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ASYMPTOTICS OF THE EXIT DISTRIBUTION FOR MARKOV JUMP PROCESSES; APPLICATION TO ATM

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ABSTRACT. We approximate the exit distribution of a Markov jump process into a set of *forbidden* states and we apply these general results to an ATM multiplexor. In this case the forbidden states represent an overloaded multiplexor. Statistics for this overload or busy period are difficult to obtain since this is such a rare event. Starting from the approximate exit distribution, one may simulate the busy period without wasting simulation time waiting for the overload to occur.

1. **Introduction.** Let $(X_t; t \geq 0)$ be a continuous-time, non-terminating, Markov jump process, on a measurable state space $(S, \mathcal{S})$, having an invariant probability measure π. We work in the setting of Iscoe and McDonald (1994) and employ much of the same notation, some of which we now briefly recall. Let $(T_t; t \geq 0)$ denote the transition semi-group of the jump process $(X_t; t \geq 0)$; T_t operates on $L^2(S; \pi)$. The inner product on $L^2(S; \pi)$ will be denoted by $(\cdot, \cdot)_\pi$; and we let ad denote the adjoint operation on operators in $L^2(S; \pi)$. Note that the adjoint operation was denoted by an asterisk superscript in Iscoe and McDonald (1994). The asterisk will be employed for a different, *induced* operation here—see the the end of this Introduction.

We denote the probability transition rate kernel by $J(x, dy)$; so the weak infinitesimal generator, $-\mathcal{L}$, of $(T_t; t \geq 0)$, is given by

$$(1.1) -\mathcal{L}u(x) = \int J(x, dy)[u(y) - u(x)], \quad u \in \mathcal{D}(\mathcal{L}) \subset \{u: S \longrightarrow \mathcal{R}^1 \mid u \text{ is bounded}\}.$$

The strong infinitesimal generator, $-L$, of $(T_t; t \geq 0)$ in $L^2(S; \pi)$ is an extension of $-\mathcal{L}$. Denoting the jump rate by $J(x) = J(x, \{x\}^c)$, we assume throughout that

$$\int_S J(x)\pi(dx) < \infty.$$

We fix a (measurable) set of *forbidden* states and, letting $B \in \mathcal{S}$ denote those states outside the forbidden region, we denote by $(T_t^B; t \geq 0)$ the semi-group of the process killed outside B; that is, the process killed when it enters the forbidden region, B^c. For

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$f \in L^2(B;\pi)$ this semigroup is defined by

$$(1.2) \qquad \begin{aligned} T_t^B f(z) &:= E_z[f(X_t); \tau > t] \\ \tau \equiv \tau^B &:= \inf\{t > 0 : X_t \notin B\}. \end{aligned}$$

Let $-L^B$ denote the (strong) infinitesimal generator associated with $(T_t^B; t \geq 0)$. We now make the second standing assumption:

$$M(B) := \pi\text{-ess}\sup_{x\in B} J(x) < +\infty.$$

We assume throughout the paper that $0 < \pi(B) < 1$ and set

$$\hat{\pi}(B) := \pi(B)^{-1}(\pi|B)$$

where $\pi|B$ is the restriction of π to B.

The notion of a *spectral gap* is central to this article. We remark that in the reversible case, in which T_t is self-adjoint on $L^2(S;\pi)$, the spectral gap (when positive) is the gap in the spectrum, $\sigma(L) \subset \mathbf{R}_+$, between the simple eigenvalue 0 and the rest of the spectrum.

DEFINITION. For the strong infinitesimal generator $-L$, *of* $(T_t; t \geq 0)$ in $L^2(S;\pi)$,

$$\mathrm{Gap}(L) := \inf\{(u, Lu)_\pi : u \in \mathcal{D}(L), \|u\|_\pi = 1, (u, \mathbf{1})_\pi = 0\}.$$

The following will be a standing assumption:

$$\mathrm{Gap}(L) > 0.$$

The main goal of Iscoe and McDonald (1994) was to estimate $P_{\hat{\pi}}(\tau > t)$ by $\exp(-\Lambda(B)t)$ as $B \longrightarrow S$, when $0 < \Lambda(B) := \inf \mathrm{Re}\{\sigma(L^B)\}$, the bottom of the real part of the spectrum of L^B; and "$B \longrightarrow S$" is a shorthand for the convergence: $\pi(B) \longrightarrow 1$. This in turn implied that τ is approximately exponentially distributed with mean $1/\Lambda(B)$. See Theorem 2.2 in Section 2 below for a precise statement. In the present paper, we study the (exit) distribution of X_τ, particularly for large B. Roughly stated, the distribution of the Markov jump process, just prior to exiting from B, is asymptotically proportional to the quasi-stationary distribution on B: For $A \subset B^c$,

$$P_{\hat{\pi}}(X_\tau \in A) \approx E_{\hat{\pi}}\tau^B \cdot \int_B \tilde{\pi}(dx) J(x, A), \quad \text{as } B \longrightarrow S,$$

where $d\tilde{\pi} \equiv \rho^B\, d\hat{\pi}$ denotes the quasi-stationary measure—see Theorems 2.1 and 2.7 for a more precise description.

In Section 2 we recall and complete our standing assumptions and recall the spectral results in Iscoe and McDonald (1994) (for non-reversible Markov jump processes). We then extend them to a full Perron-Frobenius theorem (Theorem 2.3) for the killed semigroup of a Markov jump process on a general state space under our standing assumptions. As a corollary (see Corollary 2.4), we obtain the following probabilistic interpretation of the quasi-stationary measure $\tilde{\pi}$:

$$\tilde{\pi}(\cdot) = \lim_{t \to \infty} P_{\hat{\pi}}(X_t \in \cdot \mid \tau^B > t).$$

Finally we establish Theorem 2.7. This result makes rigorous, and provides error bounds for the heuristic estimates of the mean hitting time to hit B^c and the associated hitting distribution, described in Section B17 in Aldous (1989). A more detailed discussion relating the two approaches is given at the end of Section 2.

