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OSCILLATION CRITERIA FOR SECOND ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Some oscillation criteria are given for the second order neutral delay differential equation

$$[a(t)(x(t)+p(t)x(t-\tau))']'+q(t)f(x(t-\sigma))=0,$$

where τ and σ are nonnegative constants, $a, p, q \in C([t_0,\infty);\Re)$ and $f \in C(\Re;\Re)$. These results generalize and improve some known results about both neutral and delay differential equations.

1. **Introduction.** Consider the second order neutral delay differential equation

$$\text{(E)} \qquad \Big[a(t)\big(x(t)+p(t)x(t-\tau)\big)'\Big]'+q(t)f\big(x(t-\sigma)\big)=0,$$

where $t \geq t_0$, τ and σ are nonnegative constants, $a, p, q \in C\big([t_0,\infty);\Re\big)$, $f \in C(\Re;\Re)$. Throughout this paper, we assume that

(a) $0 \leq p(t) \leq 1$, $q(t) \geq 0$, $a(t) > 0$;
(b) $\int^\infty \frac{1}{a(s)}\,ds = \infty$;
(c) $\frac{f(x)}{x} \geq \gamma > 0$ for $x \neq 0$.

Second order neutral delay differential equations have applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems (see Hale [3]).

Let $u \in C\big([t_0-\theta,\infty);\Re\big)$, where $\theta = \max\{\tau,\sigma\}$, be a given function and let y_0 be a given constant. Using the method of steps, equation (E) has a unique solution $x \in C\big([t_0-\theta,\infty);\Re\big)$ in the sense that both $x(t)+p(t)x(t-\tau)$ and $a(t)\big(x(t)+p(t)x(t-\tau)\big)'$ are continuously differentiable for $t \geq t_0$, $x(t)$ satisfies equation (E) and

$$x(s)=u(s) \qquad \text{for } s \in [t_0-\theta, t_0],$$
$$[x(t)+p(t)x(t-\tau)]'_{t=t_0}=y_0.$$

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, see Hale [3].

A solution of equation (E) is called oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory. Equation (E) is oscillatory if all its solutions are oscillatory.

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Special cases of equation (E) are the following delay equation

$$(E_1) \qquad [a(t)x'(t)]' + q(t)f\big(x(t-\sigma)\big) = 0$$

and ordinary differential equation

$$(E_2) \qquad [a(t)x'(t)]' + q(t)f\big(x(t)\big) = 0.$$

If $a(t) = 1, f\big(x(t)\big) = x(t)$, then equations (E), (E_1) and (E_2) become the following linear second order neutral delay differential equation

$$(E_3) \qquad [x(t) + p(t)x(t-\tau)]'' + q(t)x(t-\sigma) = 0,$$

delay differential equation

$$(E_4) \qquad x''(t) + q(t)x(t-\sigma) = 0,$$

and ordinary differential equation

$$(E_5) \qquad x''(t) + q(t)x(t) = 0.$$

Averaging function method is one of the most important techniques in studying oscillation. Using this technique, many oscillation criteria have been found which involve the behavior of the integrals of the coefficients. For the linear ordinary differential equation (E_5), we list some of more important oscillation criteria as follows:

1. Leighton [6]. (E_5) is oscillatory if
$$\int_{t_0}^{\infty} q(s)\,ds = \infty.$$
2. Wintner [12]. (E_5) is oscillatory if
$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds = \infty.$$
3. Hartman [4]. (E_5) is oscillatory if
$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds < \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} q(u)\,du\,ds \le \infty.$$
4. Kamenev [5]. (E_5) is oscillatory if
$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)\,ds = \infty \qquad \text{for some integer } n > 1.$$
5. Yan [14]. (E_5) is oscillatory if there exist a continuous ϕ on $[t_0, \infty)$ and an integer $n > 1$ satisfying
$$\limsup_{t\to\infty} \frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)\,ds < \infty,$$
$$\limsup_{t\to\infty} \int_{u}^{t}(t-s)^n q(s)\,ds > \phi(u) \qquad \text{for all } u \ge t_0$$

and

$$\int_{t_0}^{\infty} \phi_+(u)\,du = \infty, \quad \phi_+(u) = \max\{\phi(u), 0\}.$$

6. Philos [9]. Suppose that $H: D \equiv \{(t,s) : t \geq s \geq t_0\} \longrightarrow \Re$ is a continuous function, which is such that

$$H(t,t) = 0 \quad \text{for } t \geq t_0,$$
$$H(t,s) > 0 \quad \text{for } t > s \geq t_0$$

and has a continuous and nonpositive partial derivative on D with respect to the second variable. Let $h: D \longrightarrow \Re$ be a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D.$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)q(s) - \frac{1}{4}h^2(t,s) \right\} ds = \infty,$$

then (E_5) is oscillatory.

