

Appendix A

Notations, conventions, standard mathematical results

In this appendix, I have collected the definitions of notations and conventions that I use. In addition, I have collected some standard numerical results and formulae that are frequently used in practical QCD calculations. For normalization conventions and the like, I generally follow the conventions of the Particle Data Group (PDG) (Amsler *et al.*, 2008).







In some cases it may be quite difficult to discover some of these formulae in the literature, and the reader wishing to check them may find it easier to rederive them than do a literature search.

A.1 General notations

1. I use $\stackrel{\text{def}}{=}$ to denote the definition of a symbol, as in $Q^2 \stackrel{\text{def}}{=} -(l-l')^2$.
2. I use $\stackrel{\text{prelim}}{=}$ to indicate a preliminary early version of a definition, that is to be corrected later.

$$\text{Quantity} \stackrel{\text{prelim}}{=} \text{preliminary candidate definition.} \quad (\text{A.1})$$

3. I use $\stackrel{?}{=}$ to indicate an incorrect result: $F_L \stackrel{?}{=} 0$. Typically, this represents a result true only in some simplified situation.
4. I use $[A, B]_+$, with a subscript $+$, to denote an anticommutator: $[A, B]_+ \stackrel{\text{def}}{=} AB + BA$.
5. A hat over a symbol, e.g., \hat{k} , generally indicates that some parton-type approximator has been applied. It is also used to denote a hard scattering or a kinematic variable at a partonic level.
6. Generally a tilde, as in \tilde{f} , indicates a Mellin or a Fourier transform. It has some other rarer uses, e.g., the wave function renormalization \tilde{Z} for the Faddeev-Popov ghost field.
7. In Feynman graphs, the normal association of line types is:

	quark or lepton		Wilson line
	gluon		scalar
	Faddeev-Popov ghost		photon, W, Z

A.2 Units, and conversion factors

1. Generally I use units with $\hbar = c = \epsilon_0 = 1$, with energy in GeV. To convert to standard units, factors of \hbar , c , etc. need to be inserted according to the demands of dimensional analysis, after which the following conversion factors are useful.
2. $\hbar c = 0.197\,327\,0\text{ GeV fm}$.
3. $(\hbar c)^2 = 0.389\,379\,3\text{ GeV}^2\text{ mbarn}$, where $1\text{ barn} = 10^{-28}\text{ m}^2$.

4. The fine-structure constant is $\alpha = e^2/(4\pi) \simeq 1/137.036$, with e being the size of the charge of the electron. In SI units, $\alpha = e^2/(4\pi\hbar c\epsilon_0)$.

A.3 Acronyms and abbreviations

Common acronyms and abbreviations are:

1PI	one-particle irreducible
2PI	two-particle irreducible
ACOT	Aivazis-Collins-Olness-Tung
BFKL	Balitsky-Fadin-Kuraev-Lipatov
BJL	Bjorken-Johnson-Low
BNL	Brookhaven National Laboratory
BRST	Becchi-Rouet-Stora-Tyutin
CCFM	Catani-Ciafaloni-Fiorani-Marchesini
CERN	European Organization for Nuclear Research
CKM	Cabibbo-Kobayashi-Maskawa
CM	center-of-mass
CS	Collins-Soper
CSS	Collins-Soper-Sterman
CWZ	Collins-Wilczek-Zee
DDVCS	double deeply virtual Compton scattering
DESY	Deutsches Elektronen-Synchrotron
DGLAP	Dokshitzer-Gribov-Lipatov-Altarelli-Parisi
DIS	deeply inelastic scattering
DVCS	deeply virtual Compton scattering
DY	Drell-Yan
ELO	extended leading order
ENLO	extended next-to-leading order
FNAL	Fermi National Accelerator Laboratory
GPD	generalized parton density
HERA	Hadron-Electron Ring Accelerator (at DESY)
HQET	heavy-quark effective theory
IR	infra-red
KLN	Kinoshita-Lee-Nauenberg
LEET	low-energy effective theory
LEP	Large Electron Positron collider (at CERN)
LHC	Large Hadron Collider (at CERN)
l.h.s.	left-hand side (of equation)
LLA	leading-logarithm approximation
LO	leading order
LSZ	Lehmann-Symanzik-Zimmermann
MNS	Maki-Nakagawa-Sakata
$\overline{\text{MS}}$	modified minimal subtraction (renormalization scheme)
NLO	next-to-leading order
NNLO	next-to-next-to-leading order
OPE	operator product expansion
pdf	parton distribution function (or parton density function)
PDG	Particle Data Group
pQCD	perturbative QCD
p.s.c.	power-suppressed correction
PSS	pinch-singular surface

QCD	quantum chromodynamics
QED	quantum electrodynamics
QFT	quantum field theory
RG	renormalization group
RGE	renormalization-group equation
RHIC	Relativistic Heavy Ion Collider (at BNL)
r.h.s.	right-hand side (of equation)
SCET	soft-collinear effective theory
SIDIS	semi-inclusive deeply inelastic scattering
SLAC	Stanford Linear Accelerator Center
SM	Standard Model
TMD	transverse momentum dependent
Tr	trace
UV	ultra-violet
VEV	vacuum expectation value

A.4 Vectors, metric, etc.