In Section 3 this general result on exit distributions is applied to a model for an ATM multiplexor as in Iscoe, McDonald and Qian (1993), to which we refer for more details on notation and terminology. The ATM paper mainly discussed the time until overload of an ATM switch. Here we turn our attention to the onset of the busy period of an ATM switch. We assume that the link rate of the multiplexor is $\ell - 1$ cells per second and that n distinct, independent traffic categories (voice, text, video, *etc.*) are multiplexed together at the switch. Traffic sources in category i may be described as an alternating series of idle and bursty periods. A burst from a source in category i produces cells at a rate of d_i cells per second. (As in the ATM paper, we assume that the d_is are integers with g.c.d. equal to 1; $d := \max_i(d_i)$.) We assume that bursts of category i arrive according to a Poisson process having a rate of a_i bursts per second. We also assume that the burst periods are independent (and independent of the arrival process) and are exponentially distributed with a mean burst length of $1/b_i$.

The aggregate of the n different source categories represents the total load at the link. In particular, if we let $N_i(t)$ represent the number of bursts from category i sources being multiplexed at the link at time t, then the total load at time t may be represented by

$$N(t) := \sum_{i=1}^{n} d_i N_i(t).$$

When the load exceeds the link rate we say the multiplexor is busy. Define

$$\tau = \inf\{t \geq 0 : N(t) \geq \ell\}$$

so that τ is the waiting time until the busy period starts.

We describe the traffic at the multiplexor, up to time τ, by the Markov process $\mathbf{N}(t) := \big(N_1(t), \ldots, N_n(t)\big)$ defined on the state space $S := \{0, 1, 2, \ldots\}^n$. As such, τ is the first time the process $\mathbf{N}(t)$ reaches the forbidden region $B^c = \{x \in S : \sum_i^n d_i x_i \geq \ell\}$.

Let $\mathcal{D}_0$ denote those real-valued functions which are constant outside a finite subset of S. $\mathbf{N}(t)$ has infinitesimal generator $-L$ (having $\mathcal{D}_0$ as a core) given at $u \in \mathcal{D}_0$ by

$$-Lu(x) = \sum_{i=1}^{n}\Big([u(x+\delta_i) - u(x)]a_i + [u(x-\delta_i) - u(x)]x_i b_i\Big), \quad x = (x_1, x_2, \ldots, x_n) \in S$$

where δ_i is the i-th basis vector in S having all its components equal to 0 except the i-th, which is 1. Thus $J(x, x+\delta_i) = a_i$ and $J(x, x-\delta_i) = x_i b_i$.

We remark that for each $1 \leq i \leq n$, $N_i(\cdot)$ is reversible with respect to the stationary Poisson measure having mean $\lambda_i := a_i/b_i$. Moreover the $\big(N_i(\cdot); 1 \leq i \leq n\big)$ are independent. Hence $\mathbf{N}(t)$ is also reversible, with respect to the stationary product measure π given by

$$\pi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{\lambda_i^{x_i}}{x_i!} e^{-\lambda_i}.$$

We show, in Section 3, that $\mathrm{Gap}(L) > 0$.

The technique of induced Dirichlet forms was used in Iscoe, McDonald and Qian (1993) to generate a reversible one-dimensional induced Markov process $(N_t^*; t \geq 0)$ with induced stationary measure π^*, having support $S^* \subset \mathcal{N}_+$, defined by

$$\pi^*(r) \equiv \pi\Big(\{x : f(x) = r\}\Big) = \sum_{x: \sum_{j=1}^{n} d_j x_j = r} \pi(x), \quad \text{where } f(x) := \sum_{j=1}^{n} d_j x_j.$$

The generator $-L^*$ of the induced process N_t^* is given by

$$-L^* u(r) = \sum_{i=1}^{n} [u(r + d_i) - u(r)] a_i + \sum_{i=1}^{n} [u(r - d_i) - u(r)] \Big[\frac{a_i \pi^*(r - d_i)}{\pi^*(r)}\Big]$$

for u in the core $\mathcal{D}_0^*$ of real-valued functions which are constant outside a finite subset of the non-negative integers. As above let $-[L^B]^* \equiv -[L^*]^{B^*}$ denote the generator of the process killed on the induced forbidden set $[B^*]^c := \{\ell, \ell + 1, \ldots\}$.

The exit time τ^* of $N^*(t)$ from B^* provides a stochastic bound on the exit time of $\mathbf{N}(t)$ into the forbidden set: τ^* is stochastically smaller than the original exit time τ in the sense that, for all $\theta \geq 0$,

$$E_\pi \exp(-\theta\tau) \leq E_{\pi^*} \exp(-\theta\tau^*)$$

and

$$E_{\pi^*}\tau^* \leq E_\pi \tau.$$

The numerical results in Iscoe, McDonald and Qian (1993) yielded virtually identical values for $E_{\pi^*}\tau^*$ and $E_\pi\tau$ for even moderate values of ℓ; far better than just a lower bound. One of the main goals of this paper is to explain this extremely good approximation. The explanation lies in the fact that $E_\pi\tau$ is closely approximated by $1/\Lambda$, while $E_{\pi^*}\tau^*$ is closely approximated by $1/\Lambda^*$, the inverse of the *principal* or Perron-Frobenius eigenvalue corresponding to the killed generator for the induced process. We use the Temple-Kato theorem (*cf.* Kato (1949) or Reed and Simon (1978)) to show that as $B \longrightarrow S$, $\Lambda/\Lambda^* \longrightarrow 1$. The second goal is to provide a practical means of simulating the busy period of the ATM switch. Applying Corollary 3.13, we can start the busy period on the hyperplane $\{x : \sum_{i=1}^{n} d_i x_i = \ell\}$ with distribution e having probability mass function