For the oscillation of the nonlinear ordinary differential equation (E_2), we refer to the recent paper of Wong [13] and the references cited therein.

In [11], Waltman extended Leighton's criterion to equation (E_4) and showed that (E_4) is oscillatory if $q(t) \geq 0$ and

$$\int_{t_0}^{\infty} q(s)\,ds = \infty.$$

But, Travis [10] showed that Leighton's criterion is not enough to ensure the oscillation of (E_4). Hence, the oscillation analysis of the delay differential equations is more complicated than that of ordinary differential equations.

There has recently been an increase in the studying the oscillation for the second order neutral delay differential equations. The results of Waltman and Travis have been extended to neutral delay differential equations by Grammatikopoulos, Ladas and Meimaridou [2]. They proved that if

$$0 \leq p(t) \leq 1, \quad q(t) \geq 0$$

and

$$\int_{t_0}^{\infty} q(s)[1 - p(s-\sigma)]\,ds = \infty,$$

then (E_3) is oscillatory. In [8], by using Riccati technique and averaging functions method and following the results of Yan and Philos, Ruan [8] established some general oscillation criteria for second order neutral delay differential equation (E). One of which is as follows.

RUAN'S THEOREM. *Suppose conditions (a), (b) and (c) hold. Let* $H(t,s)$, $h(t,s)\colon D = \{(t,s) : t \geq s \geq t_0\} \longrightarrow \Re$ *be continuous with*

$$H(t,t) = 0 \quad \textit{for } t \geq t_0,$$
$$H(t,s) > 0 \quad \textit{for } t > s \geq t_0,$$

$\frac{\partial H}{\partial t}(t,s) \geq 0$, $\frac{\partial H}{\partial s}(t,s)$ *nonpositive and continuous, and such that*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \textit{for all } (t,s) \in D.$$

Assume further that

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big\{ H(t,s)\gamma q(s)[1 - p(s-\sigma)] - \frac{1}{4} a(s-\sigma) h^2(t,s) \Big\}\, ds = \infty.$$

Then the nonlinear neutral equation (E) is oscillatory.

The purpose of this paper is to improve the above mentioned oscillation criteria by using a generalized Riccati transformation due to Yu [15].

2. **Main Results.** The following theorem provides sufficient conditions for the oscillation of the nonlinear neutral delay differential equation (E).

THEOREM 1. *Assume* $D = \{(t,s) \mid t \geq s \geq t_0\}$. *Let* $H \in C(D;\Re)$ *satisfy the following two conditions:*

(i) $H(t,t) = 0$ *for* $t \geq t_0$,
$H(t,s) > 0$ *for* $t > s \geq t_0$;

(ii) H has a continuous and nonpositive partial derivative on D with respect to the second variable.

Suppose that $h\colon D \longrightarrow \Re$ is a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D.$$

If there exists a function $\phi \in C^1[t_0,\infty)$ such that

$$(C_1) \qquad \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s) a(s-\sigma) h^2(t,s) \Big]\, ds = \infty,$$

where $\Phi(s) = \exp\{-2\int^s \phi(\xi)\, d\xi\}$ and

$$\psi(s) = \Phi(s)\{\gamma q(s)[1 - p(s-\sigma)] + a(s-\sigma)\phi^2(s) - [a(s-\sigma)\phi(s)]'\},$$

then the nonlinear neutral equation (E) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (E). Without loss of generality, we may assume that $x(t) > 0$ on $[T_0 - \tau - \sigma, \infty)$, for some $T_0 \geq t_0$. Define