- 3-vectors are written in boldface: \mathbf{x} .
- In ordinary coordinates, Lorentz 4-vectors are written as, e.g., $x^\mu = (t, x, y, z) = (t, \mathbf{x})$, with a right-handed coordinate system.
- The metric is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
- The fully antisymmetric tensor $\epsilon_{\kappa\lambda\mu\nu}$ is normalized to $\epsilon_{0123} = 1$. With raised indices it has the opposite sign: $\epsilon^{0123} = -1$.
- Light-front coordinates (App. B) are defined by $x^\pm = (t \pm z)/\sqrt{2}$. A vector is written $x^\mu = (x^+, x^-, \mathbf{x}_T)$.
- The 2-dimensional antisymmetric tensor ϵ_{ij} obeys $\epsilon_{12} = \epsilon^{12} = 1$.
- Rapidity for a 4-momentum is defined by $y \stackrel{\text{def}}{=} \frac{1}{2} \ln \left| \frac{p^+}{p^-} \right|$.
- I make a clear distinction between contravariant vectors, with upper indices, and covariant vectors, with lower indices. See App. B for further details.
- An on-shell momentum $p^\mu = (E, \mathbf{p})$ for a particle of mass m obeys $p^2 = m^2$, and so $E = E_p \stackrel{\text{def}}{=} \sqrt{\mathbf{p}^2 + m^2}$.
- Hence for an on-shell particle

$$p^\pm = e^{\pm y} \sqrt{(p_T^2 + m^2)/2}, \quad (\text{A.2})$$

when the transverse momentum is $\mathbf{p}_T = (p^1, p^2)$.

A.5 Renormalization group (RG)

- I consistently write renormalization group equations (RGEs) in terms of a derivative with respect to $\ln \mu$.
- Then the anomalous dimension γ_G of a quantity G is defined as

$$\gamma_G = - \frac{d \ln G}{d \ln \mu}. \quad (\text{A.3})$$

Note the minus sign. This corresponds to the natural use of the term ‘‘anomalous dimension’’ where there is a fixed point in the coupling.

- However, certain quantities do not have the minus sign that might otherwise be expected, notably γ_m in (3.48), β in (3.44) etc., and the DGLAP kernels in (8.30).

- Furthermore, the definitions of β and the DGLAP kernels are conventionally made in terms of derivatives with respect to $\ln \mu^2$, so our definitions in terms of $d/d \ln \mu$ acquire factors of half.

A.6 Lorentz, vector, color, etc. sub- and superscripts

Generally, the symbols for indices of various kinds are taken from different ranges of letters:

- Lorentz: μ , etc.
- 3-vector: i, j , etc.
- Dirac: ρ , etc.
- Color, in adjoint representation: α , etc.
- Color, in fundamental representation: a , etc.
- Flavor, in adjoint representation: A , etc.
- Flavor, in fundamental representation: f , etc.
- Symbols for momenta tend to be taken from the list k, l, p, q , etc.
- Symbols for coordinates tend to be from the end of the roman alphabet: x, y, z .

Note that there are so many symbols needed that it is not always possible to be consistent. Also symbols may be overloaded: e.g., a common sub- or superscript index (notably e, i, ρ , and δ) may have a different, standardized meaning when not used as a sub- or superscript.

A.7 Polarization and spin

Note: there is no agreement in the literature on the normalization of quantities defined in this section.

- The Pauli-Lubański (Lubański, 1942a, b) spin vector is the operator

$$W_\mu \stackrel{\text{def}}{=} \frac{1}{2} \epsilon_{\mu\alpha\beta\gamma} J^{\alpha\beta} P^\gamma, \tag{A.4}$$

where P^γ is the momentum operator, and $J^{\alpha\beta}$ are the generators of the Lorentz group, normalized to obey commutation relations

$$[J^{\mu\nu}, J^{\alpha\beta}] = i(-g^{\mu\alpha} J^{\nu\beta} + g^{\nu\alpha} J^{\mu\beta} + g^{\mu\beta} J^{\nu\alpha} - g^{\nu\beta} J^{\mu\alpha}), \tag{A.5}$$

$$[J^{\mu\nu}, P^\alpha] = i(-g^{\mu\alpha} P^\nu + g^{\nu\alpha} P^\mu), \tag{A.6}$$

$$[P^\alpha, P^\beta] = 0. \tag{A.7}$$

- The most general state – pure or mixed – of a particle of momentum p can be written in terms of a spin density matrix $\rho_{\alpha\alpha'}$, with α and α' being labels for the possible helicities of the particle.¹ The expectation value of an operator in such a state is

$$\langle p, \rho | \text{op} | p, \rho \rangle \stackrel{\text{def}}{=} \sum_{\alpha, \alpha'} \rho_{\alpha, \alpha'} \langle p, \alpha' | \text{op} | p, \alpha \rangle. \tag{A.8}$$

The basis states $|p, \alpha\rangle$ have definite momentum p and helicity α . The density matrix ρ is Hermitian, it has trace unity, and all its eigenvalues are non-negative. An unpolarized state of a particle of spin s has $\rho_{\alpha, \alpha'} = \delta_{\alpha, \alpha'} / (2s + 1)$.

