) + (1 + \beta - \alpha) m(x_0). \end{aligned}$$

Since G is of infinite, we may let n tend to infinity, and we have

$$\frac{\#\partial B_n(x_0)}{m_V\bigl(B_n(x_0)\bigr)} \leq (\alpha - \beta) + \frac{(1 + \beta - \alpha) m(x_0)}{m_V\bigl(B_n(x_0)\bigr)} \to \alpha - \beta < 0, \quad \text{as} \quad n \to \infty,$$

which is a contradiction. ■

4.2

We prove Theorem A in this subsection.

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Let $D \subset V$ be a finite subset, and consider the Dirichlet eigenvalue problem for D:

$$\begin{cases} \Delta\varphi = \lambda\varphi, & \text{on } D, \\ \varphi = 0, & \text{on } \partial D, \end{cases}$$

where $\partial D = \{y \in V; y \notin D \text{ and } y \sim x \text{ for some } x \in D\}$, the *boundary* of D. Let λ_D be the first eigenvalue of this problem and φ_D be the corresponding eigenfunction with $\|\varphi_D\| = 1$. Applying Corollary 3.13 and taking $f_i = \varphi_D$ $(i = 1, 2)$, we have

$$\begin{aligned} 0 &\le \sum_{e\in E} (e^{-cr})^{\sim}(e) d(T\varphi_D)(e)^2 + \Big(\cosh\frac{c}{2} - 1\Big)(\Delta_0\varphi_D, \varphi_D) \\ &= \cosh\frac{c}{2}\left(\Delta\varphi_D + \left(\frac{m_+(\cdot)}{m(\cdot)}(e^{-\frac{c}{2}} - 1) + \frac{m_-(\cdot)}{m(\cdot)}(e^{\frac{c}{2}} - 1)\right)\varphi_D, \varphi_D\right). \end{aligned}$$

Therefore, we have

$$0 \le (\Delta\varphi_D, \varphi_D) + \left(\left(\frac{m_+(\cdot)}{m(\cdot)}(e^{-\frac{c}{2}} - 1) + \frac{m_-(\cdot)}{m(\cdot)}(e^{\frac{c}{2}} - 1)\right)\varphi_D, \varphi_D\right),$$

which implies

$$\begin{aligned} \lambda_D &= (\Delta\varphi_D, \varphi_D) \\ &\ge \sum_{x\in V} m(x)\left\{\frac{m_+(x)}{m(x)}(1 - e^{-\frac{c}{2}}) + \frac{m_-(x)}{m(x)}(1 - e^{\frac{c}{2}})\right\}\varphi_D(x)^2 \end{aligned} \tag{4.1}$$

for all $c \in \mathbf{R}$.

Now assume that $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = \varnothing$. By Proposition 4.1, we have $\mathbf{M}_-(G) < \mathbf{m}_+(G)$.

If $c \ge 0$, then $1 - e^{-\frac{c}{2}} \ge 0$ and $1 - e^{\frac{c}{2}} \le 0$, hence we have

$$\frac{m_+(x)}{m(x)}(1 - e^{-\frac{c}{2}}) + \frac{m_-(x)}{m(x)}(1 - e^{\frac{c}{2}}) \ge \mathbf{m}_+(G)(1 - e^{-\frac{c}{2}}) + \mathbf{M}_-(G)(1 - e^{\frac{c}{2}}),$$

for all $x \in V$. Thus, we have

$$\begin{aligned} \sup&\left\{\frac{m_+(x)}{m(x)}(1 - e^{-\frac{c}{2}}) + \frac{m_-(x)}{m(x)}(1 - e^{\frac{c}{2}}); c \ge 0\right\} \\ &\ge \mathbf{m}_+(G) + \mathbf{M}_-(G) - \inf\left\{\mathbf{m}_+(G)e^{\frac{c}{2}} + \mathbf{M}_-(G)e^{-\frac{c}{2}}; c \ge 0\right\} \\ &= \mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_+(G)} \\ &= \left(\sqrt{\mathbf{m}_+(G)} - \sqrt{\mathbf{M}_-(G)}\right)^2, \end{aligned}$$

where the infimum attains at $c = \log\big(\mathbf{m}_+(G)/\mathbf{M}_-(G)\big) > 0$. Then we have

$$\begin{aligned} \lambda_D &\ge \left(\sqrt{\mathbf{m}_+(G)} - \sqrt{\mathbf{M}_-(G)}\right)^2 \sum_{x\in V} m(x)\varphi_D(x)^2 \\ &= \left(\sqrt{\mathbf{m}_+(G)} - \sqrt{\mathbf{M}_-(G)}\right)^2. \end{aligned}$$

Thus, we obtain

$$\inf \sigma(G) \geq \left(\sqrt{\mathbf{m}_+(G)} - \sqrt{\mathbf{M}_-(G)}\right)^2.$$

We have Theorem A. ■

5 The Estimations of the Essential Spectrum (I)

In the sequel, let

$$B_r = \{x \in V; r(x) < r\},$$

and for $0 < s < t$,

$$B_{s,t} = B_t \setminus B_s = \{x \in V; s \leq r(x) < t\}.$$

Define

$$\mathbf{m}_\pm^\infty(G) = \liminf_{s,t\to\infty} \left\{ \frac{m_\pm(x)}{m(x)}; x \in B_{s,t} \right\},$$

$$\mathbf{M}_\pm^\infty(G) = \limsup_{s,t\to\infty} \left\{ \frac{m_\pm(x)}{m(x)}; x \in B_{s,t} \right\},$$

respectively. By the same way of Proposition 4.1, we have

$$\mathbf{m}_-^\infty(G) \leq \mathbf{M}_+^\infty(G).$$

Let $0 < s < t$. Let $\lambda_{s,t}$ be the first eigenvalue of the Dirichlet eigenvalue problem of Δ for $B_{s,t}$ and $\varphi_{s,t}$, the corresponding eigenfunction. Then, by making use of Corollary 3.13, we obtain

$$\begin{aligned} \lambda_{s,t} &= (\Delta\varphi_{s,t}, \varphi_{s,t}) \\ &\geq \sum_{s \in B_{s,t}} m(x) \left\{ \frac{m_+(x)}{m(x)}(1 - e^{-\frac{c}{2}}) + \frac{m_-(x)}{m(x)}(1 - e^{\frac{c}{2}}) \right\} \varphi_{s,t}(x)^2. \end{aligned} \tag{5.1}$$