$$(1) \qquad z(t) = x(t) + p(t)x(t-\tau) \quad \text{for all } t \geq T_0.$$

It follows from condition (a) that $z(t) \geq x(t) > 0$ for $t \geq T_0$ and

$$[a(t)z'(t)]' \leq 0 \quad \text{for } t \geq T_0. \tag{2}$$

Therefore, $a(t)z'(t)$ is a decreasing function. We claim that

$$z'(t) \geq 0 \quad \text{for } t \geq T_0. \tag{3}$$

Otherwise, there is a $t_1 \geq T_0$ such that $z'(t_1) < 0$. It follows from (2) that

$$z(t) \leq z(t_1) + a(t_1)z'(t_1) \int_{t_1}^{t} \frac{1}{a(s)}\, ds.$$

Hence, by condition (b), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t) > 0$ for $t \geq T_0$. Now observe that from (E) we have

$$[a(t)z'(t)]' + q(t)f\big(x(t-\sigma)\big) = 0. \tag{4}$$

Using condition (c) and (1) in (4), we get

$$[a(t)z'(t)]' + \gamma q(t)[z(t-\sigma) - p(t-\sigma)x(t-\tau-\sigma)] \leq 0,$$

which, in view of the fact that $z(t) \geq x(t)$ and $z(t)$ is increasing, yields

$$[a(t)z'(t)]' + \gamma q(t)[1 - p(t-\sigma)]z(t-\sigma) \leq 0.$$

Define

$$w(t) = \Phi(t)\Big\{\frac{a(t)z'(t)}{z(t-\sigma)} + a(t-\sigma)\phi(t)\Big\}, \tag{5}$$

then

$$\begin{aligned} w'(t) \leq\ & -2\phi(t)w(t) \\ & + \Phi(t)\Big\{-\gamma q(t)[1-p(t-\sigma)] - \frac{a(t)z'(t)z'(t-\sigma)}{z^2(t-\sigma)} + [a(t-\sigma)\phi(t)]'\Big\}. \end{aligned}$$

Using the fact that $a(t)z'(t)$ is decreasing, we get

$$a(t)z'(t) \leq a(t-\sigma)z'(t-\sigma) \quad \text{for } t \geq T_0.$$

Thus

$$\begin{aligned} w'(t) \leq\ & -2\phi(t)w(t) \\ & + \Phi(t)\Big\{-\gamma q(t)[1-p(t-\sigma)] - \frac{1}{a(t-\sigma)}\Big(\frac{a(t)z'(t)}{z(t-\sigma)}\Big)^2 + [a(t-\sigma)\phi(t)]'\Big\} \\ =\ & -2\phi(t)w(t) \\ & + \Phi(t)\Big\{-\gamma q(t)[1-p(t-\sigma)] - \frac{1}{a(t-\sigma)}\Big(\frac{w(t)}{\Phi(t)} - a(t-\sigma)\phi(t)\Big)^2 \\ & + [a(t-\sigma)\phi(t)]'\Big\} \\ =\ & -\psi(t) - \frac{w^2(t)}{a(t-\sigma)\Phi(t)}, \end{aligned} \tag{6}$$

where $\psi(t) = \Phi(t)\{\gamma q(t)[1 - p(t-\sigma)] + a(t-\sigma)\phi^2(t) - [a(t-\sigma)\phi(t)]'\}$. Hence, for all $t \geq T \geq T_0$, we have

$$\begin{aligned}
&\int_T^t H(t,s)\psi(s)\,ds \\
&\quad \leq H(t,T)w(T) - \int_T^t \Big\{ \Big(-\frac{\partial H}{\partial s}(t,s)\Big) w(s) + H(t,s)\frac{w^2(s)}{a(s-\sigma)\Phi(s)} \Big\}\,ds \\
&\quad = H(t,T)w(T) - \int_T^t \Big\{ h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)\frac{w^2(s)}{a(s-\sigma)\Phi(s)} \Big\}\,ds \\
&\quad = H(t,T)w(T) - \int_T^t \Big\{ \sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s) \Big\}^2\,ds \\
&\qquad + \frac{1}{4}\int_T^t \Phi(s)a(s-\sigma)h^2(t,s)\,ds.
\end{aligned}$$

Then, for all $t \geq T \geq T_0$

$$\begin{aligned}
&\int_T^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\
&\quad \leq H(t,T)w(T) - \int_T^t \Big\{ \sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s) \Big\}^2\,ds.
\end{aligned} \tag{7}$$

