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Two-particle billiard system with arbitrary mass ratio

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Abstract. We describe ergodic properties of the system of two hard discs with arbitrary masses moving on the two dimensional torus.

1. *Introduction*

In the present paper we study the system of two disc-like particles in the two dimensional torus T^2. This system was first studied by Sinai [**Sin1**] in the case of equal masses. We describe the structure of the system in the case of arbitrary masses. It turns out that after fixing the values of classical first integrals (the total energy and the total momentum) we obtain a T^2-isometric extension of the Sinai billiard flow on the torus with a disc removed. Since the Sinai billiard flow is Bernoulli (see [**Gal–Orn**]) we can apply the theorem of Rudolph [**Rud**] to conclude that our system is also Bernoulli if only it is weakly mixing.

We prove (Theorem) that this is the case when the ratio of the masses is irrational. In § 3 we describe our system in the case of the rational ratio of the masses. It turns out that in this case there is always a discrete component in the spectrum.

In particular in the rational case the system is or is not ergodic depending on the value of the total momentum but in the irrational case the system is Bernoulli for any value of the total momentum. Somehow this phenomenon is caused by the fact that the motion takes place on a torus and it is no longer true when the discs are sufficiently large ($r > \frac{1}{4}$).

Our paper relies heavily on the theory of Sinai dispersing billiards. In § 5 we formulate the two facts from this theory that we use. Modulo these two facts our proofs are rather nontechnical.

2. *Description of the dynamical system*

Let us consider a system of two disc-like particles of equal radius r, $0 < r < \frac{1}{4}$, and

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masses m_1 and m_2 moving on the torus $T^2 = R^2/Z^2$ We assume that the particles do not interact until they collide and the collisions are elastic.

Let $x_i \in R^2$, x_i mod 1, $i = 1, 2$, be the position of the centre of the disc with mass m_i. We choose to describe the configuration of the discs by the position of the first disc $y_1 = x_1$ and the relative position of the discs $y_2 = x_2 - x_1$ and we take both y_1 and y_2 mod 1. The condition that the two discs do not overlap means that $|y_2 + z| \geq 2r$ for every $z \in Z^2$. So the configuration space $Q = T^2 \times T_r^2$ where $T_r^2 = \{y_2 \bmod 1 \mid |y_2| \geq 2r\}$ i.e. T_r^2 is the torus T^2 with the disc $|y_2| < 2r$ removed. We will use y_1 and y_2 as coordinates in Q.

The first integrals of the system are the total energy E and the total momentum I. We have $I = m_1\dot{x}_1 + m_2\dot{x}_2 = (m_1 + m_2)\dot{y}_1 + m_2\dot{y}_2$ and

$$E = \tfrac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2) = \frac{1}{2(m_1 + m_2)}(I^2 + m_1 m_2 \dot{y}_2^2).$$

Let us fix the values of all three integrals, satisfying the compatibility condition $2(m_1 + m_2)E > I^2$. We get

$$\dot{y}_1 = \frac{I}{m_1 + m_2} - \frac{m_2}{m_1 + m_2}\dot{y}_2 \tag{1}$$

and

$$\dot{y}_2^2 = \frac{2(m_1 + m_2)E - I^2}{m_1 m_2}. \tag{2}$$

Hence the reduced phase space is the trivial circle bundle $Q \times S^1$. By ϕ^t, $\phi^t : Q \times S^1 \circlearrowleft$, $t \in R$ we denote the time flow describing the dynamics. The flow ϕ^t preserves the product measure μ = standard Lebesgue measure in $Q \times$ angular measure in S^1.

The important observation is that the dynamics in $(y_2, \dot{y}_2)$ does not depend on y_1 i.e. the flow ϕ^t factors onto the flow ψ^t, $\psi^t : T_r^2 \times S^1 \circlearrowleft$ by the natural projection $\pi : Q \times S^1 \to T_r^2 \times S^1$. By inspection one establishes that the law of ellastic collision implies that the flow ψ^t is the Sinai billiard flow on T_r^2 with the speed

$$\sqrt{\frac{2(m_1 + m_2)E - I^2}{m_1 m_2}}.$$

The flow ψ^t preserves the product measure ν = standard Lebesgue measure in $T_r^2 \times$ angular measure in S^1. Let us note that the passage from the original dynamical system to the factor flow ψ^t is a special case of the general reduction procedure for a Hamiltonian system possessing several first integrals in involution.