$$\pi(x) \sum_{i=1}^{n} b_i x_i \Big/ \sum_{i=1}^{n} a_i \pi^*(\ell - d_i).$$

2. **General results.** The semigroup of the process killed off some $B \in \mathcal{S}$ was given at (1.2). The associated transition rate kernel and killing rates are given by

$$J^B(x, dy) := \mathbf{1}_B(y)J(x, dy), \quad K^B(x) := \mathbf{1}_B(x)J(x, B^c).$$

We may also define the resuscitation rate R^B to be the Radon-Nikodym derivative of the measure $\mu(dy) := \int_{B^c} \pi(dx)J(x, dy)$ with respect to $\pi|B$, the restriction of π to B. The infinitesimal generator, $-L^B$, can then be expressed as

$$-L^B u(x) = \int_B J(x, dy)[u(y) - u(x)] - K^B(x)u(x), \quad u \in L^2(B; \pi|B);$$

so that $K^B = L^B\mathbf{1}$, where $\mathbf{1}$ denotes the constant function with value 1.

Recall our renormalization of $\pi|B$,

$$\hat{\pi} \equiv \hat{\pi}^B := [\pi(B)]^{-1}(\pi|B), \quad L^2(\hat{\pi}) \equiv L^2(B; \hat{\pi}).$$

The following three quantities were defined in Iscoe and McDonald (1994). They are the mean killing rate and standard deviations of the killing rate and resuscitation rate with respect to the probability $\hat{\pi}$ on B, *viz.*

$$\bar{\kappa} \equiv \bar{\kappa}^B := \int_B K^B(x)\hat{\pi}(dx), \quad \kappa_1 := \|K^B - \bar{\kappa}\|_{\hat{\pi}}, \quad \kappa_2 := \|R^B - \bar{\kappa}\|_{\hat{\pi}}. \tag{2.1}$$

The following will be the standing assumptions throughout this section:

- $(X_t; t \geq 0)$ is non-terminating.
- π is an invariant probability.
- $B \in \mathcal{S}$ with $0 < \pi(B) < 1$.
- $\int J(x)\pi(dx) < \infty$ and $M(B) := \pi\text{-ess}\sup_{x \in B} J(x) < \infty$.
- In the nonreversible case (*i.e.*, when π is not a reversibility measure for $(X_t; t \geq 0)$), $\kappa_1, \kappa_2 \longrightarrow 0$ as $B \longrightarrow S$.
- $\mathrm{Gap}(L) > 0$.

The following two results are taken from Iscoe and McDonald (1994). Theorem 2.1 establishes the existence of the Perron-Frobenius eigenvalue and the associated eigenfunctions of L^B provided that B is sufficiently *large*. Moreover it gives an explicit bound on the L^2 distance to the function $\mathbf{1}$ which is the limiting Perron-Frobenius eigenfunction corresponding to the eigenvalue 0 of the unkilled generator, $-L$. We note that in the the reversible case, in which L and hence L^B are self-adjoint, similar but simpler estimates hold without any restrictions on the size of B (*cf.* Iscoe and McDonald (1994): Lemma 2.12 and the estimate (3.11) in the proof of Theorem 2.13.) Also the conclusions of Theorem 2.2 below can be strengthened, again without restrictions on the size of B, in the reversible case (*cf.* Theorem 2.13 in Iscoe and McDonald (1994).)

THEOREM 2.1. *If the quantities $\bar{\kappa}$, κ_1 and κ_2 (defined at (2.1)) satisfy*

$$\bar{\kappa} < \mathrm{Gap}(L), \quad 4\kappa_1\kappa_2 < [\mathrm{Gap}(L) - \bar{\kappa}]^2,$$

then L^B and $(L^B)^{\mathrm{ad}}$, the adjoint, have a common positive, isolated eigenvalue $\Lambda(B)$ and associated (real-valued) eigenfunctions ϕ^B and ρ^B, respectively, belonging to $L^2(\hat{\pi})$ such that $\int_B \rho^B d\hat{\pi} = \int_B \phi^B \rho^B d\hat{\pi} = 1$. Moreover $0 < \Lambda(B) = \inf \mathrm{Re}\{\sigma(L^B)\}$, $\inf\big[\mathrm{Re}\{\sigma(L^B) \setminus \{\Lambda(B)\}\}\big] > 0$, and:

(i) $|\Lambda(B) - \bar{\kappa}| \le 2\kappa_1\kappa_2 / [\mathrm{Gap}(L) - \bar{\kappa}]$

(ii) $\|\rho^B - \mathbf{1}\|_{\hat{\pi}} \le 2\kappa_2 / [\mathrm{Gap}(L) - \bar{\kappa}]$

(iii) $\|\phi^B - \mathbf{1}\|_{\hat{\pi}} \le \Big\{\frac{2\sqrt{(\mathrm{Gap}(L)-\bar{\kappa})^2+4\kappa_2^2}}{[\mathrm{Gap}(L)-\bar{\kappa}]^2-4\kappa_1\kappa_2}\Big\}\kappa_1$; $\big|\int_B \phi^B\, d\hat{\pi} - 1\big| \le \frac{4\kappa_1\kappa_2}{[\mathrm{Gap}(L)-\bar{\kappa}]^2-4\kappa_1\kappa_2}$.