This implies that for every $t \geq T_0$

$$\begin{aligned}
\int_{T_0}^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds &\leq H(t,T_0)w(T_0) \\
&\leq H(t,T_0)|w(T_0)| \\
&\leq H(t,t_0)|w(T_0)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{t_0}^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\
&\quad = \int_{t_0}^{T_0} \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\
&\qquad + \int_{T_0}^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\
&\quad \leq H(t,t_0)\int_{t_0}^{T_0} |\psi(s)|\,ds + H(t,t_0)|w(T_0)| \\
&\quad = H(t,t_0)\Big\{\int_{t_0}^{T_0} |\psi(s)|\,ds + |w(T_0)|\Big\}
\end{aligned}$$

for all $t \geq T_0$. This gives

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\
&\quad \leq \int_{t_0}^{T_0} |\psi(s)|\,ds + |w(T_0)|,
\end{aligned}$$

which contradicts (C_1). This completes the proof of the Theorem.

The following corollary is an extension of Moore's criterion [7] to the second order neutral delay differential equation (E).

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COROLLARY 1. *Suppose*

$$(C_2) \qquad \liminf_{t\to\infty} g(t)\int_t^\infty \gamma q(s)[1-p(s-\sigma)]\,ds > \frac{1}{4},$$

where

$$g(t) = \int_{t_0}^t \frac{1}{a(s-\sigma)}\,ds.$$

Then the nonlinear neutral equation (E) is oscillatory.

PROOF. By (C_2), there are two numbers $T \geq t_0$ and $k > \frac{1}{4}$ such that

$$g(t)\int_t^\infty \gamma q(s)[1-p(s-\sigma)]\,ds > k \quad \text{for } t \geq T.$$

Let

$$H(t,s) = [g(t)-g(s)]^2 \quad \text{and} \quad \phi(t) = -\frac{1}{2a(t-\sigma)g(t)}.$$

Then

$$h(t,s) = 2g'(s) = \frac{2}{a(s-\sigma)} \quad \text{and} \quad \Phi(t) = g(t).$$

Thus

$$\begin{aligned} &H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s) \\ &\quad = [g(t)-g(s)]^2 g(s)\Big\{\gamma q(s)[1-p(s-\sigma)] - \frac{1}{4a(s-\sigma)g^2(s)}\Big\} - \frac{g(s)}{a(s-\sigma)}. \end{aligned}$$

Define

$$R(t) = \int_t^\infty \gamma q(s)[1-p(s-\sigma)]\,ds.$$

Then, for all $t \geq T$

$$\begin{aligned} &\int_T^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds \\ &\quad = \int_T^t [g(t)-g(s)]^2 g(s)\,d\Big(-R(s)+\frac{1}{4g(s)}\Big) - \int_T^t \frac{g(s)}{a(s-\sigma)}\,ds \\ &\quad = [g(t)-g(T)]^2 g(T)\Big(R(T)-\frac{1}{4g(T)}\Big) - \frac{1}{2}[g^2(t)-g^2(T)] \\ &\qquad + \int_T^t \Big[g(s)R(s)-\frac{1}{4}\Big]\Big(-4g(t)+3g(s)+\frac{g^2(t)}{g(s)}\Big)g'(s)\,ds \\ &\quad \geq \Big(k-\frac{1}{4}\Big)\Big[\Big(-\frac{5}{2}-\ln g(T)\Big)g^2(t)+g^2(t)\ln g(t)\Big] - \frac{1}{2}g^2(t). \end{aligned}$$

This implies

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \Big[H(t,s)\psi(s) - \frac{1}{4}\Phi(s)a(s-\sigma)h^2(t,s)\Big]\,ds = \infty.$$

By condition (b), it is equivalent to (C_1). It follows from Theorem 1 that equation (E) is oscillatory.

EXAMPLE 1. Consider the following second order neutral delay differential equation

$$(E_6) \qquad \left(x(t)+\frac{1}{\sqrt{t-1}}x(t-1)\right)''+\frac{\lambda}{\sqrt{t^3(t-1)}}x(t-1)=0, \quad t\geq 2.$$

Then

$$\liminf_{t\to\infty} t\int_t^\infty \frac{\lambda}{\sqrt{s^3(s-1)}}\left(1-\frac{1}{\sqrt{s-2}}\right)ds=\lambda.$$

Hence, by Corollary 1, equation (E_6) is oscillatory if $\lambda > \frac{1}{4}$. However, Theorem 1 of [2] and Theorem 1 of [8] fail to apply equation (E_6). We note that equation (E_6) has a nonoscillatory solution $x(t)=\sqrt{t}$ if $\lambda=\frac{1}{4}$.