In view of (1) we conclude that ϕ^t is an isometric T^2-extension of ψ^t, for the appropriate definition see [**Rud**].

It was proved by Sinai ([**Sin1, Bun–Sin, Gal, Kel, Kub**]) that the flow ψ^t is a K-flow and by Gallavotti and Ornstein ([**Gal–Orn**]) that it is a Bernoulli flow. Hence ϕ^t is an isometric T^2-extension of a Bernoulli flow and so by the theorem of Rudolph ([**Rud**]) ϕ^t is also Bernoulli provided it is weakly mixing. We will prove the following theorem.

THEOREM. *If m_1/m_2 is irrational then ϕ^t is a K-flow and hence also a Bernoulli flow.*

Note that we do not assume anything about the total momentum I. When m_1/m_2 is rational then ϕ^t has discrete component in the spectrum and it is or it is not ergodic depending on the rationality of I_1/I_2 where $I=(I_1, I_2)$.

The flows ϕ^t for different values of E and I are related in the following way. Let ϕ_0^t be the flow in $Q\times S^1$ for $I=0$ and

$$E=\frac{m_1m_2}{2(m_1+m_2)}$$

(the factor flow ψ^t is the billiard flow with the speed 1). Further let $q_I^t: Q\times S^1 \circlearrowleft$ be the quasiperiodic flow

$$\begin{cases} \dot{y}_1 = I/(m_1+m_2) \\ \dot{y}_2 = 0 \\ \dot{\alpha} = 0. \end{cases}$$

Where α is the angular coordinate in S^1.

The flows ϕ_0^t and q_I^t commute and by (1) and (2) we have for given values of E and I that

$$\phi^t = q_I^t \phi_0^{at} \quad \text{where} \quad a=\sqrt{\frac{2(m_1+m_2)E-I^2}{m_1m_2}}. \tag{3}$$

This construction allows the reduction of the general case to the study of the flow ϕ_0^t.

3. *Lorentz fibres*

We will take advantage of the fact that the centre of mass is preserved in our system when $I=0$. Because we are on a torus we prefer not to define what a centre of mass is.

An immersed 3-dimensional submanifold of $Q\times S^1$ defined by the equations $m_1x_1+m_2x_2=$ constant or equivalently $(m_1+m_2)y_1+m_2y_2=$ constant will be called a Lorentz fibre. More precisely a Lorentz fibre is a 3-dimensional immersed submanifold defined by an immersion $l: \boldsymbol{R}_r^2\times S^1\to Q\times S^1$ given by the formula

$$l(x,\alpha)=\left(y_1-\frac{m_2}{m_1+m_2}x, x, \alpha\right)$$

where y_1 is chosen arbitrarily, α is the angular coordinate on S^1 and

$$\boldsymbol{R}_r^2=\{x\in\boldsymbol{R}^2 \mid |x+z|\geq 2r \quad \text{for every } z\in\boldsymbol{Z}^2\}.$$

It follows straightforwardly from (1) that a Lorentz fibre is invariant under the flow ϕ_0^t and (3) implies that ϕ^t takes a Lorentz fibre into a Lorentz fibre.

Locally a Lorentz fibre projects 1-1 onto $\boldsymbol{T}_r^2\times S^1$ i.e. $\pi\circ l$ is a covering. Hence the flow ϕ_0^t restricted to a Lorentz fibre is the billiard flow on a factor space of $\boldsymbol{R}_r^2$.