Notice that for the essential spectrum $\sigma_{\mathrm{ess}}(G)$,

$$\inf \sigma_{\mathrm{ess}}(G) = \lim_{s,t\to\infty} \lambda_{s,t}. \tag{5.2}$$

Thus, by the same way as the proof of Theorem A, we obtain

Theorem 5.3 *Let $G = (V, E)$ be an infinite graph and assume that two closed intervals $[\mathbf{m}_-^\infty(G), \mathbf{M}_-^\infty(G)]$ and $[\mathbf{m}_+^\infty(G), \mathbf{M}_+^\infty(G)]$ have no intersection. Then we obtain*

$$\inf \sigma_{\mathrm{ess}}(G) \geq \left(\sqrt{\mathbf{m}_+^\infty(G)} - \sqrt{\mathbf{M}_-^\infty(G)}\right)^2.$$

(The proof is omitted.)

Therefore, we obtain

Corollary 5.4 *Let $G = (V, E)$ be an infinite bipartite graph and assume that two closed intervals $[\mathbf{m}_-^\infty(G), \mathbf{M}_-^\infty(G)]$ and $[\mathbf{m}_+^\infty(G), \mathbf{M}_+^\infty(G)]$ have no intersection. Then we have*

$$\sigma_{\mathrm{ess}}(G) \subset \left[\left(\sqrt{\mathbf{m}_+^\infty(G)} - \sqrt{\mathbf{M}_-^\infty(G)}\right)^2, \left(\sqrt{\mathbf{m}_+^\infty(G)} + \sqrt{\mathbf{M}_-^\infty(G)}\right)^2\right].$$

6 The Estimations of the Essential Spectrum (II)

For $c \in \mathbf{R}$, let

(6.1)
$$U_c(t) = \sum_{x \in B_t} m(x) e^{-cr(x)}, \quad (0 < t \leq \infty)$$

(possibly $U_c(\infty) = \infty$), where $B_t = \{x \in V; r(x) < t\}$. Then we obtain

Theorem 6.2 *Assume that there exist positive constants A and B and $c \in \mathbf{R}$ satisfying the following two conditions:*

(6.3)
$$\lim_{s\to\infty} \limsup_{t\to\infty} \frac{1}{U_c(t) - U_c(s)} \sum_{x \in B_t \setminus B_s} m(x) \left\{ \left(\frac{m_+(x)}{m(x)} - B \right)^2 + \left(\frac{m_-(x)}{m(x)} - A \right)^2 \right\} e^{-cr(x)} = 0,$$

where $e^c = \frac{B}{A}$, and either in the case $U_c(\infty) = \infty$,

(6.4)
$$\lim_{t\to\infty} U_c(t) e^{-\epsilon t} = 0, \quad (\textit{for all } \epsilon > 0),$$

or in the case $U_c(\infty) < \infty$,

(6.5)
$$\lim_{t\to\infty} \frac{1}{U_c(\infty) - U_c(t)} e^{-\epsilon t} = 0, \quad (\textit{for all } \epsilon > 0).$$

Then we have

$$\sigma_{\mathrm{ess}}(G) \supset \left[\left(\sqrt{A} - \sqrt{B}\right)^2, \left(\sqrt{A} + \sqrt{B}\right)^2 \right].$$

Remark 6.6 (a) This theorem can be regarded as a discrete version of Theorem 1.1 in [3].

(b) The first condition (6.3) in Theorem 6.2 is satisfied if $\frac{m_+(x)}{m(x)} \to B$ and $\frac{m_-(x)}{m(x)} \to A$ uniformly as $r(x) \to \infty$. The second conditions (6.4) and (6.5) are stronger than the condition $U_c(t)$ and $\frac{1}{U_c(\infty) - U_c(t)}$ grow to ∞ subexponentially, that is,

$$\limsup_{t\to\infty} t^{-1} \log U_c(t) = 0,$$

and

$$\limsup_{t\to\infty} t^{-1} \log \left(\frac{1}{U_c(\infty) - U_c(t)} \right) = 0.$$

Example 6.7 Let T_d ($d \geq 3$) be the infinite homogeneous regular tree. For fixed x_0, let $r(x) = d(x, x_0)$, ($x \in T_d$). Since $m_+(x) = d - 1$ and $m_-(x) = 1$ for any $x \neq x_0$, we have

$$\frac{m_+(x)}{m(x)} = B = \frac{d-1}{d}, \quad \frac{m_-(x)}{m(x)} = A = \frac{1}{d} \quad (\forall x \neq x_0).$$

For $c \in \mathbf{R}$, we have

$$U_c(t) = \sum_{x \in B_t} d e^{-cr(x)} = d \sum_{r=0}^{t-1} s_r e^{-cr},$$

where $s_r = \# B_r = d(d-1)^{r-1}$. Thus, we have for any $c \in \mathbf{R}$,

$$U_c(t) = \frac{d^2}{d-1} \frac{e^{t(\log(d-1)-c)} - 1}{e^{\log(d-1)-c} - 1},$$

and if we choose c in such a way that $e^c = \frac{B}{A} = d - 1$,

$$U_c(t) = \frac{d^2}{d-1} t.$$

Therefore, we can apply for the regular homogeneous tree T_d, Theorem 6.2 which implies that

$$\begin{aligned} \sigma_{\text{ess}}(T_d) &\supset \left[\left(\sqrt{\frac{1}{d}} - \sqrt{\frac{d-1}{d}} \right)^2, \left(\sqrt{\frac{1}{d}} + \sqrt{\frac{d-1}{d}} \right)^2 \right] \\ &= \left[1 - \frac{2\sqrt{d-1}}{d}, 1 + \frac{2\sqrt{d-1}}{d} \right]. \end{aligned}$$