- Helicity is a particle’s spin angular momentum projected on its direction of motion. Thus for a spin- $\frac{1}{2}$ particle its possible values are $\pm \frac{1}{2}$.

¹ Another basis could be chosen for the spin states, but the helicity basis is most convenient for our purposes.

4. The helicity basis states are simultaneous eigenvectors of the momentum operators and a suitable projection of the Pauli-Lubański vector.
5. To specify a general spin state of a spin- $\frac{1}{2}$ particle, it is also possible to use a Bloch vector \mathbf{b} , which is a real-valued 3-vector obeying $|\mathbf{b}| \leq 1$. The correspondence to a 2×2 density matrix is

$$\rho = \frac{1}{2}(1 + \mathbf{b} \cdot \boldsymbol{\sigma}). \quad (\text{A.9})$$

6. For a spin- $\frac{1}{2}$ particle moving in the $+z$ direction, we write the Bloch vector as

$$\mathbf{b} = (\mathbf{b}_T, \lambda). \quad (\text{A.10})$$

Here λ is twice the average helicity of a state, and \mathbf{b}_T is twice the average transverse spin. We call these normalized helicity and transverse spin; their maximum values are unity.

7. The spin vector S^μ of a single-particle state is twice the expectation value of the Pauli-Lubański vector:

$$S^\mu = 2 \langle \psi(p), \rho | W^\mu | \psi(p), \rho \rangle. \quad (\text{A.11})$$

The factor of 2 is to agree with a standard normalization (Amsler *et al.*, 2008) of S^μ . Here $|\psi(p), \rho\rangle$ denotes a normalized state whose momentum is closely centered on p , and whose helicity density matrix is ρ .

8. In the rest frame of a spin- $\frac{1}{2}$ particle, the Bloch vector corresponds exactly to the Bloch vector concept in non-relativistic spin physics, and $S^\mu = M(0, \mathbf{b})$. Thus S^μ is a Lorentz-covariant generalization of the Bloch vector.
9. If the particle is moving in the z direction with 4-momentum $p = (p^0, 0, 0, p^z)$, the spin and the Bloch vectors are related by

$$S = (S^0, S^x, S^y, S^z) = (\lambda p^z \text{sign}(p^z), M b_T^x, M b_T^y, \lambda p^0 \text{sign}(p^z)). \quad (\text{A.12})$$

The factors of $\text{sign } p^z$ show that the (\mathbf{b}_T, λ) representation is not ideal for a non-relativistic particle. But the factor of M with the transverse components shows that the spin vector cannot correctly represent the general spin state of a massless spin- $\frac{1}{2}$ particle.

10. For a massive spin- $\frac{1}{2}$ particle of definite momentum p , the most general spin state is determined by the spin vector S^μ . For a spin- $\frac{1}{2}$ particle of mass M , the spin vector obeys
 - (a) $S \cdot p = 0$.
 - (b) For a general state $0 \geq S \cdot S \geq -M^2$.
 - (c) For a pure state $S \cdot S = -M^2$.

The helicity density matrix can be deduced from the spin vector S , and therefore we also write the matrix element in (A.8) as

$$\langle p, S | \text{op} | p, S \rangle = \langle p, \rho(S) | \text{op} | p, \rho(S) \rangle. \quad (\text{A.13})$$

A.8 Structure functions

Definitions of structure functions for various processes are as follows:

1. F_1, F_2, g_1 , and g_2 for electromagnetic DIS, in (2.20).
2. For unpolarized weak interaction DIS, in (7.3).
3. For one-particle-inclusive e^+e^- annihilation, in (12.5).
4. For two-particle-inclusive e^+e^- annihilation, in (13.9).
5. For Drell-Yan, see Lam and Tung (1978); Mirkes (1992); Ralston and Soper (1979); Donohue and Gottlieb (1981).

A.9 States, cross sections, integrals over particle momentum

1. The normalization of single particle states is

$$\begin{aligned} \delta_{pp'} &\stackrel{\text{def}}{=} \langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^3 2E_p \delta^{(3)}(\mathbf{p} - \mathbf{p}') \\ &= (2\pi)^3 2p^+ \delta(p^+ - p'^+) \delta^{(2)}(\mathbf{p}_T - \mathbf{p}'_T) \\ &= (2\pi)^3 2 \delta(y - y') \delta^{(2)}(\mathbf{p}_T - \mathbf{p}'_T). \end{aligned} \tag{A.14}$$

In the last two lines, light-front coordinates and rapidity were used, as defined in Sec. A.4.