4. *Rational* m_1/m_2

Let

$$\frac{m_2}{m_1+m_2}=\frac{p}{q}$$

where p and q are relatively prime. The immersion l factors naturally to the embedding l_q defined by the following commutative diagram

$$\begin{tikzcd} R_r^2 \times S^1 \arrow[d] \arrow[dr, "l"] & \\ & Q \times S^1 \\ R_r^2/qZ^2 \times S^1 \arrow[ur, "l_q"'] & \end{tikzcd}$$

Hence $Q \times S^1$ is foliated by compact Lorentz fibres and on each Lorentz fibre ϕ_0^t is the billiard flow on the R_r^2/qZ^2 - a torus with q^2 discs removed. This is a dispersing billiard system and hence the flow is Bernoulli [**Sin1, Gal–Orn**].

We may also obtain this conclusion formally from properties of the flow ψ^t. Indeed $\pi \circ l_q$ is a finite covering (q^2 to 1) and so the restriction of ϕ_0^t to a Lorentz fibre is a finite extension of ψ^t. Using again the theorem of Rudolph [**Rud**] the extension is Bernoulli if only it is weakly mixing. We may then prove that the extension has the K-property by the method used in the following in the proof of Proposition 2.

To describe the flow ϕ^t we introduce new coordinates (z_1, z_2) mod 1 in Q defined by the following formulas

$$y_1 = z_1 - pz_2$$
$$y_2 = qz_2.$$

Now a Lorentz fibre is defined by $z_1 = \text{constant}$. This change of coordinates is actually a q^2 to 1 covering. In these coordinates we obtain from (1) that

$$\dot{z}_1 = \frac{I}{m_1 + m_2}.$$

So the dynamics is the product of the quasiperiodic flow in z_1 and the billiard flow in z_2 - the billiard table being the torus with q^2 discs removed.

Hence the flow ϕ^t is a finite factor of the product of a quasiperiodic flow in T^2 and a Bernoulli flow. In particular if I_1/I_2 is irrational, where $I = (I_1, I_2)$, then the quasiperiodic flow is ergodic and so is the flow ϕ^t.

The functions $f_k = e^{2\pi i\langle k, qy_1 + py_2\rangle}$, $k \in Z^2$ are eigenfunctions for the flow ϕ^t, so that ϕ^t is never weakly mixing. Indeed by (1) we get

$$\frac{d}{dt} f_k = 2\pi q i \langle k, I\rangle (m_1 + m_2)^{-1} f_k.$$

Further if I_1/I_2 is rational then ϕ^t has a first integral (f_k such that $\langle k, I\rangle = 0$) and so it is not ergodic.

5. *Irrational* m_1/m_2

In this case l is a 1-1 immersion so that ϕ_0^t restricted to a Lorentz fibre is the billiard flow on the infinite billiard table R_r^2, preserving the infinite Lebesgue measure. Every Lorentz fibre is dense in $Q \times S^1$ and the foliation into Lorentz fibres is ergodic in the following sense.

PROPOSITION 1. *A measurable subset of* $Q \times S^1$ *which is a union of Lorentz fibres* mod 0 *has measure zero or its complement has measure zero.*

Proof. Let us consider an auxiliary $\boldsymbol{R}^2$ action A on $\boldsymbol{T}^2 \times \boldsymbol{T}^2$ defined by the formula

$$A^x(y_1, y_2) = \left(y_1 - \frac{m_2}{m_1 + m_2} x, y_2 + x \right).$$

It is well known that such an action is minimal and ergodic.

Let $\tilde{Y} \subset Q \times S^1$ be a measurable union of Lorentz fibres. We have that $\tilde{Y} = Y \times S^1$ and $Y \subset Q$ is a measurable union of orbits of the action A restricted to Q. For sufficiently small ε

$$Y_1 = \bigcup_{x \in \boldsymbol{R}^2} A^x Y = \bigcup_{x \in \varepsilon \boldsymbol{Z}^2} A^x Y$$

so that Y_1 is measurable, $Y_1 \cap Q = Y$ and Y_1 is a union of the orbits of the action A. Hence either Y_1 has measure zero or its complement (in $\boldsymbol{T}^2 \times \boldsymbol{T}^2$) has measure zero. But then the same is true about Y and $\tilde{Y}$. □

6. *Some facts about the flow* ψ^t

In the proof of the Theorem we will rely on some facts about the Sinai billiard flow $\psi^t : \boldsymbol{T}_r^2 \times S^1 \circlearrowleft$ which we are now going to formulate. The details and proofs can be found (at least in principle) in [**Sin1, Bun–Sin, Kub, Kub–Mur**].