Together with Theorems A and B, we have the following well-known fact (*cf.* [7])

$$\sigma(T_d) = \sigma_{\text{ess}}(T_d) = \left[1 - \frac{2\sqrt{d-1}}{d}, 1 + \frac{2\sqrt{d-1}}{d} \right].$$

To prove Theorem 6.2, we need some preparations:

Lemma 6.8 *Let $F(r)$, $r = 0, 1, \ldots$ and $f(x) = F\big(r(x)\big)$, $x \in V$, where $r(x)$ is the distance function from a fixed point $p \in V$. Then the Laplacian Δf is given by*

$$\Delta f(x) = \frac{m_+(x)}{m(x)} \big(F(r) - F(r+1) \big) + \frac{m_-(x)}{m(x)} \big(F(r) - F(r-1) \big).$$

Proof This follows from by definitions of f and Δ. ■

We have immediately

Lemma 6.9 *Assume that $\frac{m_+(x)}{m(x)} = B$ and $\frac{m_-(x)}{m(x)} = A$ for all $x \in B_k$, where $B_k = \{x \in V; r(x) < k\}$. The eigenvalue problem for B_k,*

$$\Delta f(x) = \lambda f(x), \quad x \in B_k,$$

with the eigenvalue λ satisfying that $(\sqrt{A} - \sqrt{B})^2 < \lambda < (\sqrt{A} + \sqrt{B})^2$, has the following radial solutions $f(x)$, i.e., which is of the form $f(x) = F\big(r(x)\big)$, $x \in V$ is given by

$$F(r) = C_1 f_1(r) + C_2 f_2(r),$$

where C_1 and C_2 are two arbitrary constants, and $f_i(r)$, $i = 1, 2$ are given by

$$f_1(r) = e^{-\frac{c}{2}r}\sin(\lambda_c r); \quad f_2(r) = e^{-\frac{c}{2}r}\cos(\lambda_c r).$$

Here $e^{-\frac{c}{2}} = \sqrt{\frac{A}{B}}$ and λ_c is given by

$$\cos\lambda_c = -\frac{\lambda - A - B}{2\sqrt{AB}}.$$

Proof of Theorem 6.2 Now let us begin the proof of Theorem 6.2. For positive integers $0 < s < t$, let $h(r)$ be a function in $r = 0, 1, \dots$ satisfying that

$$\begin{cases} 0 \le h(r) \le 1, & r = 0, 1, 2, \dots \\ h(r) = 1, & s \le r \le t-1, \\ h(r) = 0, & 0 \le r \le s-1 \text{ or } t \le r < \infty. \end{cases}$$

For two positive integers $0 < s < t$, let $g_{s,t}$ be a function on V defined by

$$g_{s,t}(x) = \begin{cases} h\big(r(x)\big) f_1\big(r(x)\big), & \text{if } \sum_{s \le r(x) < t-1} m(x)\sin^2\big(\lambda_c r(x)\big)e^{-cr(x)} \\ & \qquad \ge \frac{1}{2}\{U_c(t-1) - U_c(s)\}, \\ h\big(r(x)\big) f_2\big(r(x)\big), & \text{otherwise.} \end{cases}$$

Then we have

Lemma 6.10 *The function $g_{s,t}$ satisfies*

$$\|g_{s,t}\|^2 = \sum_{x \in V} m(x) g_{s,t}(x)^2 \ge \frac{1}{2}\{U_c(t-1) - U_c(t)\}.$$

Proof In the case

$$\sum_{s \le r(x) < t-1} m(x)\sin^2\big(\lambda_c r(x)\big)e^{-cr(x)} \ge \frac{1}{2}\{U_c(t-1) - U_c(s)\},$$

we have

$$\begin{aligned} \|g_{s,t}\|^2 &= \sum_{x \in V} m(x) h\big(r(x)\big)^2 f_1\big(r(x)\big)^2 \\ &= \sum_{s \le r(x) < t-1} m(x)\sin^2\big(\lambda_c r(x)\big)e^{-cr(x)} \\ &\ge \frac{1}{2}\{U_c(t-1) - U_c(t)\}. \end{aligned}$$

In the remained case, that is,

$$\sum_{s \le r(x) < t-1} m(x)\sin^2\big(\lambda_c r(x)\big)e^{-cr(x)} < \frac{1}{2}\{U_c(t-1) - U_c(s)\},$$

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we have

$$\begin{aligned}\|g_{s,t}\|^2 &= \sum_{x\in V} m(x)h\big(r(x)\big)^2 f_2\big(r(x)\big)^2\\ &= \sum_{x\in V} m(x)h\big(r(x)\big)^2 e^{-cr(x)}\cos^2\big(\lambda_c r(x)\big)\\ &= \sum_{s\le r(x)<t-1} m(x)h\big(r(x)\big)^2 - \sum_{s\le r(x)<t-1} m(x)h\big(r(x)\big)^2 e^{-cr(x)}\sin^2\big(\lambda_c r(x)\big)\\ &\ge U_c(t-1)-U_c(s)-\frac{1}{2}\{U_c(t-1)-U_c(s)\}\\ &= \frac{1}{2}\{U_c(t-1)-U_c(s)\},\end{aligned}$$

we have Lemma 6.10. ∎

Let $\lambda \in \mathbf{R}$ satisfy

$$\big(\sqrt{A}-\sqrt{B}\big)^2 < \lambda < \big(\sqrt{A}+\sqrt{B}\big)^2.$$

To show Theorem 6.2, it suffices to prove

$$\inf_{2<s+1<t} \frac{\|(\Delta-\lambda)g_{s,t}\|^2}{\|g_{s,t}\|^2} = 0.$$

Note that

$$(6.11)\qquad \begin{aligned}(\Delta-\lambda)g_{s,t} = \Delta(h\circ r)(f_i\circ r) - 2\langle d(h\circ r), d(f_i\circ r)\rangle + (h\circ r)\Delta(f_i\circ r)\\ - \lambda(h\circ r)(f_i\circ r).\end{aligned}$$