Note, the non-negativity of ϕ^B and ρ^B is established in Theorem 2.3 below, under addition hypotheses on B.

THEOREM 2.2. *If $\pi(B^c)$ is sufficiently small then for all $t \ge 0$*

$$|P_{\hat{\pi}}(\tau > t) - e^{-\Lambda(B)t}| \le \beta(B) e^{-\Lambda(B)t} \tag{2.2}$$

where

$$\beta(B) := \frac{4}{\big(\mathrm{Gap}(L) - \bar{\kappa}\big)^2 - 4\kappa_1\kappa_2}\left[1 + \frac{\sqrt{\big(\mathrm{Gap}(L) - \bar{\kappa}\big)^2 + 4\kappa_2^2}}{\mathrm{Gap}(L) - \bar{\kappa}}\right]\kappa_1\kappa_2. \tag{2.3}$$

More precisely, $\pi(B^c)$ is sufficiently small if, in addition to the assumptions in Theorem 2.1:

$$\varepsilon(B) := \bar{\kappa} + \frac{2\kappa_1\kappa_2}{\mathrm{Gap}(L) - \bar{\kappa}} + \varepsilon_0(B) < \mathrm{Gap}(L). \tag{2.4}$$

where

$$\varepsilon_0(B) := \left[8\big(\mathrm{Gap}(L) - \bar{\kappa}\big)^{-1}\left[\mathrm{Gap}(L) + 2M(B)\left(1 + \frac{2\kappa_2}{\mathrm{Gap}(L) - \bar{\kappa}}\right)\right]\right] \cdot \kappa_2. \tag{2.5}$$

Let $\|\ \|_\infty$ denote the total-variation norm of a (signed) measure or the sup-norm of a function, as appropriate. The following result extends Theorem 2.2 to a full Perron-Frobenius theorem for T_t^B. See Definition 4.2 in Iscoe and McDonald (1994) for Γ_{ρ^B}, Γ_{ϕ^B}.

THEOREM 2.3. *Denote $\int_B \phi^B\, d\hat{\pi}$ by $s \equiv s(B)$ and set $d\tilde{\pi} \equiv \rho^B\, d\hat{\pi}$. For sufficiently large B, $\rho^B \ge 0$, $\phi^B \ge 0$ ($\hat{\pi}$-a.e.) and*

$$\|\hat{\pi}T_t^B - s\tilde{\pi}T_t^B\|_\infty \le 2\|\mathbf{1} - s\rho^B\|_{\hat{\pi}} e^{-\Gamma(B)t}.$$

where $\Gamma(B) := \min(\Gamma_{\rho^B}, \Gamma_{\phi^B})$; $\Gamma_{\rho^B} \ge \mathrm{Gap}(L) - \varepsilon_0(B)$ and $\Gamma_{\phi^B} \ge \mathrm{Gap}(L) - \varepsilon_0^*(B)$, where $\varepsilon_0(B), \varepsilon_0^*(B) = 0$ in the selfadjoint case and are given in general by (2.5) and

$$\varepsilon_0^*(B) := 6c\{\mathrm{Gap}(L) + M(B)[2 + 3c\kappa_1]\}\kappa_1$$

where

$$c := 2\sqrt{\big(\mathrm{Gap}(L) - \bar{\kappa}\big)^2 + 4\kappa_2^2} \Big/ \Big[\big(\mathrm{Gap}(L) - \bar{\kappa}\big)^2 - 4\kappa_1\kappa_2\Big].$$

More precisely, B is sufficiently large (in the non-selfadjoint case) if, in addition to the assumptions of Theorem 2.1, (2.4) and its analogue, with $\varepsilon_0(B)$ replaced by $\varepsilon_0^*(B)$, hold. In the selfadjoint case, B is sufficiently large if $\bar{\kappa} < \mathrm{Gap}(L)$.

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PROOF.

$$
\begin{aligned}
\|\hat{\pi}T_t^B - s\tilde{\pi}T_t^B\|_\infty &= \sup_{\|f\|_\infty=1}\left|\int_B [\mathbf{1}-s\rho^B]T_t^B f\, d\hat{\pi}\right| \\
&= \sup_{\|f\|_\infty=1}\Big[\Big|\int_B [\mathbf{1}-s\rho^B]T_t^B[f-(f,\rho^B)_{\hat{\pi}}]\, d\hat{\pi} \\
&\qquad +(f,\rho^B)_{\hat{\pi}}\cdot\int_B [\mathbf{1}-s\rho^B]T_t^B\mathbf{1}\, d\hat{\pi}\Big|\Big] \\
&\le \|\mathbf{1}-s\rho^B\|_{\hat{\pi}} \sup_{\|f\|_\infty=1}\|T_t^B[f-(f,\rho^B)_{\hat{\pi}}]\|_{\hat{\pi}} \\
&\qquad +(\mathbf{1},|\rho^B|)_{\hat{\pi}}\|(T_t^B)^{\mathrm{ad}}[\mathbf{1}-s\rho^B]\ \|_{\hat{\pi}} \\
&\le \exp(-\Gamma_{\rho^B}t)\|\mathbf{1}-s\rho^B\|_{\hat{\pi}} \sup_{\|f\|_\infty=1}\|f-(f,\rho^B)_{\hat{\pi}}\|_{\hat{\pi}} \\
&\qquad +\exp(-\Gamma_{\phi^B}t)(\mathbf{1},|\rho^B|)_{\hat{\pi}}\|\mathbf{1}-s\rho^B\|_{\hat{\pi}} \\
&\le e^{-\Gamma(B)t}[1+(\mathbf{1},|\rho^B|)_{\hat{\pi}}]\|\mathbf{1}-s\rho^B\|_{\hat{\pi}} \qquad (2.6)
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality and Proposition 4.4 of Iscoe and McDonald (1994), first with $\rho \equiv \rho^B$, $\mu \equiv \hat{\pi}$, $T_t \equiv T_t^B$ and $A \equiv L^B$ along with the orthogonality condition:

$$\left(\rho^B, f-(f,\rho^B)_{\hat{\pi}}\right)_{\hat{\pi}} = 0$$

to estimate the first term; and then with $\rho \equiv \phi^B$, $\mu \equiv \hat{\pi}$, $T_t \equiv (T_t^B)^{\mathrm{ad}}$ and $A \equiv (L^B)^{\mathrm{ad}}$ along with the orthogonality condition:

$$(\phi^B, \mathbf{1}-s\rho^B)_{\hat{\pi}} = 0$$

to estimate the second term. The estimate on Γ_{ρ^B} is just Lemma 4.11 of Iscoe and McDonald (1994); that for Γ_{ϕ^B} can be derived similarly. Indeed, one simply replaces ρ^B everywhere in the proof by $s^{-1}\phi^B$. In the selfadjoint case, both Γs coincide and are bounded below by Gap(L) by the classical minimax theorem [*cf.* Theorem XIII.3 in Reed and Simon (1978)] as in the proof of Lemma 2.12 in Iscoe and McDonald (1994).

It only remains to show that $\rho^B \ge 0$; for then $(\mathbf{1},|\rho^B|)_{\hat{\pi}} = (\mathbf{1},\rho^B)_{\hat{\pi}} = 1$. Now, applying the inequality (2.6) to $f = \mathbf{1}_A$, with $A \in \mathcal{S}\cap B$, and using the fact that $\tilde{\pi}T_t^B = e^{-\Lambda(B)t}\tilde{\pi}$, we derive that

$$\left|e^{\Lambda(B)t}\hat{\pi}T_t^B(\mathbf{1}_A) - s\int_A \rho^B\, d\hat{\pi}\right| \le e^{-[\Gamma(B)-\Lambda(B)]t}[1+(\mathbf{1},|\rho^B|)_{\hat{\pi}}]\|\mathbf{1}-s\rho^B\|_{\hat{\pi}} \longrightarrow 0,$$

as $t \longrightarrow \infty$, since by (2.4) and Theorem 2.1(i), for large B: $\Gamma(B)-\Lambda(B) > 0$. (In the selfadjoint case, $\Gamma(B) \ge \mathrm{Gap}(L) > \bar{\kappa} \ge \Lambda(B)$.) Therefore

$$\int_A \rho^B d\hat{\pi} = s^{-1}\lim_{t\to\infty} e^{\Lambda(B)t}\int_B T_t^B(\mathbf{1}_A)\, d\hat{\pi} \ge 0.$$

Since A was arbitrary we conclude that $\rho^B \ge 0$, π-a.e. Repeating the previous argument, with $f = \max\big(-1, \min(\phi^B, 1)\big)\cdot \mathbf{1}_A$, yields the non-negativity of ϕ^B. ∎

REMARK. We can obtain an explicit bound on the quantity $\|\mathbf{1} - s\rho^B\|_{\hat{\pi}}$ as follows:

$$\|\mathbf{1} - s\rho^B\|_{\hat{\pi}} \leq s\|\mathbf{1} - \rho^B\|_{\hat{\pi}} + |s - 1|$$

and bounds for each of the terms on the right-hand side are given in Theorem 2.1. However, introduction of these estimates into $s\tilde{\pi}$ will increase the the error bound to order $\exp\bigl(-\Lambda(B)t\bigr)$. (Similarly, introduction of these estimates into s in (2.7) below would essentially reproduce (2.2).) ■

As an immediate corollary we have the following probabilistic description (2.8) of the quasi-stationary measure $\tilde{\pi}$.

COROLLARY 2.4. *Let $B, s,$ and $\Gamma(B)$ be as in Theorem 2.3. Then for $A \in \mathcal{S} \cap B$*

$$\left| P_{\hat{\pi}}(X_t \in A, \tau^B > t) - \int_B \phi^B \, d\hat{\pi} \cdot \int_A \rho^B \, d\hat{\pi} \cdot e^{-\Lambda(B)t} \right| \leq 2\|\mathbf{1} - s\rho^B\|_{\hat{\pi}} e^{-\Gamma(B)t}.$$

In particular, when $A = B$ we obtain the estimate

$$\left| P_{\hat{\pi}}(\tau^B > t) - \int_B \phi^B \, d\hat{\pi} \cdot e^{-\Lambda(B)t} \right| \leq 2\|\mathbf{1} - s\rho^B\|_{\hat{\pi}} e^{-\Gamma(B)t}. \tag{2.7}$$

Moreover,

$$\tilde{\pi}(\cdot) = \lim_{t \to \infty} P_{\hat{\pi}}(X_t \in \cdot \mid \tau^B > t), \tag{2.8}$$

the limit being in total-variation norm for sufficiently large B; uniformly with respect to B if $\varepsilon_0(B), \varepsilon_0^*(B) \longrightarrow 0$ as $B \longrightarrow S$.

