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# A CANONICAL FACTORIZATION FOR GRAPH HOMOMORPHISMS

BARRY FAWCETT

**Introduction.** The graphs are undirected, without loops or multiple edges. The *edge set* $E(X)$ of a graph $X$ is a set of certain unordered pairs $[x, y]$ of distinct elements of the vertex set $V(X)$. For $x \in V(X)$ we denote by $E(x; X)$ the edges of $X$ incident with $x$. A (*homo*)*morphism* $\phi : X \rightarrow Y$ is a function from $V(X)$ to $V(Y)$ which preserves edges; thus it induces $\phi^{\#} : E(X) \rightarrow E(Y)$ by $\phi^{\#}[x, x'] = [\phi x, \phi x']$. $\phi$ is *strong* if and only if $\phi^{\#}$ is injective. $\phi X$ is the graph with vertex set $\phi V(X)$ and edge set $\phi^{\#}E(X)$. $\phi$ is *full* if and only if $\phi X$ is a *section* (i.e., an induced subgraph) of $Y$.

A *congruence* of $X$ is an equivalence relation on $V(X)$ which does not identify the endpoints of edges. The quotient graph $X/R$ has the set of $R$-equivalence classes for vertex set; $[Rx, Ry]$ is an edge of $X/R$ if and only if some edge $[x', y']$ joins these classes in $X$. $\mathscr{G}$ denotes the category of graphs and morphisms; $\mathscr{H}$ denotes the full subcategory of graphs without isolated vertices.

*Definition* 1. A *proper morphism* is a surjective morphism $\phi : X \rightarrow Y$ such that

(a) $\phi$ is full;

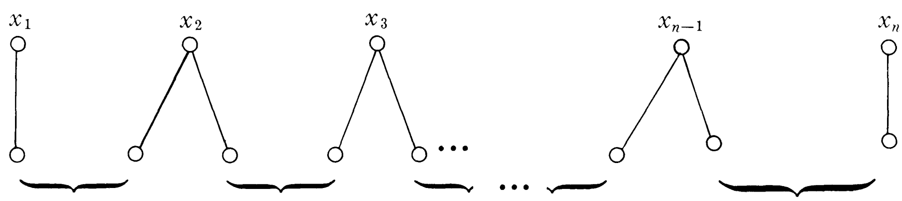
(b) whenever $\phi x = \phi x'$ and $x \neq x'$ there exists a finite sequence $x = x_1, x_2, \ldots, x_n = x'$ satisfying

(*) $\phi x_i = \phi x \quad i = 1, 2, \ldots, n$ and

(**) $\phi^{\#}E(x_i; X) \cap \phi^{\#}E(x_{i+1}; X) \neq \phi, \quad i = 1, 2, \ldots, n-1.$

*Remarks.* In the special case that all the sequences $(x_i)$ may be taken to be of length 2, condition (b) states that whenever $x$ and $x'$ are identified, some edge incident with $x$ is identified with some edge incident with $x'$.

In the general case, there exists a sequence of edges in $X$ as diagrammed below, with parentheses indicating those pairs of edges identified by $\phi^{\#}$.




Chromatic (i.e. minimal) colourings of graphs without isolated vertices are proper morphisms onto complete graphs. (To see this, note that in all minimal

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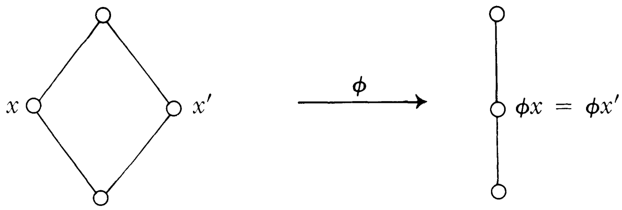
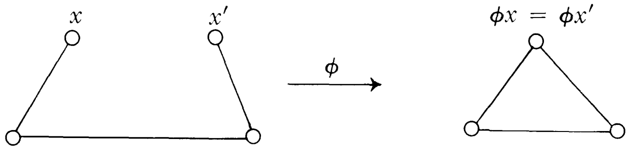
colourings each colour-class contains a vertex adjacent to all other colour-classes.)