We have

Lemma 6.12 *We have*

$$(6.13)\qquad \|\Delta(h\circ r)f_i\circ r\|^2 \le U_c(s)-U_c(s-1)+U_c(t)-U_c(t-1),$$

$$(6.14)\qquad \|\langle d(h\circ r), d(f_i\circ r)\rangle\|^2 \le (e^{|c|}+1)\{U_c(s)-U_c(s-1)+U_c(t)-U_c(t-1)\},$$

and

$$(6.15)\qquad \begin{aligned}&\|(h\circ r)\Delta(f_i\circ r)-\lambda(h\circ r)(f_i\circ r)\|\\ &\le 4(e^{|c|}+1)\sum_{x\in B_t-B_s} m(x)\left\{\left(\frac{m_+(x)}{m(x)}-B\right)^2+\left(\frac{m_-(x)}{m(x)}-A\right)^2\right\}e^{-cr(x)}.\end{aligned}$$

Continued Proof of Theorem 6.2 By Lemma 6.12, we obtain

$$\frac{\|(\Delta-\lambda)g_{s,t}\|^2}{\|g_{s,t}\|^2} \le 2(e^{|c|}+2)\frac{U_c(s)-U_c(s-1)+U_c(t)-U_c(t-1)}{U_c(t-1)-U_c(s)}.$$

By virtue of the condition of (6.4) or (6.5), there exist sequences $\{s_m\}_{m=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ which are divergent to infinity as $m \to \infty$ or $n \to \infty$, satisfying that

$$(1) \quad \lim_{m\to\infty} \frac{U_c(s_m) - U_c(s_m - 1)}{U_c(\infty) - U_c(s_m)} = 0, \quad \text{and} \quad (2) \quad \lim_{n\to\infty} \frac{U_c(t_n) - U_c(t_n - 1)}{U_c(t_n - 1)} = 0.$$

Because, if $U_c(\infty) = \infty$, (1) is trivial. We assume that (2) does not hold for any sequence $\{t_n\}$ divergent to infinity. Then there exists a positive number $\delta > 0$ such that

$$\frac{U_c(n) - U_c(n-1)}{U_c(n-1)} \geq \delta \quad (\forall n = N, N+1, \dots).$$

Then we have

$$U_c(n) \geq (1+\delta) U_c(n-1), \quad (\forall n = N, N+1, \dots),$$

which implies that there exists a positive constant $C_1 > 0$ such that

$$U_c(n) \geq C_1 (1+\delta)^n, \quad (\forall n = N, N+1, \dots).$$

But, if we choose $0 < \epsilon < \log(1+\delta)$,

$$U_c(n) e^{-\epsilon n} \geq C_1 e^{n(\log(1+\delta) - \epsilon)} \to \infty \quad (n \to \infty),$$

which contradicts (6.4).

In the case $U_c(\infty) < \infty$, (2) is trivial. In order to show (1), we take $V_c(t) = U_c(\infty) - U_c(t)$, which satisfies that $V_c(t)$ converges to 0 when t tends to infinity. The condition (6.5) is equivalent to the one that

$$\lim_{t\to\infty} \frac{1}{V_c(t)} e^{-\epsilon t} = 0, \quad (\forall \epsilon > 0).$$

The statement (1) is equivalent to that there exists a sequence $\{s_m\}_{i=1}^{\infty}$ satisfying that

$$(1') \quad \lim_{m\to\infty} \frac{V_c(s_m - 1) - V_c(s_m)}{V_c(s_m)} = 0,$$

which can be shown by a similar way as the first case. We omit its proof.

Hence we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{U_c(s_m) - U_c(s_m - 1) + U_c(t_n) - U_c(t_n - 1)}{U_c(t_n - 1) - U_c(s_m)} = 0.$$

Therefore, we obtain the desired conclusion:

$$\inf_{2<s+1<t} \frac{\|(\Delta - \lambda) g_{s,t}\|^2}{\|g_{s,t}\|^2} = 0. \qquad \blacksquare$$

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We give here the proof of Lemma 6.12.

Proof of (6.13) If either $r(x) < s-1$, $s < r(x) < t-1$ or $t < r(x)$, we have

$$\Delta(h\circ r)(x) = (h\circ r)(x) - \sum_{z\sim x}(h\circ r)(z) = 0. \tag{6.16}$$

Because, if $z\sim x$, it holds that

$$\begin{cases} h\big(r(z)\big) = h\big(r(x)\big) = 0, & \text{if } r(x) < s-1,\\ h\big(r(z)\big) = h\big(r(x)\big) = 1, & \text{if } s-1 < r(x) < t,\\ h\big(r(z)\big) = h\big(r(x)\big) = 0, & \text{if } t < r(x). \end{cases}$$

Therefore, the left hand side of (6.13) coincides with

$$\sum_{r(x)=s-1} m(x)\big(\Delta(h\circ r)(x) f_i\circ r(x)\big)^2 + \sum_{r(x)=t-1} m(x)\big(\Delta(h\circ r)(x) f_i\circ r(x)\big)^2.$$

