A SKEW HADAMARD MATRIX OF ORDER 36

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Hadamard matrices exist for infinitely many orders $4m$, $m \geqq 1$, m integer, including all $4m < 100$, cf. [3], [2]. They are conjectured to exist for all such orders. Skew Hadamard matrices have been constructed for all orders $4m < 100$ except for 36, 52, 76, 92, cf. the table in [4]. Recently Szekeres [6] found skew Hadamard matrices of the order $2(p^t+1) \equiv 12 \pmod{16}$, p prime, thus covering the case 76. In addition, Blatt and Szekeres [1] constructed one of order 52. The present note contains a skew Hadamard matrix of order 36 (and one of order 52), thus leaving 92 as the smallest open case.

The unit matrix of any order is denoted by I. The square matrices Q and R of order m are defined by their only nonzero elements

$$q_{i,i+1} = q_{m,1} = 1,\ i = 1, \cdots, m-1;\ r_{i,m-i+1} = 1,\ i = 1, \cdots, m$$

We have

$$Q^m = I,\ R^2 = I,\ RQ = Q^T R.$$

Any square matrix A of order m is symmetric if $A = A^T$, skew if $A + A^T = 0$, circulant if $AQ = QA$. Hence, for circulant A we have

$$A = \sum_{i=0}^{m-1} a_i Q^i,\ RA = A^T R.$$

Any square matrix H of order $4m$ is *skew Hadamard* if its elements are 1 and -1 (we write $+$and $-$) and

$$HH^T = 4mI,\ H + H^T = 2I.$$

THEOREM 1. *If A, B, C, D are square circulant matrices of order m, if A is skew, and if*

$$AA^T + BB^T + CC^T + DD^T = (4m-1)I,$$

then

$$H = \begin{bmatrix} A+I & BR & CR & DR \\ -BR & A+I & -D^T R & C^T R \\ -CR & D^T R & A+I & -B^T R \\ -DR & -C^T R & B^T R & A+I \end{bmatrix}$$

satisfies $HH^T = 4mI$, $H + H^T = 2I$.

PROOF. By straightforward verification.

REMARK. If, in addition, B, C, and D are symmetric, then H may be written in terms of the quaternion matrices K_4, L_4, M_4 and the Kronecker product $\otimes$ as follows:

$$H = I_4 \otimes (I+A) + K_4 \otimes BR + L_4 \otimes CR + M_4 \otimes DR,$$

hence looking much like a Williamson-type matrix, cf. [7].

THEOREM 2. *There exist skew Hadamard matrices of orders* 36 *and* 52.

PROOF. We apply theorem 1 with the following circulant matrices of order 9:

$$A = (0 \ + \ + \ - \ + \ - \ + \ - \ -),\ B = (+ \ - \ + \ + \ - \ - \ + \ + \ -),$$
$$C = (- \ - \ + \ + \ + \ + \ + \ + \ -),\ D = (+ \ + \ + \ - \ + \ + \ - \ + \ +).$$

By inspection the skew A and the symmetric B, C, D are seen to satisfy the hypotheses. Hence a skew Hadamard matrix of order 36 is obtained. Secondly, we consider the following circulant matrices of order 13:

$$A = (0 \ + \ + \ + \ - \ + \ + \ - \ - \ + \ - \ - \ -),$$
$$B = (- \ + \ - \ + \ + \ - \ - \ - \ - \ + \ + \ - \ +),$$
$$C = D = (- \ - \ + \ - \ + \ + \ + \ + \ + \ - \ + \ + \ +).$$

Application of theorem 1 to A, B, C, D yields a skew Hadamard matrix of order 52 since

$$AA^T = 15I - J + 2B,\ BB^T = 12I - J - 2B,\ CC^T = DD^T = 12I + J.$$

REMARK. The positive elements of B indicate the quadratic residues mod 13. The matrix of order 26

$$\begin{bmatrix} B+I & C \\ C^T & -B-I \end{bmatrix}$$

is an orthogonal matrix with zero diagonal, cf. [2] p. 1007. The matrix A describes the unique tournament of order 13 having no transitive subtournament of order 5, which was recently found by Reid and Parker [5].

References

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