*Examples.*




$\phi$ proper, not strong



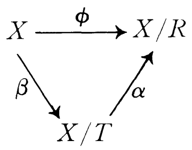

$\phi$ strong, not proper.

THEOREM 1. *Every morphism in $\mathscr{G}$ may be factored as a strong morphism preceded by a proper morphism; likewise in $\mathscr{H}$.*

*Proof.* Since inclusion preceded by a strong morphism is strong, it may be assumed that $\phi : X \to Y$ is onto and that $Y = X/R$ (where $R = R_\phi$, the congruence induced by $\phi$). Define a relation $T$ (or $T_\phi$ if there are more morphisms about) as follows:

$(x, x') \in T$ if and only if $x = x'$, or $x \neq x'$ and there exists a finite sequence $x = x_1, x_2, \ldots, x_n = x'$ satisfying (*) and (**) of Definition 1.

$T$ is a congruence of $X$. (Transitivity follows from the observation that two sequences satisfying (*) and (**) may be spliced.) Consider the natural factorization:



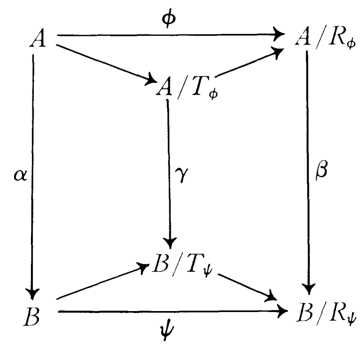

$\beta: x \hookrightarrow Tx$

$\alpha: Tx \hookrightarrow Rx$

$\beta$ is a morphism since $T \subseteq R$; like all projections to quotients it is full; it is proper by definition of $T$. $\alpha$ is a well-defined morphism since $T \subseteq R$. To show that it is strong, suppose that $\alpha^\#$ identifies some edges $[Tx, Ty]$, $[Tx', Ty']$ of $X/T$, say $Rx = Rx'$, $Ry = Ry'$. Thus an edge in $X$ between $Tx$, $Ty$ is identified by $\phi^\#$ with an edge between $Tx'$, $Ty'$. It follows that $(x, x') \in T$ and $Tx = Tx'$ and $Ty = Ty'$.

Note that all identifications of isolated points occur in the stage $X/T \to X/R$ and none in $X \to X/T$.

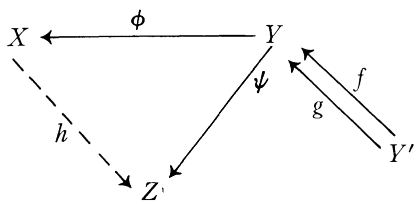
*Remark.* The morphisms which are both strong and proper are exactly the isomorphisms. The factorization is natural in the sense that, in the diagram below, whenever the outer square is commutative ($\beta\phi = \psi\alpha$), there exists a unique $\gamma$ which makes the diagram commutative.




In fact $\gamma(T_\phi a) = T_\psi(\alpha a)$. The above observations imply that the factorization of Theorem 1 is an $\mathscr{E}$-$\mathscr{M}$ factorization in the sense of [**3**, p. 187].

We require the notions of strict and extremal epimorphism. Other categorical notions are used as in [**3**]. In what follows, it is possible to work with coequalizers in place of strict and extremal epimorphisms. The details are similar.

*Definition* 2. An epimorphism $\phi : Y \to X$ is *strict* if and only if whenever $\psi : Y \to Z$ has the property that $\phi f = \phi g$ implies $\psi f = \psi g$ for all pairs of morphisms $f, g$: $Y' \to Y$ (that is, $\psi$ equalizes any pair of morphisms equalized by $\phi$), there exists $h$: $X \to Z$ such that $h\phi = \psi$.