2. The Lorentz-invariant integral over particle momentum is

$$\sum_p \dots \stackrel{\text{def}}{=} \int \frac{d^3 \mathbf{p}}{(2\pi)^3 2E_p} \dots = \int \frac{dp^+ d^2 \mathbf{p}_T}{2p^+ (2\pi)^3} \dots = \int \frac{dy d^2 \mathbf{p}_T}{2(2\pi)^3} \dots \tag{A.15}$$

Notice that the formula with ordinary Cartesian coordinates is explicitly dependent on the particle mass, in E_p , but the formulae with light-front coordinates or rapidity are not.

3. The differential cross section for a $2 \rightarrow n$ process with incoming momenta \mathbf{p}_1 and \mathbf{p}_2 , and outgoing momenta $\mathbf{q}_1, \dots, \mathbf{q}_n$ is

$$d\sigma = (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_j q_j\right) \prod_{j=1}^n \frac{d^3 \mathbf{q}_j}{(2\pi)^3 2E_{q_j}} \frac{|\mathcal{M}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}_1, \dots, \mathbf{q}_n)|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}. \tag{A.16}$$

The matrix element \mathcal{M} is normalized so that it corresponds to an amputated, on-shell, connected Green function (supplemented by residue factors from the LSZ reduction formula beyond tree approximation), with the overall $(2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_j q_j)$ factor for momentum conservation removed. See Sterman (1993) for details.

4. The integral over “final-state phase space” is defined by

$$\int d\text{fsps} \dots = \prod_{j=1}^n \int \frac{d^3 \mathbf{q}_j}{(2\pi)^3 2E_{q_j}} (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - \sum_j q_j\right) \dots \tag{A.17}$$

A.10 Dirac, or gamma, matrices

Here I summarize results on Dirac matrices. They can be gleaned from a standard QFT textbook. When there are competing conventions, I normally follow Sterman (1993).

1. The anticommutator is $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} I$, where I is a unit matrix.
2. The hermiticity relation is $(\gamma^\mu)^\dagger = \gamma_\mu$.
3. $\gamma_5 \stackrel{\text{def}}{=} i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{1}{4!} i\gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu \epsilon_{\kappa\lambda\mu\nu}$, where $\epsilon_{\kappa\lambda\mu\nu}$ is the totally antisymmetric tensor obeying $\epsilon_{0123} = 1$.
4. In the antisymmetric combination $\sigma^{\mu\nu} \stackrel{\text{def}}{=} \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ only 6 cases are independent, in 4 space-time dimensions.
5. When the normal space-time dimension is 4, the dimensionally regulated Dirac matrices (in $n = 4 - 2\epsilon$ space-time dimensions) are normalized to have $\text{Tr } I = 4$ for all n .
6. The contraction of γ^μ and a vector is written $\not{v} \stackrel{\text{def}}{=} \gamma^\mu v_\mu$.
7. The Dirac conjugate of a matrix is defined by $\overline{\Gamma} \stackrel{\text{def}}{=} \gamma^0 \Gamma^\dagger \gamma^0$. The basic matrices obey $\overline{\gamma^\mu} = \gamma^\mu$ and $\overline{\gamma_5} = -\gamma_5$.

8. Useful identities:

$$\text{Tr odd number of } \gamma^\mu\text{s} = 0, \tag{A.18}$$

$$\text{Tr } \gamma^\mu \gamma^\nu = 4g^{\mu\nu}, \tag{A.19}$$

$$\text{Tr } \gamma_5 \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu = 4i \epsilon^{\kappa\lambda\mu\nu} = -4i \epsilon_{\kappa\lambda\mu\nu}, \tag{A.20}$$

$$\gamma^\mu \gamma_\mu = (4 - 2\epsilon)I, \tag{A.21}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(2 - 2\epsilon)\gamma^\nu. \tag{A.22}$$

9. In 4 space-time dimensions, Dirac matrices are 4×4 , and a 16-dimensional basis for them is given by $1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma_5, \gamma_5$.

10. Thus a general 4×4 matrix M can be written as

$$\Gamma = S + \gamma_5 P + \gamma_\mu V^\mu + \gamma_\mu \gamma_5 A^\mu + \frac{1}{2} \sigma_{\mu\nu} T^{\mu\nu}, \tag{A.23}$$

where we assume we are in 4 space-time dimensions. (Otherwise generalization is needed.) If Γ obeys the normal Lorentz-transformation properties of a matrix on Dirac spinor space, then the coefficients $S, P, V^\mu, A^\mu,$ and $T^{\mu\nu}$ have respectively the transformation rules of: scalar, pseudo-scalar, vector, axial-vector, and second rank antisymmetric tensor. The factor of $\frac{1}{2}$ in the tensor term is introduced because both $\sigma_{\mu\nu}$ and $T^{\mu\nu}$ are antisymmetric, so that each independent term appears twice in the sum over μ and ν .