Note that

$$f_i\big(r(x)\big) = e^{-\frac{c}{2}r(x)}\sin^2\big(\lambda_c r(x)\big), \quad \text{or} \quad e^{-\frac{c}{2}r(x)}\cos^2\big(\lambda_c r(x)\big),$$

and if $r(x) = s-1$ or $r(x) = t-1$,

$$|\Delta(h\circ r)(x)| \le 1,$$

by the same reason of (6.16). Therefore, the left hand side of (6.13) is smaller than or equal to

$$\sum_{r(x)=s-1,\text{ or } r(x)=t-1} m(x)e^{-cr(x)} = U_c(s) - U_c(s-1) + U_c(t) - U_c(t-1).$$

We have (6.13). ∎

Proof of (6.14) By definition of $\langle\ ,\ \rangle$, we have

$$\begin{aligned}
&\|\langle d(h\circ r), d(f_i\circ r)\rangle\|^2\\
&\quad= \sum_{x\in V}\frac{1}{m(x)}\Big\{\sum_{z\sim x}\Big(h\big(r(z)\big) - h\big(r(x)\big)\Big)\Big(f_i\big(r(z)\big) - f_i\big(r(x)\big)\Big)\Big\}^2\\
&\quad\le \sum_{\substack{r(x)=s-1\text{ or}\\ r(x)=t-1}}\frac{1}{m(x)}\Big\{\sum_{z\sim x}\big|f_i\big(r(z)\big) - f_i\big(r(x)\big)\big|^2\Big\}\\
&\quad\le \sum_{\substack{r(x)=s-1\text{ or}\\ r(x)=t-1}}\Big\{\sum_{z\sim x}\big|f_i\big(r(z)\big) - f_i\big(r(x)\big)\big|^2\Big\}\\
&\quad= \sum_{\substack{r(x)=s-1\text{ or}\\ r(x)=t-1}}\left\{\sum_{z\sim x}\left|e^{-\frac{c}{2}r(z)}\begin{Bmatrix}\sin\lambda_c r(z)\\ \cos\lambda_c r(z)\end{Bmatrix} - e^{-\frac{c}{2}r(x)}\begin{Bmatrix}\sin\lambda_c r(x)\\ \cos\lambda_c r(x)\end{Bmatrix}\right|^2\right\}
\end{aligned}$$

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$$\leq \sum_{\substack{r(x)=s-1 \text{ or} \\ r(x)=t-1}} m(x)(e^{|c|}+1)e^{-cr(x)}$$

$$\leq (e^{|c|}+1)\{U_c(s)-U_c(s-1)+U_c(t)-U_c(t-1)\}.$$

We have (6.14). ■

Proof of (6.15) To show (6.15), we put

$$\Delta_0(f\circ r) = B\big(f(r)-f(r+1)\big) + A\big(f(r)-f(r-1)\big)$$

for a function f on V. Then we have

$$\|(h\circ r)\Delta(f_i\circ r)-\lambda(h\circ r)(f_i\circ r)\|^2 \leq 2\|(h\circ r)\Delta(f_i\circ r)-(h\circ r)\Delta_0(f_i\circ r)\|^2 + 2\|(h\circ r)\Delta_0(f_i\circ r)-\lambda(h\circ r)(f_i\circ r)\|^2.$$

The first term of the right hand side can be estimated as follows:

$$\begin{aligned}
&\|(h\circ r)\Delta(f_i\circ r)-(h\circ r)\Delta_0(f_i\circ r)\|^2 \\
&= \sum_{x\in B_t-B_s} m(x)\big(\Delta(f_i\circ r)(x)-\Delta_0(f_i\circ r)(x)\big)^2 \\
&= \sum_{x\in B_t-B_s} m(x)\Bigg\{\left(\frac{m_+(x)}{m(x)}-B\right)\Big(f_i\big(r(x)\big)-f_i\big(r(x)+1\big)\Big) \\
&\qquad + \left(\frac{m_-(x)}{m(x)}-A\right)\Big(f_i\big(r(x)\big)-f_i\big(r(x)-1\big)\Big)\Bigg\}^2 \qquad (6.17)\\
&\leq \sum_{x\in B_t-B_s} m(x)\left\{\left(\frac{m_+(x)}{m(x)}-B\right)^2+\left(\frac{m_-(x)}{m(x)}-A\right)^2\right\} \\
&\qquad \times\left\{\Big(f_i\big(r(x)\big)-f_i\big(r(x)+1\big)\Big)^2+\Big(f_i\big(r(x)\big)-f_i\big(r(x)-1\big)\Big)^2\right\} \\
&\leq 2(e^{|c|}+1)\sum_{x\in B_t-B_s} m(x)\left\{\left(\frac{m_+(x)}{m(x)}-B\right)^2+\left(\frac{m_-(x)}{m(x)}-A\right)^2\right\}e^{-cr(x)}
\end{aligned}$$

since, by definition of f_i, we have

$$\Big(f_i\big(r(x)\big)-f_i\big(r(x)+1\big)\Big)^2+\Big(f_i\big(r(x)\big)-f_i\big(r(x)-1\big)\Big)^2 \leq 2(e^{|c|}+1)e^{-cr(x)}.$$

On the other hand, the second term vanishes because of Lemmas 6.8 and 6.9, and definition of f_i. Thus, we have (6.15). ■

Thus, we obtain Theorem 6.2. ■

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7 Examples

Example 1 Let $G = (V, E)$ be an infinite tree satisfying

$$3 \le k \le m(x) \le \ell, \quad \forall x \in V.$$

Since $m_0(x) = 0$, $m_-(x) = 1$ and $m_+(x) = m(x) - 1$ $(x \ne x_0)$,

$$\mathcal{M}_-(G) = \left\{ \frac{m_-(x)}{m(x)}; x \in V - \{x_0\} \right\} \subset \left[\frac{1}{\ell}, \frac{1}{k} \right] \subset \left[0, \frac{1}{3} \right],$$

and

$$\mathcal{M}_+(G) = \left\{ \frac{m_+(x)}{m(x)}; x \in V - \{x_0\} \right\} \subset \left[1 - \frac{1}{k}, 1 - \frac{1}{\ell} \right] \subset \left[\frac{2}{3}, 1 \right],$$

whence we have

$$\mathbf{m}_-(G) \ge \frac{1}{\ell}, \mathbf{m}_+(G) \ge 1 - \frac{1}{k}, \mathbf{M}_-(G) \le \frac{1}{k}, \mathbf{M}_+(G) \le 1 - \frac{1}{\ell}.$$

Since G is bipartite, by virtue of Corollary B, we have

$$\sigma(G) \subset \left[1 - \frac{2\sqrt{k-1}}{k}, 1 + \frac{2\sqrt{k-1}}{k} \right],$$

which is known (see for example [7]).