PROOF. The first result follows from Theorem 2.3 applied to the function $f = \mathbf{1}_A$. Set $\beta_1(B) := 2\|\mathbf{1} - s\rho^B\|_{\hat{\pi}}$. Now, by Theorem 2.3, for each (measurable) f such that $\|f\|_\infty = 1$, there exists θ_1 (depending on f, t, B), with $|\theta_1| \leq 1$, such that

$$\hat{\pi} T_t^B f = s\tilde{\pi} f e^{-\Lambda(B)t} + \theta_1 \beta_1(B) e^{-\Gamma(B)t}$$

and by (2.7) there exists θ_2 (depending on t, B), with $|\theta_2| \leq 1$, such that

$$P_{\hat{\pi}}(\tau^B > t) = s e^{-\Lambda(B)t} + \theta_2 \beta_1(B) e^{-\Gamma(B)t}.$$

Therefore

$$\begin{aligned} E_{\hat{\pi}}[f(X_t) \mid \tau^B > t] &= \frac{\hat{\pi} T_t^B f}{P_{\hat{\pi}}(\tau^B > t)} \\ &= \frac{s\tilde{\pi} f e^{-\Lambda(B)t} + \theta_1 \beta_1(B) e^{-\Gamma(B)t}}{s e^{-\Lambda(B)t} + \theta_2 \beta_1(B) e^{-\Gamma(B)t}} \end{aligned}$$

Hence for sufficiently large B

$$\left| E_{\hat{\pi}}[f(X_t) \mid \tau^B > t] - \tilde{\pi} f \right| \leq 2 s^{-1} \beta_1(B) [e^{[\Gamma(B) - \Lambda(B)]t} - s^{-1} \beta_1(B)]^{-1} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty$$

since, for sufficiently large B, $\Lambda(B) < \Gamma(B)$ and $\beta_1(B) < s$. Indeed, the former holds by the reasoning in the last paragraph of the proof of Theorem 2.3. The latter holds because $\lim_{B\to S} s(B) = 1$, and $\lim_{B\to S}\beta_1(B) = 0$. Finally, under the stated hypotheses, the limit is uniform with respect to large B because $\lim_{B\to S}\Lambda(B) = 0$ and by Theorem 2.3, $\liminf_{B\to S}\Gamma(B) - \Lambda(B) \geq \mathrm{Gap}(L) > 0$. ∎

Integrating the estimate (2.7) with respect to t we get

COROLLARY 2.5. *For sufficiently large B (as in Theorem 2.3),*

$$\left|E_{\hat{\pi}}\tau^B - \Lambda(B)^{-1}\int_B \phi^B d\hat{\pi}\right| \leq 2\|\mathbf{1} - s\rho^B\|_{\hat{\pi}}/\Gamma(B).$$

with s and $\Gamma(B)$ as in Theorem 2.3.

THEOREM 2.6. *Under the hypotheses of Theorem 2.1, the probability of hitting a subset $A \subset B^c$ when the Markov jump process first exits B, having started in the conditional stationary distribution $\hat{\pi}$, is given by*

$$P_{\hat{\pi}}(X_\tau \in A) = \int_0^\infty \int_B (\hat{\pi}T_t^B)(dx)J(x,A)\,dt$$
$$= \int_B [L^B]^{-1}J(x,A)\hat{\pi}(dx) \equiv \hat{\pi}[L^B]^{-1}J(x,A).$$

PROOF. For a fixed measurable $A \subset B^c$ and $\varepsilon > 0$, set $f_\varepsilon = \varepsilon^{-1}\int_0^\varepsilon T_t \mathbf{1}_A\,dt$. Since, by Corollary 2.10 of Iscoe and McDonald (1994), $E_{\hat{\pi}}\tau < \infty$, then for π-a.e. $x \in B$, $E_x\tau < \infty$. Denoting the weak infinitesimal generator of $(T_t; t \geq 0)$ by $-\mathcal{L}$, we may apply *Dynkin's formula* (*cf.* the Corollary to Theorem 5.1 in Dynkin (1965)) to obtain

$$E_x[f_\varepsilon(X_\tau)] - f_\varepsilon(x) = E_x\left[\int_0^\tau -\mathcal{L}f_\varepsilon(X_t)\,dt\right], \quad \pi\text{-a.e. } x \in B \tag{2.9}$$

(Note that X_τ is measurable by the result 3.17A on p. 98 of Dynkin (1965) and the remark preceding it.)

By Theorem 5.4 in Dynkin (1965), $(T_t; t \geq 0)$ is weakly continuous. Therefore $f_\varepsilon \longrightarrow \mathbf{1}_A$ pointwise and boundedly on S, as $\varepsilon \longrightarrow 0$. Therefore the left-hand side of (2.9) tends to $P_x(X_\tau \in A)$, as $\varepsilon \longrightarrow 0$, for π-a.e. $x \in B$. Also, by (1.1), for π-a.e. $x \in B$

$$-\mathcal{L}f_\varepsilon(x) = \int_S [f_\varepsilon(y) - f_\varepsilon(x)]J(x,dy) \tag{2.10}$$
$$\longrightarrow J(x,A), \quad \text{as } \varepsilon \longrightarrow 0 \tag{2.11}$$

by the bounded convergence theorem. (Recall that, for π-a.e. $x \in B$, $J(x) \leq M(B) < \infty$.)

In addition, the convergence in (2.11) takes place boundedly on B since (2.10) implies that

$$|\mathcal{L}f_\varepsilon(x)| \leq 2M(B), \quad \pi\text{-a.e. } x \in B. \tag{2.12}$$

Therefore the inner integral on the right-hand side of (2.9) converges, by the bounded convergence theorem, to $\int_0^\tau J(X_t,A)\,dt$, as $\varepsilon \longrightarrow 0$. To justify the interchange of limit and expectation on the right-hand side of (2.9), we can apply the dominated convergence theorem since, by (2.12),

$$\left|\int_0^\tau -\mathcal{L}f_\varepsilon(X_t)\,dt\right| \leq 2M(B)\tau$$

and $E_x\tau < \infty$.