The coefficients can be obtained from Γ as

$$\begin{aligned} S &= \frac{1}{4} \text{Tr } \Gamma, & P &= \frac{1}{4} \text{Tr } \Gamma \gamma_5, & V^\mu &= \frac{1}{4} \text{Tr } \Gamma \gamma^\mu, \\ A^\mu &= \frac{1}{4} \text{Tr } \Gamma \gamma_5 \gamma^\mu, & T^{\mu\nu} &= \frac{1}{4} \text{Tr } \Gamma \sigma^{\mu\nu}. \end{aligned} \tag{A.24}$$

11. In cross sections we encounter combinations $u\bar{u}$ and $v\bar{v}$ of Dirac spinors for on-shell particles. An average over independent spin states for a Dirac particle of mass M gives

$$\frac{1}{2} \sum_{\text{spin}} u\bar{u} = \frac{1}{2}(\not{p} + M), \quad \frac{1}{2} \sum_{\text{spin}} v\bar{v} = \frac{1}{2}(\not{p} - M). \tag{A.25}$$

For a Dirac particle with non-trivial spin, we have instead

$$(\not{p} + M) \frac{1}{2} (1 + \gamma_5 \not{S} / M), \quad (\not{p} - M) \frac{1}{2} (1 + \gamma_5 \not{S} / M), \tag{A.26}$$

where S is the particle's spin vector, normalized (Amsler *et al.*, 2008) to a maximum of $-S^2 \leq M^2$. In the case of a massless particle we use a helicity variable λ and a transverse spin variable \mathbf{b}_T , normalized to have a maximum values unity, $\lambda^2 + |\mathbf{b}_T|^2 \leq 1$ (Sec. A.7) to give

$$\frac{1}{2} \not{p} \left(1 - \lambda \gamma_5 - \sum_{j=1,2} \gamma_5 \gamma^j b_T^j \right) \quad \text{for quark}, \tag{A.27a}$$

$$\frac{1}{2} \not{p} \left(1 + \lambda \gamma_5 - \sum_{j=1,2} \gamma_5 \gamma^j b_T^j \right) \quad \text{for antiquark}. \tag{A.27b}$$

A.11 Group theory

1. For SU(3), the definition of the structure constants $f_{\alpha\beta\gamma}$ and the representation matrices in the fundamental (i.e., triplet) representation t_α are the standard ones, with $t_\alpha = \lambda_\alpha / 2$. Here λ_α are the Gell-Mann matrices, as defined in Amsler *et al.* (2008, p. 338).
2. The commutation relations are $[t_\alpha, t_\beta] = i f_{\alpha\beta\gamma} t_\gamma$.

3. $f_{\alpha\beta\gamma}$ are totally antisymmetric.
4. Combinations of representation matrices and structure constants:

$$\text{Tr}(t_\alpha t_\beta) = T_F \delta_{\alpha\beta}, \tag{A.28}$$

$$t_\alpha t_\alpha = C_F I, \tag{A.29}$$

$$f_{\alpha\gamma\delta} f_{\beta\gamma\delta} = C_A \delta_{\alpha\beta}, \tag{A.30}$$

where repeated indices are summed, I is the unit matrix, and the t_α s are in the fundamental representation. Useful values with standard conventions:

Symbol	SU(n)	SU(3)
T_F	$\frac{1}{2}$	$\frac{1}{2}$
C_F	$\frac{n^2-1}{2n}$	$\frac{4}{3}$
C_A	n	3

(A.31)

5. Combinations useful in calculations:

$$t_\beta t_\alpha t_\beta = t_\alpha (C_F - \frac{1}{2} C_A), \tag{A.32}$$

$$f_{\delta\alpha\epsilon} f_{\epsilon\beta\phi} f_{\phi\gamma\delta} = -\frac{1}{2} C_A f_{\alpha\beta\gamma}. \tag{A.33}$$

A.12 Dimensional regularization and $\overline{\text{MS}}$: basics

See Collins (1984, Ch. 4) for a systematic mathematical treatment of dimensional regularization.

1. The space-time dimension is $n = 4 - 2\epsilon$.
2. Rotationally symmetric Euclidean integral in d dimensions:

$$\int d^d \mathbf{k} f(k^2) = \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk^2 (k^2)^{d/2-1} f(k^2). \tag{A.34}$$

This is often used for the transverse dimensions, with $d = 2 - 2\epsilon$.