Example 2 Let $G = \mathbf{Z}^d$ be the integer lattice graph in $\mathbf{R}^d$ $(d \ge 1)$. In this case, it is known that

$$\sigma(G) = \sigma_{\mathrm{ess}}(G) = [0, 2]$$

and every estimate should collapse. We have to see our estimate in Theorem A is compatible in this case: For all $x \in V - \{x_0\}$, $m(x) = 2d$, and $m_0(x) = 0$,

$$m_+(x) = 2d - 2, \quad d, \quad 2d - 1, \quad d + 1,$$
$$m_-(x) = 2, \quad d, \quad 1, \quad d - 1,$$

respectively. Thus, we obtain

$$\mathbf{m}_-(G) = \frac{1}{2d}, \quad \mathbf{M}_-(G) = \frac{1}{2}, \quad \mathbf{m}_+(G) = \frac{1}{2}, \quad \text{and} \quad \mathbf{M}_+(G) = \frac{2d-1}{2d}.$$

In this case, we have $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = \{1/2\}$ and

$$\mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_-(G)} = 0.$$

Example 3 Let $G = (V, E)$ be the triangle lattice in $\mathbf{R}^2$. In this case, for all $x \in V - \{x_0\}$, $m_0(x) = 0$, and

$$m_+(x) = 2, 3; \quad m_-(x) = 2, 1,$$

respectively. Thus, we have

$$\mathcal{M}_-(G) = \left\{ \frac{m_-(x)}{m(x)}; x \in V - \{x_0\} \right\} = \left\{ \frac{2}{6}, \frac{1}{6} \right\},$$
$$\mathcal{M}_+(G) = \left\{ \frac{m_+(x)}{m(x)}; x \in V - \{x_0\} \right\} = \left\{ \frac{2}{6}, \frac{3}{6} \right\},$$

and

$$\mathbf{m}_-(G) = \frac{1}{6}, \quad \mathbf{M}_-(G) = \frac{2}{6}, \quad \mathbf{m}_+(G) = \frac{2}{6}, \quad \mathbf{M}_+(G) = \frac{3}{6}.$$

Hence we have $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = \{2/6\}$ and

$$\mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_-(G)} = 0.$$

Indeed, it is known that

$$\sigma(G) = \left[0, \frac{3}{2}\right].$$

Example 4 Let $G = (V, E)$ be the Sierpinski gasket (*cf.* [12]). In this case, $m(x_0) = 2$ and for all $x \in V - \{x_0\}$, $m(x) = 4$ and

$$m_0(x) = 1, 2; \quad m_+(x) = 2, 0; \quad m_-(x) = 1, 2,$$

respectively. Therefore, we have

$$\mathcal{M}_+(G) = \left\{ 0, \frac{2}{4} \right\}, \quad \mathcal{M}_-(G) = \left\{ \frac{1}{4}, \frac{2}{4} \right\},$$

hence we have

$$\mathbf{m}_-(G) = \frac{1}{4}, \quad \mathbf{M}_-(G) = \frac{2}{4}, \quad \mathbf{m}_+(G) = 0, \quad \mathbf{M}_+(G) = \frac{2}{4}.$$

Thus, we have $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = [1/4, 2/4]$, and

$$\mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_-(G)} = \frac{1}{2}.$$

Indeed, it is known that

$$\inf \sigma(G) = 0.$$

Example 5 Let G be a regular infinite graph each vertex of which is the intersection of d triangles ($d \geq 2$). In this case,

$$m(x) = 2d, \quad m_-(x) = 1, \quad m_+(x) = 2d - 2, \quad m_0(x) = 1.$$

Thus, we have

$$\mathcal{M}_+(G) = \left\{ \frac{m_+(x)}{m(x)}; x \in V - \{x_0\} \right\} = \left\{ \frac{2d-2}{2d} \right\},$$
$$\mathcal{M}_-(G) = \left\{ \frac{m_-(x)}{m(x)}; x \in V - \{x_0\} \right\} = \left\{ \frac{1}{2d} \right\}.$$

We have

$$\mathbf{m}_+(G) = \mathbf{M}_+(G) = 1 - \frac{1}{d}, \quad \mathbf{m}_-(G) = \mathbf{M}_-(G) = \frac{1}{2d},$$

thus we have

$$\mathbf{m}_+(G) + \mathbf{M}_-(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_-(G)} = 1 - \frac{1}{2d} - \frac{\sqrt{2(d-1)}}{d}.$$

By Theorem A, we obtain

$$\inf \sigma(G) \geq 1 - \frac{1}{2d} - \frac{\sqrt{2(d-1)}}{d}.$$

We can apply Theorem 6.2, which implies that

$$\sigma_{\mathrm{ess}}(G) \supset \left[\left(\sqrt{\frac{1}{2d}} - \sqrt{\frac{2d-2}{2d}} \right)^2, \left(\sqrt{\frac{1}{2d}} + \sqrt{\frac{2d-2}{2d}} \right)^2 \right].$$

Therefore, we obtain

$$\inf \sigma(G) = \inf \sigma_{\mathrm{ess}}(G) = \left(\sqrt{\frac{1}{2d}} - \sqrt{\frac{2d-2}{2d}} \right)^2.$$

On the other hand, we do not know the upper estimate of $\sup \sigma(G)$ since G is not bipartite.