*Definition* 3. An epimorphism $\phi : Y \to X$ is *extremal* if and only if in any factorization $\phi = \mu\psi$, $\mu$ mono $\Rightarrow \mu$ iso.

It is known (see [**1**; **2**]) that epis are surjective and monos are injective in $\mathscr{G}$. The surjectivity of epis in $\mathscr{H}$ goes through without changes. The situation for monomorphisms is somewhat different.

THEOREM 2. *In $\mathscr{H}$, the monomorphisms are the strong morphisms.*

*Proof.* Assume $\phi: X \to Y$ is strong and that $\phi f = \phi g$.

$$Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{\phi} Y, \quad E(Z) \underset{g^{\#}}{\overset{f^{\#}}{\rightrightarrows}} E(X) \xrightarrow{\phi^{\#}} E(Y)$$

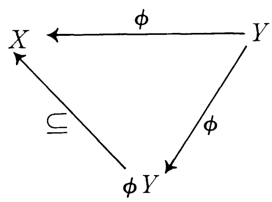
One has $\phi^{\#} f^{\#} = \phi^{\#} g^{\#}$ and $f^{\#} = g^{\#}$ (injectivity of $\phi^{\#}$). Let $a \in V(Z)$. Since we are in $\mathscr{H}$ there is an edge $[a, b] \in E(a; Z)$. Since $f^{\#}\ [a, b] = g^{\#}\ [a, b]$ there are two possibilities viz., (i) $fa = ga$ or (ii) $fa = gb$. The second case cannot arise since it would imply that $\phi fa = \phi gb = \phi ga$, contradicting that $\phi g$ is a morphism. Thus $f = g$ and $\phi$ is mono.

Conversely, if $\phi: X \to Y$ is not strong, $\phi^{\#} e = \phi^{\#} e'$ for distinct edges $e, e' \in E(X)$. One easily constructs a pair of morphisms $f, g: K_2 \to X$ satisfying $\phi f = \phi g$ and $f \neq g$. (Take $f^{\#} E(K_2) = \{e\}$ and $g^{\#} E(K_2) = \{e'\}$). Thus $\phi$ is not mono.

THEOREM 3. *In $\mathscr{G}$, these are equivalent:*
1) *$\phi$ is a strict epimorphism;*
2) *$\phi$ is an extremal epimorphism;*
3) *$\phi$ is a full epimorphism.*

*Proof.* (1) $\Rightarrow$ (2) holds in any category and is well-known.

(2) $\Rightarrow$ (3). Suppose $\phi: Y \to X$ is extremal epi and consider the factorization below:




Since inclusion is mono and $\phi$ is extremal, this inclusion is iso. $\phi Y \simeq X$ and $\phi$ is full.

(3) $\Rightarrow$ (1). Taking $Y'$ as a one-point graph in the definition of strict, it is immediate that the condition on $\psi$ may be strengthened to read: $\psi$ identifies any two vertices identified by $\phi$. Thus the function $h: V(X) \to V(Z)$ given by $h: \phi y \to \psi y$ is well defined. Since $\phi$ is full, it is a morphism.

THEOREM 4. *In $\mathscr{H}$, these are equivalent:*
1) *$\phi$ is a strict epimorphism;*
2) *$\phi$ is an extremal epimorphism;*
3) *$\phi$ is a proper epimorphism.*

*Proof.* (2) $\Rightarrow$ (3). Condition a) follows as in Theorem 3.

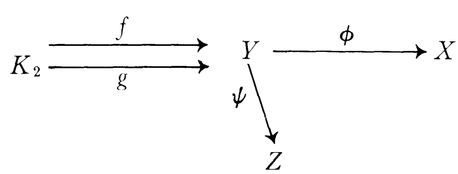
Since $\phi$: $Y \to X$ is full and surjective it may be represented as $\phi$: $Y \to Y/R$. In the factorization of Theorem 1 we have $\alpha$: $Y/T \to Y/R$ is mono by Theorem 2. Since $\phi$ is extremal, $\alpha$ is iso and $Y/T \simeq Y/R$. Since $T \subseteq R$, $Y/T = Y/R$ and $\beta = \phi$. So $\phi$ is proper.