3. The Lorentz-invariant integral over particle momentum is

$$\sum_p \dots \stackrel{\text{def}}{=} \int \frac{d^{3-2\epsilon} \mathbf{p}}{(2\pi)^{3-2\epsilon} 2E_p} \dots \tag{A.35}$$

4. Decomposition of integration over a spatial $3 - 2\epsilon$ -dimensional variable into integrals over radius, a polar angle, and an azimuthal angle:

$$\int d^{3-2\epsilon} \mathbf{k} f(k) = \int_0^\infty dk k^{2-2\epsilon} \int_{-1}^1 d \cos \theta (\sin \theta)^{-2\epsilon} \int d\Omega_T f(\mathbf{k}), \tag{A.36}$$

where $d\Omega_T$ represents an integral over a $1 - 2\epsilon$ -dimensional angle in the transverse dimensions, which would be $d\phi$ in a 3-dimensional space, i.e., at $\epsilon = 0$. The normalization of the angular integral is

$$\int d\Omega_T = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}. \tag{A.37}$$

These results can be proved by decomposing \mathbf{k} into a z component $k \cos \theta$ and a $2 - 2\epsilon$ -dimensional transverse vector, and then using (A.34) to get the normalization of the azimuthal integral. See Sec. A.14 for the Gamma function.

5. The normalization of single particle states is

$$\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^{3-2\epsilon} 2E_p \delta^{(3-2\epsilon)}(\mathbf{p} - \mathbf{p}') = (2\pi)^{3-2\epsilon} 2p^+ \delta(p^+ - p'^+) \delta^{(2-2\epsilon)}(\mathbf{p}_T - \mathbf{p}'_T). \tag{A.38}$$

6. Loop-momentum integrals are

$$\int \frac{d^{4-2\epsilon} k}{(2\pi)^{4-2\epsilon}} \dots \tag{A.39}$$

7. Momentum-conservation delta functions are

$$(2\pi)^{4-2\epsilon} \delta^{(4-2\epsilon)}(k_1 + \dots). \tag{A.40}$$

8. Dirac matrices are defined to obey $\text{Tr } I = 4$ for all n .

9. $\overline{\text{MS}}$ definition:

- (a) The lowest-order bare coupling is defined to be $g_0 = \mu^\epsilon g$, with g dimensionless for all n .
- (b) Counterterms have a factor S_ϵ for each loop, where

$$S_\epsilon = (4\pi)^\epsilon / \Gamma(1 - \epsilon). \tag{A.41}$$

See (3.16) and (3.17) for examples. This definition differs from the more conventional one, $S_\epsilon = (4\pi e^{-\gamma_E})^\epsilon \simeq (7.056)^\epsilon$, but only by terms of order ϵ^2 . It can be shown that differences of order ϵ^2 do not affect the values of ordinary renormalized Green functions at any order (problem 3.3). However, the definition given here is preferable for $\overline{\text{MS}}$ renormalization of the collinear factors defined in Chs. 10 and 13.

A.13 Dimensional regularization: standard integrals

1. “Scale-invariant” integrals, i.e., integrals of a power of the integration momentum are zero:

$$\int d^n k (k^2)^{-\alpha} = 0. \tag{A.42}$$

2. Rotationally invariant phase-space integrals for massless particles:

(a) Two bodies:

$$\begin{aligned} & \int \prod_{i=1}^2 \frac{d^{3-2\epsilon} \mathbf{k}_i}{(2\pi)^{3-2\epsilon} 2|\mathbf{k}_i|} (2\pi)^{4-2\epsilon} \delta^{(4-2\epsilon)}(q - k_1 - k_2) f(\mathbf{k}_1, \mathbf{k}_2) \\ &= \frac{Q^{-2\epsilon}}{2^{4-4\epsilon} \pi^{1/2-\epsilon} \Gamma(\frac{3}{2} - \epsilon)} \times \text{angular average of } f\left(\frac{1}{2} Q\mathbf{n}, -\frac{1}{2} Q\mathbf{n}\right), \end{aligned} \tag{A.43}$$

in the center-of-mass, with $Q = \sqrt{q^2}$.

(b) Three bodies:

$$\begin{aligned} & \int \prod_{i=1}^3 \frac{d^{3-2\epsilon} \mathbf{k}_i}{(2\pi)^{3-2\epsilon} 2|\mathbf{k}_i|} (2\pi)^{4-2\epsilon} \delta^{(4-2\epsilon)}(q - k_1 - k_2 - k_3) f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ &= \frac{Q^{2-4\epsilon}}{2^{8-6\epsilon} \pi^{5/2-2\epsilon} \Gamma(\frac{3}{2} - \epsilon) \Gamma(1 - \epsilon)} \\ & \times \text{ang. avg.} \int_0^1 \prod_{i=1}^3 dy_i \delta\left(1 - \sum y_i\right) (y_1 y_2 y_3)^{-\epsilon} f(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \end{aligned} \tag{A.44}$$

Here the spatial momenta $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, add up to $\mathbf{0}$ in the center-of-mass frame, and the sizes are given by dimensionless variables y_i defined by $|\mathbf{k}_i| = (1 - y_i)Q/2$.