Let us consider the standard section map T of the flow ψ^t. T is the first return map defined on the set Σ of unit tangent vectors attached at the boundary of $\boldsymbol{T}_r^2$ and pointing inwards. $T : \Sigma \to \Sigma$ is piecewise differentiable and it preserves a smooth measure ρ. For almost every $v \in \Sigma$ we can construct two smooth curves $\gamma^c(v)$ and $\gamma^e(v)$ which are the local contracting and expanding fibres (l.c.f. and l.e.f.) respectively i.e.

$$\operatorname{dist}(T^n v_1, T^n v_2) \to 0 \quad \text{when } n \to +\infty (n \to -\infty)$$

for every $v_1, v_2 \in \gamma^c(v)(\gamma^e(v))$ and the decay is exponential.

Let ζ_{loc}^c and ζ_{loc}^e be the measurable partitions into the l.c.f. and l.e.f. respectively.

Fact 1.

$$\zeta_{\mathrm{loc}}^c \wedge \zeta_{\mathrm{loc}}^e \text{ is the trivial partition}$$

The consequence is that T has the K-property and further that ψ^t is ergodic. To establish the K-property for the flow ψ^t we need to construct the l.c.f. and the l.e.f. for the flow ψ^t itself. This can be done at almost every point v of the phase space $\boldsymbol{T}_r^2 \times S^1$ and we obtain two smooth curves $\delta^c(v)$ and $\delta^e(v)$. Let ξ_{loc}^c and ξ_{loc}^e be the measurable partitions into these curves respectively.

The curve $\delta^c(v)$ (and $\delta^e(v)$) in $\boldsymbol{T}_r^2 \times S^1$ can be described as a field of vectors normal to a convex curve in $\boldsymbol{T}_r^2$. The convexity makes the pair of partitions ξ_{loc}^c and ξ_{loc}^e 'nonintegrable'. More precisely the following holds.

Fact 2. $\xi_{\mathrm{loc}}^c \wedge \xi_{\mathrm{loc}}^e$ contains an atom and this is a local property, i.e. for any neighbourhood U in $\boldsymbol{T}_r^2 \times S^1$

$$\xi_{\mathrm{loc}}^c|_U \wedge \xi_{\mathrm{loc}}^e|_U \quad \text{contains an atom.}$$

This is essentially proved in [**Kub–Mur**, pp. 19–20].

Let $C:\boldsymbol{R}_r^2\times S^1\to \boldsymbol{R}_r^2/\boldsymbol{Z}^2\times S^1=\boldsymbol{T}_r^2\times S^1$ be the natural covering map. Using C we lift the billiard flow ψ^t to $\tilde{\psi}^t:\boldsymbol{R}_r^2\times S^1 \circlearrowleft$ and the section map T to $\tilde{T}:\Sigma\times\boldsymbol{Z}^2\circlearrowleft$ (clearly $C^{-1}(\Sigma)$ can be naturally identified with $\Sigma\times\boldsymbol{Z}^2$). ψ^t and T preserve the lifted infinite measures $\tilde{\nu}$ and $\tilde{\rho}$ respectively. Let us consider the lifted l.c.f and l.e.f. for $\tilde{\psi}^t$ and $\tilde{T}$ which we will denote by $\tilde{\delta}^{c(e)}(v)$ and $\tilde{\gamma}^{c(e)}(v)$. Let $\tilde{\xi}^{c(e)}_{\text{loc}}$ and $\tilde{\zeta}^{c(e)}_{\text{loc}}$ be the respective measurable partitions. It follows immediately from Fact 1 that $\tilde{\zeta}^c_{\text{loc}}\wedge\tilde{\zeta}^e_{\text{loc}}$ is the partition into individual copies of Σ, $\{\Sigma\times\{z\}, z\in\boldsymbol{Z}^2\}$. We introduce partitions into global contracting and expanding fibres for ψ^t

$$\tilde{\xi}^{c(e)}=\bigwedge_{t\in\boldsymbol{R}}\tilde{\psi}^t\tilde{\xi}^{c(e)}_{\text{loc}}.$$