Example 6 Let $G = (V, E)$ be a regular infinite graph each of vertex which is the intersection of n $2m-$gons ($n \geq 2$, $m \geq 2$). In this case, $m(x) = 2n$, and for all $x \in V - \{x_0\}$, $m_0(x) = 0$, and

$$m_+(x) = 2n-2, 2n-1,$$
$$m_-(x) = 2, 1,$$

respectively. Thus, we obtain

$$\mathbf{m}_-(G) = \frac{1}{2n}, \quad \mathbf{M}_-(G) = \frac{2}{2n}, \quad \mathbf{m}_+(G) = \frac{2n-2}{2n}, \quad \mathbf{M}_+(G) = \frac{2n-1}{2n}.$$

Since G is bipartite, by Theorem A and Corollary B, we have

$$\sigma(G) \subset \left[1 - \frac{2\sqrt{n-1}}{n}, 1 + \frac{2\sqrt{n-1}}{n} \right].$$

Example 7 Let $G = (V, E)$ be a regular infinite graph each vertex of which is the intersection of n $(2m+1)-$gons $(n \geq 2, m \geq 1)$. In this case, $m(x) = 2n$, and for all $x \in V - \{x_0\}$,

$$m_+(x) = 2n-2, 2n-1; \quad m_-(x) = 1, 1; \quad m_0(x) = 1, 0,$$

respectively. Then we have

$$\mathcal{M}_+(G) = \left\{\frac{2n-2}{2n}, \frac{2n-1}{2n}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{2n}\right\}.$$

Thus, we have

$$\mathbf{m}_+(G) = \frac{2n-2}{2n}, \quad \mathbf{M}_+(G) = \frac{2n-1}{2n}, \quad \mathbf{m}_-(G) = \mathbf{M}_-(G) = \frac{1}{2n}.$$

By Theorem A, we obtain

$$\inf \sigma(G) \geq \frac{1}{2n} + \frac{2n-2}{2n} - 2\sqrt{\frac{1}{2n}\frac{2n-2}{2n}} = 1 - \frac{1}{2n} - \frac{\sqrt{2n-2}}{n}.$$

Since G is not bipartite, we do not know the estimate of $\sup \sigma(G)$.

Example 8 Let $G = (V, E)$ be the distance regular graph, $D_{m,s}$ $(m, s \geq 2)$, *i.e.*, each vertex is the intersection of m copies of the complete graph K_s. In this case, for all $x \in V - \{x_0\}$,

$$m(x) = (s-1)m, \quad m_+(x) = (s-1)(m-1), \quad m_-(x) = 1.$$

Then we have

$$\mathcal{M}_+(G) = \left\{\frac{m-1}{m}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{(s-1)m}\right\}.$$

I.e., we have $\mathbf{m}_+(G) = \mathbf{M}_+(G) = \frac{m-1}{m}$ and $\mathbf{m}_-(G) = \mathbf{M}_-(G) = \frac{1}{(s-1)m}$. By Theorem A,

$$\inf \sigma(G) \geq \left(\sqrt{\frac{m-1}{m}} - \sqrt{\frac{1}{(s-1)m}}\right)^2.$$

By Theorem 6.2,

$$\sigma_{\text{ess}}(G) \supset \left[\left(\sqrt{\frac{m-1}{m}} - \sqrt{\frac{1}{(s-1)m}}\right)^2, \left(\sqrt{\frac{m-1}{m}} + \sqrt{\frac{1}{(s-1)m}}\right)^2\right].$$

Therefore, we obtain

$$\inf \sigma(G) = \inf \sigma_{\text{ess}}(G) = \left(\sqrt{\frac{m-1}{m}} - \sqrt{\frac{1}{(s-1)m}}\right)^2.$$

On the other hand, we do not know the upper estimate of $\sup\sigma(G)$ since G is not bipartite.

Example 9 Let $G = (V, E)$ be the free product of $K_r * K_s$ $(r \geq s \geq 2)$ of two complete graphs K_r and K_s. In this case, $m(x) = r + s - 2$, for all $x \in V - \{x_0\}$,

$$m_+(x) = s-1, r-1; \quad m_-(x) = 1, 1; \quad m_0(x) = r-1, s-2,$$

respectively. Thus, we have

$$\mathcal{M}_+(G) = \left\{\frac{s-1}{r+s-2}, \frac{r-1}{r+s-2}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{r+s-2}\right\}.$$

Then, we have

$$\mathbf{m}_+(G) = \frac{r-1}{r+s-2}, \quad \mathbf{M}_+(G) = \frac{s-1}{r+s-2}, \quad \mathbf{m}_-(G) = \mathbf{M}_-(G) = \frac{1}{r+s-2}.$$

By Theorem A, we have

$$\inf\sigma(G) \geq \left(\sqrt{\frac{r-1}{r+s-2}} - \sqrt{\frac{1}{r+s-2}}\right)^2.$$

In the case $r = s$, we can apply Theorem 6.2, and we have

$$\sigma_{\text{ess}}(G) \supset \left[\left(\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2(r-1)}}\right)^2, \left(\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2(r-1)}}\right)^2\right],$$

and then

$$\inf\sigma(G) = \inf\sigma_{\text{ess}}(G) = \left(\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2(r-1)}}\right)^2,$$

but we do not know the upper bound of $\sigma(G)$ since G is not bipartite.

Example 10 Let $G = (V, E)$ be the free product of $C_r * C_s$ $(r \geq s \geq 2)$ of circles C_r, C_s of length r, s, respectively.