3. Integral used in Fourier transformations on transverse momenta:

$$\int \frac{e^{i\mathbf{k}_T \cdot \mathbf{b}_T}}{(k_T^2)^\alpha} d^{2-2\epsilon} \mathbf{k}_T = \left(\frac{b_T^2}{4\pi}\right)^{\epsilon+\alpha-1} \frac{\pi^\alpha \Gamma(1 - \epsilon - \alpha)}{\Gamma(\alpha)}. \tag{A.45}$$

A proof can be made by converting the \mathbf{k}_T integral to a Gaussian, by the use of $(k_T^2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-xk_T^2} dx$.

4. For the case that the integrand has one or more powers of $\ln k_T^2$, the result is found by differentiating (A.45) with respect to α .

A.14 Properties of Γ function

1. Definition:

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty dt t^{z-1} e^{-t}. \tag{A.46}$$

2. Integer values: $\Gamma(n + 1) = n!$.

3. $\Gamma(z + 1) = z\Gamma(z)$.

4. Expansion about $z = 0$:

$$\Gamma(z) = \frac{1}{z} e^{-\gamma_E z} \left[1 + \frac{\pi^2}{12} z^2 + O(z^3) \right], \tag{A.47}$$

where $\gamma_E = 0.5772\dots$ is the Euler constant.

5. Expansion about $z = \frac{1}{2}$:

$$\Gamma\left(\frac{1}{2} + z\right) = \pi^{1/2} e^{-(\gamma_E + \ln 4)z} \left[1 + \frac{\pi^2}{4} z^2 + O(z^3) \right]. \tag{A.48}$$

6. We often use

$$\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \tag{A.49}$$

$$\int_0^\infty dx \frac{x^{\alpha-1}}{(A+x)^\beta} = A^{\alpha-\beta} \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)}. \tag{A.50}$$

These and other useful formulae can be found in or deduced from results in Abramowitz and Stegun (1964). Some commonly used integrals have integrands with factors of logarithms of x , $1 - x$ or $A + x$ relative to (A.49) or (A.50); these can be found by differentiation with respect to α or β .

A.15 Plus distributions, etc.

We define the general plus distribution $(\ln^n(1-x)/(1-x))_+$ by its integral with an arbitrary smooth test function $f(x)$:

$$\int_0^1 dx \left(\frac{\ln^n(1-x)}{1-x}\right)_+ f(x) \stackrel{\text{def}}{=} \int_0^1 dx \frac{[f(x) - f(1)] \ln^n(1-x)}{1-x}. \tag{A.51}$$

When $n = 0$, and there is a smooth function (e.g., a polynomial) in the numerator, we will also write

$$\int_0^1 dx \frac{A(x)}{(1-x)_+} f(x) \stackrel{\text{def}}{=} \int_0^1 dx \frac{A(x)f(x) - A(1)f(1)}{1-x}. \tag{A.52}$$

In calculations of structure functions with dimensionally regulated divergences, we find integrals in which plus distributions appear as a limit of regulated integrals. The following derivation shows both a result that is useful in itself, and a general method. The factor $[z/(1-z)]^\epsilon$ in the integrand arises in the phase-space integral for DIS: Sec. 9.9. The integral is regulated if $\epsilon < 0$.

$$\begin{aligned} \int_0^1 dz \frac{z^\epsilon}{(1-z)^{1+\epsilon}} f(z) &= \int_0^1 dz \frac{z^\epsilon f(z) - f(1)}{(1-z)^{1+\epsilon}} + f(1) \int_0^1 dz \frac{1}{(1-z)^{1+\epsilon}} \\ &= \int_0^1 dz \left\{ \frac{f(z) - f(1)}{1-z} + \epsilon \frac{f(z) \ln z - [f(z) - f(1)] \ln(1-z)}{1-z} \right\} \\ &\quad + O(\epsilon^2) - \frac{f(1)}{\epsilon}. \end{aligned} \tag{A.53}$$

The expansion in powers of ϵ in the second line is allowed because the subtracted integrand is well behaved as $\epsilon \rightarrow 0$.

This can be treated as an expansion of $z^\epsilon/(1-z)^{1+\epsilon}$ in powers of ϵ , interpreted in the standard sense of the limit of a generalized function/distribution:

$$\frac{z^\epsilon}{(1-z)^{1+\epsilon}} = -\frac{\delta(z-1)}{\epsilon} + \frac{1}{(1-z)_+} + \epsilon \left[\frac{\ln z}{1-z} - \left(\frac{\ln(1-z)}{1-z} \right)_+ \right] + O(\epsilon^2). \tag{A.54}$$

A.16 Feynman parameters

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax + B(1-x)]^{\alpha+\beta}}. \tag{A.55}$$

A.17 Orders of magnitude, estimation, etc.

We will frequently need to estimate sizes of Feynman graphs, the sizes of errors in approximations, etc. A correct use of appropriate mathematical notation keeps the arguments precise and reliable; I use the definitions given by Knuth (1976). As Knuth points out, it is quite common to misuse the definitions, and this results in a loss of precision of the arguments.