THEOREM 2. *Let H(t,s) and h(t,s) be as in Theorem 1, and let*

$$(C_3) \qquad 0<\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\leq\infty.$$

Suppose that there exist two functions $\phi\in C^1[t_0,\infty)$ and $A\in C[t_0,\infty)$ satisfying

$$(C_4) \qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t a(s-\sigma)\Phi(s)h^2(t,s)\,ds<\infty,$$

$$(C_5) \qquad \int_{t_0}^\infty \frac{A_+^2(s)}{a(s-\sigma)\Phi(s)}\,ds=\infty$$

and for every $T\geq t_0$

$$(C_6) \qquad \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left\{H(t,s)\psi(s)-\frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\right\}ds\geq A(T),$$

where $\Phi(s)=\exp\{-2\int^s\phi(\xi)d\xi\}$, $A_+(s)=\max\{A(s),0\}$ and

$$\psi(s)=\Phi(s)\{\gamma q(s)[1-p(s-\sigma)]+a(s-\sigma)\phi^2(s)-[a(s-\sigma)\phi(s)]'\}.$$

Then the nonlinear neutral equation (E) is oscillatory.

PROOF. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t)>0$ on $[T_0-\tau-\sigma,\infty)$, for some $T_0\geq t_0$. Set $z(t)=x(t)+p(t)x(t-\tau)$, as in the proof of Theorem 1, (7) holds for all $t\geq T\geq T_0$. Hence, for $t>T\geq T_0$, we have

$$\begin{aligned}&\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\psi(s)-\frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\right]ds\\&\quad\leq w(T)-\frac{1}{H(t,T)}\int_T^t\left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2ds.\end{aligned}$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[H(t,s)\psi(s) - \frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\Big]\, ds$$
$$\le w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2 ds.$$

for all $T \ge T_0$. Thus, by (C_6),

$$w(T) \ge A(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2 ds$$

for all $T \ge T_0$. This shows that

$$w(T) \ge A(T) \tag{8}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2 ds < \infty$$

for all $T \ge T_0$. Then

$$\begin{aligned}&\liminf_{t\to\infty}\left\{\frac{1}{H(t,T_0)}\int_{T_0}^t \frac{H(t,s)}{a(s-\sigma)\Phi(s)}w^2(s)\,ds + \frac{1}{H(t,T_0)}\int_{T_0}^t h(t,s)\sqrt{H(t,s)}w(s)\,ds\right\}\\ &\quad\le \liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t\left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2 ds\\ &\quad< \infty.\end{aligned} \tag{9}$$

Define

$$u(t) = \frac{1}{H(t,T_0)}\int_{T_0}^t \frac{H(t,s)}{a(s-\sigma)\Phi(s)}w^2(s)\,ds$$

and

$$v(t) = \frac{1}{H(t,T_0)}\int_{T_0}^t h(t,s)\sqrt{H(t,s)}w(s)\,ds$$

for all $t \ge T_0$. Then (9) implies that

$$\liminf_{t\to\infty}[u(t)+v(t)] < \infty. \tag{10}$$

Now, we claim that

$$\int_{T_0}^\infty \frac{w^2(s)}{a(s-\sigma)\Phi(s)}\,ds < \infty. \tag{11}$$

Suppose to the contrary that

$$\int_{T_0}^\infty \frac{w^2(s)}{a(s-\sigma)\Phi(s)}\,ds = \infty. \tag{12}$$

By (C_3), there is a positive constant M_1 satisfying

$$\inf_{s \geq t_0} \left\{ \liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right\} > M_1 > 0. \tag{13}$$

Let M_2 be any arbitrary positive number. Then it follows from (12) that there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{w^2(s)}{a(s-\sigma)\Phi(s)}\, ds \geq \frac{M_2}{M_1} \qquad \text{for all } t \geq T_1.$$