Case 1: $r \geq s \geq 4$ ***and*** r ***and*** s ***are even.*** In this case, $m(x) = 4$, and $m_+(x) = 2, 3$ and $m_-(x) = 1, 2$, respectively, and $m_0(x) = 0$ for all $x \in V - \{x_0\}$. Then we have

$$\mathcal{M}_+(G) = \left\{\frac{2}{4}, \frac{3}{4}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{4}, \frac{2}{4}\right\},$$

and

$$\mathbf{m}_+(G) = \frac{2}{4}, \quad \mathbf{M}_+(G) = \frac{3}{4}, \quad \mathbf{m}_-(G) = \frac{1}{4}, \quad \mathbf{M}_-(G) = \frac{2}{4}.$$

Thus, we have $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = \{2/4\}$, and

$$\mathbf{m}_+(G) + \mathbf{M}_+(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_+(G)} = 0.$$

Case 2: $r \geq s = 2$ ***and*** r***, even.*** In this case, $m(x) = 3$, $m_+(x) = 2, 1$, $m_-(x) = 1, 2$, respectively, and $m_0(x) = 0$ for all $x \in V - \{x_0\}$. Then we have

$$\mathcal{M}_+(G) = \left\{\frac{1}{3}, \frac{2}{3}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{3}, \frac{2}{3}\right\},$$

and

$$\mathbf{m}_+(G) = \frac{1}{3}, \quad \mathbf{M}_+(G) = \frac{2}{3}, \quad \mathbf{m}_-(G) = \frac{1}{3}, \quad \mathbf{M}_-(G) = \frac{2}{3}.$$

In this case, $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = [1/3, 2/3]$.

Case 3: $r \geq s \geq 2$***, and*** r ***and*** s ***are odd.*** In this case, $m(x) = 4$, $m_+(x) = 3, 2$, $m_-(x) = 1, 1$, and $m_0(x) = 0, 1$, respectively, for all $x \in V - \{x_0\}$. Thus, we have

$$\mathcal{M}_+(G) = \left\{\frac{2}{4}, \frac{3}{4}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{4}\right\},$$

and

$$\mathbf{m}_+(G) = \frac{2}{4}, \quad \mathbf{M}_+(G) = \frac{3}{4}, \quad \mathbf{m}_-(G) = \mathbf{M}_-(G) = \frac{1}{4}.$$

Thus, we have

$$\mathbf{m}_+(G) + \mathbf{M}_+(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_+(G)} = 1 - \frac{1 + 2\sqrt{2}}{4},$$

by Theorem A,

$$\inf \sigma(G) \geq 1 - \frac{1 + 2\sqrt{2}}{4}.$$

Since G is not bipartite, we do not know the estimation of $\sup \sigma(G)$.

We notice here that Proposition 8.4 and its proof in [9] should read as follows: The Green kernel of the free product $C_r * C_s$ with $r \geq s \geq 3$ and r, s odd, is estimated by

$$G_{T_4}(\tilde{x}, \tilde{y}) \leq G(x, y) \leq G_{T_3}(\tilde{x}', \tilde{y}'),$$

for all vertices x, y of G, $\tilde{x}$, $\tilde{y}$ of T_4 and $\tilde{x}'$, $\tilde{y}'$ of T_3, with $\rho(x, y) = \rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}', \tilde{y}')$.

Case 4: $r \geq s \geq 2$***, either*** r ***even,*** s ***odd, or*** s ***odd,*** s ***even.*** In this case, $m(x) = 4$, $m_+(x) = 3, 2, 2$, $m_-(x) = 1, 2, 1$ and $m_0(x) = 0, 0, 1$, respectively.

$$\mathcal{M}_+(G) = \left\{\frac{2}{4}, \frac{3}{4}\right\}, \quad \mathcal{M}_-(G) = \left\{\frac{1}{4}, \frac{2}{4}\right\},$$

and

$$\mathbf{m}_+(G) = \frac{2}{4}, \quad \mathbf{M}_+(G) = \frac{3}{4}, \quad \mathbf{m}_-(G) = \frac{1}{4}, \quad \mathbf{M}_-(G) = \frac{2}{4}.$$

Thus, we have $[\mathbf{m}_-(G), \mathbf{M}_-(G)] \cap [\mathbf{m}_+(G), \mathbf{M}_+(G)] = \{2/4\}$, and

$$\mathbf{m}_+(G) + \mathbf{M}_+(G) - 2\sqrt{\mathbf{m}_+(G)\mathbf{M}_+(G)} = 0.$$

References

[1] N. Alon, *Eigenvalues and expanders.* Combinatorica **6**(1986), 83–96.
[2] J. Dodziuk and W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality.* In: From Local Times to Global Geometry, Control and Physics (ed. K. D. Elworthy), Longman Scientific and Technical, 1986, 68–75.
[3] K. D. Elworthy and F-Y. Wang, *On the essential spectrum of the Laplacian on Riemannian manifolds.* To appear.
[4] K. Fujiwara, *On the bottom of the spectrum of the Laplacian on graphs.* In: Geometry and Its Applications (ed. T. Nagano *et al.*), World Scientific, Singapore. 1993, 21–27.
[5] H. Kumura, *On the essential spectrum of the Laplacian on complete manifolds.* J. Math. Soc. Japan **49**(1997), 1–14.
[6] B. Mohar, *Isoperimetric inequalities, growth, and the spectrum of graphs.* Linear Alg. Appl. **103**(1988), 119–131.
[7] B. Mohar and W. Woess, *A survey on spectra of infinite graph.* Bull. London Math. Soc. **21**(1989), 209–234.
[8] J. Tan, *On Cheeger inequalities of a graph.* To appear.
[9] H. Urakawa, *Heat kernel and Green kernel comparison theorems for infinite graphs.* J. Funct. Anal. **146**(1997), 206–235.
[10] ———, *The eigenvalue comparison theorems of the discrete Laplacians for a graph.* Geom. Dedicata **74**(1999), 95–112.
[11] ———, *Laplacian and Networks.* Shokabo, Tokyo, 1996 (Japanese).
[12] J. Kigami, *A harmonic calculus on the Sierpinski spaces.* Japan J. Appl. Math. **6**(1989), 259–290.

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