A.17.1 “Order at most”: big-O

The most commonly used notation is

$$f(Q) = O(g(Q)) \text{ when } Q \rightarrow \infty, \tag{A.56}$$

which means that there is a constant C such that

$$\left| \frac{f(Q)}{g(Q)} \right| < C \text{ for all large enough } Q. \tag{A.57}$$

It is often useful to replace the limit by some more precise specification of the range of Q (or whatever other variable is used). An example would be

$$\frac{\sin x}{\sqrt{x^2 - 1}} = O\left(\frac{1}{x}\right) \text{ for } x \geq 2. \quad (\text{A.58})$$

Although this notation is commonly used to indicate that the left-hand side is asymptotically of the order of magnitude of the right-hand side, this is not actually a correct usage. For this case Knuth's Θ notation should be used: Sec. A.17.2. The big- O notation is most appropriate when stating error estimates, for example, since the standard definition allows the left-hand side to have zeros, as in (A.58), or to go to zero relative to the right-hand side, as in

$$\frac{1}{x^2} = O\left(\frac{1}{x}\right) \text{ as } x \rightarrow \infty. \quad (\text{A.59})$$

A.17.2 "Exact order": Θ

Power-counting and error estimates are often made using what we often call order-of-magnitude estimates. We replace an exact quantity by a crude approximation that is valid up to a factor. For this we use the symbol " Θ ":

$$f(Q) = \Theta(g(Q)) \text{ when } Q \rightarrow \infty, \quad (\text{A.60})$$

which means that there are two positive *non-zero* constants C_1 and C_2 such that

$$C_1 < \left| \frac{f(Q)}{g(Q)} \right| < C_2 \text{ for all large enough } Q. \quad (\text{A.61})$$

(The use of this definition requires that $g(Q)$ is non-zero for large Q .)

An example of the use of this notation would be if we added 2 to the $\sin x$ in (A.58). The numerator of the fraction now oscillates between 1 and 3, instead of between -1 and 1, so that we have

$$\frac{2 + \sin x}{\sqrt{x^2 - 1}} = \Theta\left(\frac{1}{x}\right) \text{ for } x \geq 2. \quad (\text{A.62})$$

This is a typical use in estimation of integrals: the right-hand side can be integrated analytically, the left-hand side at best with difficulty.

We will frequently apply this notation to denominators of Feynman propagators, in which case it is important that (A.60) also applies to the reciprocal functions. That is, (A.60) implies that

$$\frac{1}{f(Q)} = \Theta\left(\frac{1}{g(Q)}\right) \text{ when } Q \rightarrow \infty. \quad (\text{A.63})$$

A.17.3 Little- o

Sometimes we simply wish to state that something becomes arbitrarily much smaller than something else in a limit, without wishing to say by how much. In that case we use the little- o notation

$$f(Q) = o(g(Q)) \text{ when } Q \rightarrow \infty, \quad (\text{A.64})$$

which means simply that

$$\frac{f(Q)}{g(Q)} \rightarrow 0 \text{ when } Q \rightarrow \infty. \quad (\text{A.65})$$

Unlike the previous cases, it makes no sense to specify a range of Q ; only the limit matters. However, if there is another parameter involved, it makes sense to specify that (A.65) applies uniformly in the other parameter. See below for an example.

A.17.4 Asymptotic equality: \sim

This notation is frequently used when the Θ notation should be used. The standard definition is the much stronger statement that

$$f(Q) \sim g(Q) \text{ when } Q \rightarrow \infty \quad (\text{A.66})$$

means

$$\lim_{Q \rightarrow \infty} \frac{f(Q)}{g(Q)} = 1. \quad (\text{A.67})$$

Both this and the Θ notation have essential uses, so that it is important not to confuse them.

A.17.5 Uniformity

Frequently we will obtain order-of-magnitude estimates of some function that has parameters. (Often the function is the difference between some exact quantity and an approximation.) It is important to know whether the estimates can be made independent of the parameters.

For example, define

$$f_1(Q; a) = \frac{1}{a^2 + Q^2}. \quad (\text{A.68})$$

Then as $Q \rightarrow \infty$,

$$f_1(Q; a) = O(1/Q^2). \quad (\text{A.69})$$

We can set the quantity C in the definition of $O(\dots)$, (A.57), to be unity (or larger), independently of the parameter a . Moreover, the application of (A.57) works with the same minimum value of Q for all a . In that case we say that (A.69) holds uniformly in a .

But if instead we used

$$f_2(Q; a) = \frac{1}{1 + a^2 Q^2}, \quad (\text{A.70})$$

then we could still say that

$$f_2(Q; a) = O(1/Q^2). \quad (\text{A.71})$$

But this would not be uniform in a . When a is made small, the quantity C in (A.57) has to be made large. A symptom of this non-uniformity is that when $a = 0$, $f_2 = O(1)$ instead of $O(1/Q^2)$.