PROPOSITION 2. *The measurable partition $\tilde{\xi}^c\wedge\tilde{\xi}^e$ is trivial.*

Proof. It follows from Fact 2 that $\tilde{\xi}^c_{\text{loc}}\wedge\tilde{\xi}^e_{\text{loc}}$ contains an atom and hence also $\tilde{\xi}^c\wedge\tilde{\xi}^e$ contains an atom. The last partition though is $\tilde{\psi}^t$ invariant and hence any of its atoms is $\tilde{\psi}^t$ invariant. Let A be such an atom. In particular $\tilde{\nu}(A)$ is positive, possibly infinite. Since A is $\tilde{\psi}^t$ invariant it follows that $B=A\cap C^{-1}(\Sigma)$ has positive $\tilde{\rho}$ measure and it is $\tilde{T}$ invariant. Moreover the structure of l.c.f. and l.e.f. for T and ψ^t implies that B is measurable with respect to $\tilde{\zeta}^c_{\text{loc}}\wedge\tilde{\zeta}^e_{\text{loc}}$. Hence B is a nonempty union of copies of Σ. But the image of an individual copy of Σ under $\tilde{T}$ has positive $\tilde{\rho}$ measure intersections with all the neighbouring copies of Σ. It follows that B contains with every copy of Σ all its neighbours so that actually $B=C^{-1}(\Sigma)$ mod 0. This implies that $A=\boldsymbol{R}_r^2\times S^1$ mod 0. □

7. *Proof of the Theorem*

The l.c.f.'s $\delta^c(v)$ and the l.e.f.'s $\delta^e(v)$ for ψ^t can be lifted to every Lorentz fibre. So for almost every $x\in Q\times S^1$ we have two smooth curves $\tilde{\delta}^c(x)$ and $\tilde{\delta}^e(x)$. These curves are clearly local contracting and expanding fibres for ϕ_0^t. It follows from (3) that they are also local contracting and expanding fibres for ϕ^t no matter what is the value of E and I. Let $\tilde{\xi}^c_{\text{loc}}$ and $\tilde{\xi}^e_{\text{loc}}$ be the measurable partitions of $Q\times S^1$ into l.c.f.'s $\tilde{\delta}^c(x)$ and l.e.f.'s $\tilde{\delta}^e(x)$, $x\in Q\times S^1$. By the standard methods (see **[Sin2]**) we get that

$$\tilde{\xi}^c_{\text{loc}}>\Pi(\phi^t)\quad\text{and}\quad\tilde{\xi}^e_{\text{loc}}>\Pi(\phi^t),$$

where $\Pi(\phi^t)$ is the Pinsker partition for the flow ϕ^t. Hence also for the partitions into global contracting and expanding fibres

$$\tilde{\xi}^c=\bigwedge_{t\in\boldsymbol{R}}\phi^t\tilde{\xi}^c_{\text{loc}}\quad\text{and}\quad\tilde{\xi}^e=\bigwedge_{t\in\boldsymbol{R}}\phi^t\tilde{\xi}^e_{\text{loc}}$$

we have

$$\tilde{\xi}^c>\Pi(\phi^t)\quad\text{and}\quad\tilde{\xi}^e>\Pi(\phi^t).$$

So to establish the K-property for the flow ϕ^t it is sufficient to show that $\tilde{\xi}^c\wedge\tilde{\xi}^e$ is the trivial partition. By Proposition 2 we get that any subset of $Q\times S^1$ measurable with respect to $\tilde{\xi}^c\wedge\tilde{\xi}^e$ must be a union of Lorentz fibres. Further by Proposition 1 any measurable union of Lorentz fibres has measure zero or full measure. □

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