In summary, for π-a.e. $x \in B$,

$$\begin{aligned} P_x(X_\tau \in A) &= E_x\Big[\int_0^\tau J(X_t, A)\,dt\Big] \\ &= \int_0^\infty E_x[J(X_t, A); \tau > t]\,dt \\ &\equiv \int_0^\infty T_t^B[J(\cdot, A)](x)\,dt \qquad (2.13) \\ &= \int_0^\infty T_t^B[J(\cdot, A)]\,dt(x) \qquad (2.14) \\ &= [L^B]^{-1}J(\cdot, A)(x). \qquad (2.15) \end{aligned}$$

Note that, in (2.14), we are permitted to interpret the integral as an element of $L^2(B, \hat{\pi})$ by the estimates:

$$J(x) \le M(B), \quad \pi\text{- a.e. } x \in B,$$
$$\|T_t^B(J)\|_{\hat{\pi}} \le M(B) \times \text{const.} \times \exp(-\Lambda(B)t);$$

the latter following from (2.2). Also, in (2.15), $[L^B]^{-1}$ exists because $0 < \Lambda(B) = \inf \operatorname{Re}\{\sigma(L^B)\}$. Integrating (2.13) and (2.15) with respect to $\hat{\pi}$ yields the result. ∎

In our next theorem, we show how this hitting probability may be approximated by replacing $\hat{\pi}$ with the eigenmeasure $\tilde{\pi}$, where $d\tilde{\pi} = \rho^B d\hat{\pi}$.

DEFINITION. For a fixed measurable $A \subset B^c$, let H denote the function $J(x, A)$, $x \in B$, and let $\bar{H} = (H, \rho^B)_{\hat{\pi}}$.

THEOREM 2.7. *Under the full hypotheses of Theorem 2.2 (in the non-selfadjoint case, and without any restriction on B in the selfadjoint case), if $A \subset B^c$ then,*

$$\Big|P_{\hat{\pi}}(X_\tau \in A) - E_{\hat{\pi}}\tau \cdot \int_B \tilde{\pi}(dx)J(x, A)\Big| \le \frac{\|1 - \rho^B\|_{\hat{\pi}}\|H - \bar{H}\|_{\hat{\pi}}}{\operatorname{Gap}(L) - \varepsilon_o(B)}.$$

(In the selfadjoint case, $\varepsilon_o(B) = 0$.)

PROOF. By the definition of H and $\bar{H}$,

$$(\rho^B, H - \bar{H})_{\hat{\pi}} = 0$$

and by Theorem 2.6

$$P_{\hat{\pi}}(X_\tau \in A) = \hat{\pi}[L^B]^{-1}H.$$

Also

$$\begin{aligned} \hat{\pi}[L^B]^{-1}\bar{H} &= \bar{H}\int_0^\infty \hat{\pi}T_t^B(B)\,dt \\ &= \bar{H}\int_0^\infty P_{\hat{\pi}}(\tau > t)\,dt \\ &= \bar{H}E_{\hat{\pi}}\tau. \end{aligned}$$

Since $\tilde{\pi}$ is an eigenmeasure for $-L^B$ and hence for $[-L^B]^{-1}$,

$$\tilde{\pi}[L^B]^{-1}H = \Lambda(B)^{-1}\tilde{\pi}H \equiv \Lambda(B)^{-1}\bar{H}$$
$$\tilde{\pi}[L^B]^{-1}\bar{H} = \Lambda(B)^{-1}\tilde{\pi}\bar{H} \equiv \Lambda(B)^{-1}\bar{H}.$$

Therefore $\tilde{\pi}[L^B]^{-1}(H-\bar{H}) = 0$ and

$$\begin{aligned}\left|P_{\hat{\pi}}(X_\tau \in A) - E_{\hat{\pi}}\tau \cdot \int_B \tilde{\pi}(dx)J(x,A)\right| &= |\hat{\pi}[L^B]^{-1}(H-\bar{H})| \\ &= |(\hat{\pi}-\tilde{\pi})[L^B]^{-1}(H-\bar{H})| \\ &= \left|\int_B d\hat{\pi}(1-\rho^B)[L^B]^{-1}(H-\bar{H})\right| \\ &\le \|1-\rho^B\|_{\hat{\pi}}\|[L^B]^{-1}(H-\bar{H})\|_{\hat{\pi}}.\end{aligned}$$

Now, since $(\rho^B, H-\bar{H})_{\hat{\pi}} = 0$

$$\|[L^B]^{-1}(H-\bar{H})\|_{\hat{\pi}} \le \|H-\bar{H}\|_{\hat{\pi}} \sup_{g\perp\rho^B,\|g\|_{\hat{\pi}}=1} \|[L^B]^{-1}g\|_{\hat{\pi}}.$$

For each g satisfying $g \perp \rho^B$ and $\|g\|_{\hat{\pi}} = 1$ define $f_g := [L^B]^{-1}g/\|[L^B]^{-1}g\|_{\hat{\pi}}$. We see immediately that $f_g \perp \rho^B$ and $\|f_g\|_{\hat{\pi}} = 1$. Clearly for any such g

$$\begin{aligned}\|[L^B]^{-1}g\|_{\hat{\pi}} &= \frac{1}{\|L^B f_g\|_{\hat{\pi}}} \\ &\le \sup_{f\perp\rho^B,\|f\|_{\hat{\pi}}=1} [\|L^B f\|_{\hat{\pi}}]^{-1} \\ &= \Big[\inf_{f\perp\rho^B,\|f\|_{\hat{\pi}}=1} \|L^B f\|_{\hat{\pi}}\Big]^{-1}\end{aligned}$$

and consequently

$$\sup_{g\perp\rho^B,\|g\|_{\hat{\pi}}=1} \|[L^B]^{-1}g\|_{\hat{\pi}} \le \Big[\inf_{f\perp\rho^B,\|f\|_{\hat{\pi}}=1} \|L^B f\|_{\hat{\pi}}\Big]^{-1}.$$