(3) $\Rightarrow$ (1). If $\psi$ satisfies the equalization condition of Definition 2 with respect to an *arbitrary* $\phi$, we can assert: $\psi^{\#}$ identifies any pair of edges identified by $\phi^{\#}$. (Supposing $\phi^{\#}e = \phi^{\#}e'$, use $K_2$ as the $Y'$ in the definition. Construct morphisms $f$, $g$: $K_2 \to Y$ such that $\phi f = \phi g$, $f^{\#}E(K_2) = \{e\}$ and $g^{\#}E(K_2) = \{e'\}$. Then $\psi^{\#}f^{\#} = \psi^{\#}g^{\#}$ so $\psi^{\#}e = \psi^{\#}e'$.)

Suppose now that $\phi$: $Y \to X$ is *proper* and that $\psi$ satisfies the equalization condition. Define $h$: $V(X) \to V(Z)$ as before by $h$: $\phi y \mapsto \psi y$. Since $\phi$ is full, $h$ will be a morphism, provided that it is well defined. Suppose that $y_1 \neq y_2$ and $\phi y_1 = \phi y_2$. For $\phi$ arbitrary we cannot in general assert that $\psi y_1 = \psi y_2$. However, supposing that $\phi^{\#}$ identifies an edge incident with $y_1$ with an edge incident with $y_2$, then we may assert that $\psi y_1 = \psi y_2$. For if $[y_1, w_1]$ and $[y_2, w_2]$ were such edges, with $\psi^{\#}[y_1, w_1] = \psi^{\#}[y_2, w_2]$, then, as in the proof of Theorem 2, the possibility $\psi y_1 = \psi w_2$ can be eliminated. Indeed, two morphisms

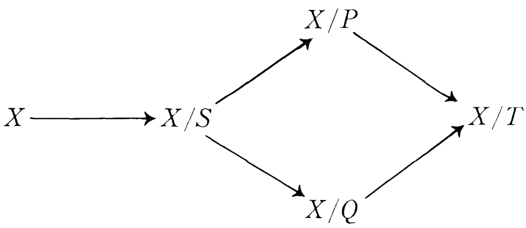
$$K_2 \overset{f}{\underset{g}{\rightrightarrows}} Y$$

could be constructed so that $\phi f = \phi g$ and $\psi f \neq \psi g$. This would violate the condition on $\psi$.




Suppose that $\phi y = \phi y'$. It may be assumed that $y \neq y'$. Since $\phi$ is proper there is a finite sequence $y = y_1, y_2, \ldots, y_n = y'$ as in Definition 1. Applying the argument above to the terms of the sequence in pairs, and using the accompanying edges (as illustrated after Definition 1) one obtains $\psi y = \psi y_1 = \psi y_2 = \ldots = \psi y_n = \psi y'$. This completes the proof.

*Remark.* Making the obvious modifications, the factorization goes through in the category $\mathscr{D}$ of loopless digraphs. There the option presents itself to define *two* congruence relations $P$ and $Q$, obtained from $T$ by restricting the finite sequence of edges to be all outdirected or, respectively, all indirected from the $X_i$'s. Thus $T$ is the smallest congruence containing $P$ and $Q$. If we denote by $S$ the congruence $P \cap Q$ it is possible to factorize as follows:




The new quotients, however, are apparently not associated with morphisms which are of significance in the category $\mathscr{D}$; they are all identified with $X/T$ by the functor $\mathscr{D} \to \mathscr{G}$ which forgets the direction of edges. The factorization can be established in many other categories of loopless graphs (e.g. multi-graphs); the type of congruence relation involved in the factorization is apparently unsuitable for categories of graphs which admit loops.

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## References

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**3.** H. Schubert, *Categories* (Academic Press, Berlin, 1972).

*Laurentian University,*
*Sudbury, Ontario*