Therefore,

$$\begin{aligned}
u(t) &= \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) d\left\{ \int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi-\sigma)\Phi(\xi)}\, d\xi \right\} \\
&= \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left\{ \int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi-\sigma)\Phi(\xi)}\, d\xi \right\} ds \\
&\geq \frac{1}{H(t,T_0)} \int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left\{ \int_{T_0}^{s} \frac{w^2(\xi)}{a(\xi-\sigma)\Phi(\xi)}\, d\xi \right\} ds \\
&\geq \frac{M_2}{M_1 H(t,T_0)} \int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) ds \\
&= \frac{M_2 H(t,T_1)}{M_1 H(t,T_0)}
\end{aligned}$$

for all $t \geq T_1$. By (13), there is a $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq M_1 \qquad \text{for all } t \geq T_2,$$

this implies

$$u(t) \geq M_2 \qquad \text{for all } t \geq T_2.$$

Since M_2 is arbitrary,

$$\lim_{t \to \infty} u(t) = \infty. \tag{14}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} [u(t_n) + v(t_n)] = \liminf_{t\to\infty} [u(t) + v(t)].$$

It follows from (10) that there exists a number M such that

$$u(t_n) + v(t_n) \leq M \qquad \text{for } n = 1, 2, 3, \ldots. \tag{15}$$

It follows from (14) that

$$\lim_{n\to\infty} u(t_n) = \infty. \tag{16}$$

This and (15) give

$$\lim_{n\to\infty} v(t_n) = -\infty. \tag{17}$$

Then, by (15) and (16),

$$1 + \frac{v(t_n)}{u(t_n)} \le \frac{M}{u(t_n)} < \frac{1}{2} \quad \text{for } n \text{ large enough.}$$

Thus,

$$\frac{v(t_n)}{u(t_n)} < -\frac{1}{2} \quad \text{for all large } n.$$

This and (17) imply that

$$\lim_{n\to\infty} \frac{v^2(t_n)}{u(t_n)} = \infty. \tag{18}$$

On the other hand, by the Schwarz inequality, we have

$$\begin{aligned}
v^2(t_n) &= \left\{ \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} h(t_n,s)\sqrt{H(t_n,s)}w(s)\,ds \right\}^2 \\
&\le \left\{ \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} \frac{H(t_n,s)}{a(s-\sigma)\Phi(s)} w^2(s)\,ds \right\} \left\{ \frac{1}{H(t_n,T_0)} \int_{T_0}^{t_n} a(s-\sigma)\Phi(s)h^2(t_n,s)\,ds \right\} \\
&\le u(t_n) \left\{ \frac{1}{H(t_n,T_0)} \int_{t_0}^{t_n} a(s-\sigma)\Phi(s)h^2(t_n,s)\,ds \right\}
\end{aligned}$$

for any positive integer n. Consequently,

$$\frac{v^2(t_n)}{u(t_n)} \le \frac{1}{H(t_n,T_0)} \int_{t_0}^{t_n} a(s-\sigma)\Phi(s)h^2(t_n,s)\,ds \quad \text{for all large } n.$$

But, (13) guarantees that

$$\liminf_{t\to\infty} \frac{H(t,T_0)}{H(t,t_0)} > M_1.$$

This means that there exists a $T_3 \ge T_0$ such that

$$\frac{H(t,T_0)}{H(t,t_0)} \ge M_1 \quad \text{for all } t \ge T_3.$$

Thus,

$$\frac{H(t_n,T_0)}{H(t_n,t_0)} \ge M_1 \quad \text{for } n \text{ large enough}$$

and therefore

$$\frac{v^2(t_n)}{u(t_n)} \le \frac{1}{M_1 H(t_n,t_0)} \int_{t_0}^{t_n} a(s-\sigma)\Phi(s)h^2(t_n,s)\,ds \quad \text{for all large } n.$$

It follows from (18) that

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} a(s-\sigma)\Phi(s)h^2(t_n,s)\,ds = \infty. \tag{19}$$

This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}a(s-\sigma)\Phi(s)h^2(t,s)\,ds=\infty,$$

which contradicts (C_4). Then (11) holds. Hence, by (8),

$$\int_{T_0}^{\infty}\frac{A_+^2(s)}{a(s-\sigma)\Phi(s)}\,ds\le\int_{T_0}^{\infty}\frac{w^2(s)}{a(s-\sigma)\Phi(s)}\,ds<\infty,$$

which contradicts (C_5). This completes our proof.

THEOREM 3. *Let $H(t,s)$ and $h(t,s)$ be as in Theorem 1, and let (C_3) hold. Suppose that there exist two functions $\phi\in C^1[t_0,\infty)$ and $A\in C[t_0,\infty)$ such that (C_5) and the following conditions hold:*

$$(C_7)\qquad \liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\psi(s)\,ds<\infty,$$

and for every $T\ge t_0$

$$(C_8)\qquad \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left\{H(t,s)\psi(s)-\frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\right\}ds\ge A(T),$$

where $\Phi(s)=\exp\{-2\int^s\phi(\xi)\,d\xi\}$, $A_+(s)=\max\{A(s),0\}$ and

$$\psi(s)=\Phi(s)\{\gamma q(s)[1-p(s-\sigma)]+a(s-\sigma)\phi^2(s)-[a(s-\sigma)\phi(s)]'\}.$$

Then the nonlinear neutral equation (E) is oscillatory.

PROOF. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t)>0$ on $[T_0-\tau-\sigma,\infty)$ for some $T_0\ge t_0$. Set $z(t)=x(t)+p(t)x(t-\tau)$, as in the proof of Theorem 1, (7) holds for all $t\ge T\ge T_0$. Hence, for $t>T\ge T_0$, we have

$$\begin{aligned}&\frac{1}{H(t,T)}\int_{T}^{t}\Big[H(t,s)\psi(s)-\frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\Big]ds\\&\qquad\le w(T)-\frac{1}{H(t,T)}\int_{T}^{t}\left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2ds.\end{aligned}$$

Consequently,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Big[H(t,s)\psi(s)-\frac{1}{4}a(s-\sigma)\Phi(s)h^2(t,s)\Big]ds\\&\qquad\le w(T)-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2ds.\end{aligned}$$

for all $T\ge T_0$. It follows from (C_8) that

$$w(T)\ge A(T)+\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left\{\sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}}w(s)+\frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s)\right\}^2ds$$

for all $T \geq T_0$. Hence, (8) holds and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{ \sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}} w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s) \right\}^2 ds < \infty$$

for all $T \geq T_0$. This implies that

$$\begin{aligned}
&\limsup_{t\to\infty} [u(t) + v(t)] \\
&\quad = \limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left\{ \frac{H(t,s)}{a(s-\sigma)\Phi(s)} w^2(s) + h(t,s)\sqrt{H(t,s)}w(s) \right\} ds \\
&\quad \leq \limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \left\{ \sqrt{\frac{H(t,s)}{a(s-\sigma)\Phi(s)}} w(s) + \frac{1}{2}\sqrt{\Phi(s)a(s-\sigma)}h(t,s) \right\}^2 ds \\
&\quad < \infty,
\end{aligned} \tag{20}$$

where $u(t)$ and $v(t)$ are defined as in the proof of Theorem 2. By (C_8),

$$\begin{aligned}
A(t_0) &\leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left\{ H(t,s)\psi(s) - \frac{1}{4} a(s-\sigma)\Phi(s)h^2(t,s) \right\} ds \\
&\leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\psi(s)\, ds \\
&\qquad - \frac{1}{4} \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t a(s-\sigma)\Phi(s)h^2(t,s)\, ds.
\end{aligned}$$

This and (C_7) imply that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \Phi(s)a(s-\sigma)h^2(t,s)\, ds < \infty.$$

Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in (t_0,∞) with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \Phi(s)a(s-\sigma)h^2(t_n,s)\, ds \\
&\qquad = \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \Phi(s)a(s-\sigma)h^2(t,s)\, ds < \infty.
\end{aligned} \tag{21}$$

Now, suppose that (12) holds. Using the procedure of the proof of Theorem 2, we conclude that (14) is satisfied. It follows from (20) that there exists a constant M such that (15) is fulfilled. Then, as in the proof of Theorem 2, we see that (19) holds, which contradicts (21). This contradiction proves that (12) fails. Since the remainder of the proof is similar to that of Theorem 2, so we omit the detail.

Now, let the function $H(t,s)$ be defined by

$$H(t,s) = (t-s)^n, \quad t \geq s \geq t_0,$$

where n is an integer with $n > 1$. Then H is continuous on $D = \{(t,s) : t \geq s \geq t_0\}$ and satisfies

$$\begin{aligned}
H(t,t) &= 0 \quad \text{for } t \geq t_0, \\
H(t,s) &> 0 \quad \text{for } t > s \geq t_0.
\end{aligned}$$

Moreover, H has a continuous and nonpositive partial derivative on D with respect to the second variable. Clearly, the function

$$h(t,s) = n(t-s)^{\frac{n}{2}-1}, \qquad t \geq s \geq t_0$$

is continuous and satisfies

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)}, \qquad t \geq s \geq t_0.$$

We see that (C_3) holds because for every $s \geq t_0$

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^n}{(t-t_0)^n} = 1.$$

Then, by Theorems 1 and 2, we have following two corollaries.

COROLLARY 2. *Let n be an integer with $n > 1$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} \left\{ (t-s)^n \psi(s) - \frac{n^2}{4}(t-s)^{n-2} \Phi(s) a(s-\sigma) \right\} ds = \infty,$$

where $\Phi(s) = \exp\{-2\int^s \phi(\xi)\,d\xi\}$ and

$$\psi(s) = \Phi(s)\{\gamma q(s)[1-p(s-\sigma)] + a(s-\sigma)\phi^2(s) - [a(s-\sigma)\phi(s)]'\}.$$

Then the nonlinear neutral equation (E) *is oscillatory.*

COROLLARY 3. *Let $n > 1$ be an integer and suppose that there exists two functions $\phi \in C^1[t_0,\infty)$ and $A \in C[t_0,\infty)$ satisfying (C_5) and*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^{n-2} a(s-\sigma)\Phi(s)\,ds < \infty$$

and for every $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{T}^{t} \left\{ (t-s)^n \psi(s) - \frac{n^2}{4}(t-s)^{n-2} a(s-\sigma)\Phi(s) \right\} ds \geq A(T),$$

where $\Phi(s) = \exp\{-2\int^s \phi(\xi)\,d\xi\}$ and

$$\psi(s) = \Phi(s)\{\gamma q(s)[1-p(s-\sigma)] + a(s-\sigma)\phi^2(s) - [a(s-\sigma)\phi(s)]'\}.$$

Then the nonlinear neutral equation (E) *is oscillatory.*

REMARK 1. Corollary 2 gives an extension of Kamenev's criterion [5] to the second order nonlinear neutral differential equation (E).

REMARK 2. If $f\big(x(t)\big) = x(t)$ and $a(t) = 1$, then Corollary 2 improves Theorem 1 of Grammatikopoulos, Ladas and Meimaridou [2].

REMARK 3. If $p(t) = 0$, then Corollary 2 improves the results of Waltman [11] and Travis [10].

EXAMPLE 2. Consider the following second order neutral delay differential equation

$$(E_7) \qquad \left[\frac{1}{t+\pi}\left(x(t)+\frac{1}{t+\pi}x(t-2\pi)\right)'\right]'+\frac{\lambda}{t^3}x(t-\pi)=0, \quad t\geq 2\pi.$$

It is clear that Ruan's theorem cannot be applied to (E_7). Let $\phi(t)=-\frac{1}{t}$. Then $\Phi(t)=t^2$ and

$$\begin{aligned}\psi(t)&=\Phi(t)\{q(t)[1-p(t-\pi)]+a(t-\pi)\phi^2(t)-[a(t-\pi)\phi(t)]'\}\\&=t^2\left\{\frac{\lambda}{t^3}\left(1-\frac{1}{t}\right)-\frac{1}{t^3}\right\}=\frac{\lambda-1}{t}-\frac{\lambda}{t^2}.\end{aligned}$$

Choose $n=2$. Then we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t^n}\int_{2\pi}^{t}\left\{(t-s)^n\psi(s)-\frac{n^2}{4}\Phi(s)a(s-\pi)(t-s)^{n-2}\right\}ds\\&=\limsup_{t\to\infty}\frac{1}{t^2}\int_{2\pi}^{t}\left\{(t-s)^2\left(\frac{\lambda-1}{s}-\frac{\lambda}{s^2}\right)-s^2\cdot\frac{1}{s}\right\}ds\\&=\limsup_{t\to\infty}(\lambda-1)\ln\frac{t}{2\pi}+\text{ constant}\\&=\infty, \qquad \text{if } \lambda>1.\end{aligned}$$

It follows from Corollary 2 that equation (E_7) is oscillatory if $\lambda>1$.

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