However, in the non-selfadjoint case, by Lemma 4.11 in Iscoe and McDonald (1994), for B sufficiently large (and by Theorem XIII.3 in Reed and Simon (1978), in the selfadjoint case—without any restriction on B), $\Gamma_{\rho^B}(L^B) \ge \mathrm{Gap}(L) - \varepsilon_o(B) > 0$ where

$$\begin{aligned}\Gamma_{\rho^B}(L^B) &:= \inf_{f\perp\rho^B,\|f\|_{\hat{\pi}}=1} (f, L^B f)_{\hat{\pi}} \\ &\le \inf_{f\perp\rho^B,\|f\|_{\hat{\pi}}=1} \|L^B f\|_{\hat{\pi}}.\end{aligned}$$

Therefore,

$$\|[L^B]^{-1}(H-\bar{H})\|_{\hat{\pi}} \le \frac{\|H-\bar{H}\|_{\hat{\pi}}}{\Gamma_{\rho^B}(L^B)} \le \frac{\|H-\bar{H}\|_{\hat{\pi}}}{\mathrm{Gap}(L)-\varepsilon_o(B)}.$$

■

The estimate of Theorem 2.1(ii) can be used for $\|1-\rho^B\|_{\hat{\pi}}$ in the general case. In the selfadjoint case, we can use the estimate (3.11) in Iscoe and McDonald (1994).

The estimate for $P_{\hat{\pi}}(X_\tau \in A)$ given in Theorem 2.7 is related to the heuristic formula in Section B17 in Aldous (1989). Consider the time-reversed process $\overleftarrow{X_t}$ whose jump kernel $\overleftarrow{J}(x,dy)$ satisfies

$$\pi(dx)\overleftarrow{J}(x,dy)=\pi(dy)J(y,dx).$$

Let $\overleftarrow{n}(T)$ denote the number of times before time T that the backwards process leaves B^c from A into B and then does not "immediately" return to B^c. By stationarity

$$\lim_{T\to\infty}\frac{1}{T}\overleftarrow{n}(T)=\int_A\pi(dx)\int_B\overleftarrow{J}(x,dy)\alpha(y) \tag{2.16}$$

where $\alpha(y)$ is the probability the process $\overleftarrow{X}$ starting at $y\in B$ does not "immediately" return to B^c. The meaning of $\alpha(y)$ can be made precise if there exists a subset $\mathbf{0}\subset B$, such that $\pi(\mathbf{0})>0$, to which the process returns again and again. (In the ATM example take $\mathbf{0}=\{(0,0,\ldots,0)\}$.) In this case $\alpha(y)$ may be defined as the probability of hitting $\mathbf{0}$ before returning to B^c. In this case α satisfies

$$\begin{aligned}L^{\mathrm{ad}}\alpha(y)&=0 \quad \text{for } y\in B\setminus\mathbf{0}\\ \alpha(y)&=1 \quad \text{for } y\in\mathbf{0}\\ \alpha(y)&=0 \quad \text{for } y\in B^c\end{aligned}$$

(This may be derived in a manner similar to (2.15) in the proof of Theorem 2.6.)

Now, ρ satisfies

$$\begin{aligned}L^{\mathrm{ad}}\rho(y)&=\Lambda(B)\rho(y) \quad \text{for } y\in B\\ \rho(y)&=0 \quad \text{for } y\in B^c\end{aligned}$$

and hence is a close approximation to α, as may be verified using Theorem 4.9 in Iscoe and McDonald (1994). Therefore

$$\int_A\pi(dx)\int_B\overleftarrow{J}(x,dy)\alpha(y)\approx\int_A\pi(dx)\int_B\overleftarrow{J}(x,dy)\rho(y)=\int_B\rho(y)\pi(dy)J(y,A).$$

Let $n(T)$ denote the number of times before time T that the forward process first enters into B^c from B at A after a return to $\mathbf{0}$. Treating successive returns to $\mathbf{0}$ *after* visiting B^c as cycles, we have from renewal theory that

$$\lim_{T\to\infty}\frac{1}{T}n(T)=\frac{P_{\mathbf{0}}(X_\tau\in A)}{\mu}$$

where μ is the mean cycle length. This follows since once each cycle the forward process enters B for the first time that cycle.

By time reversal, $\lim_{T\to\infty}n(T)/T=\lim_{T\to\infty}\overleftarrow{n}(T)/T$. Also, the mean cycle length is approximately $E_{\mathbf{0}}\tau\approx E_{\hat{\pi}}\tau$. We conclude

$$\frac{P_{\mathbf{0}}(X_\tau\in A)}{E_{\hat{\pi}}\tau}\approx\frac{P_{\mathbf{0}}(X_\tau\in A)}{\mu}\approx\int_B\rho(y)\pi(dy)J(y,A)\approx\int_B\rho(y)\hat{\pi}(dy)J(y,A).$$

Thus the heuristic results in Section B17 in Aldous (1989) coincide asymptotically with ours.

3. **Application to** ATM. For the ATM multiplexor, $M(B) \equiv M(B_\ell) \leq \sum_{i=1}^{n}(a_i + \ell b_i)$. Since this process is also reversible we have $\varepsilon_o(B) = 0$ in Theorem 2.7 and Corollary 2.5. Moreover the quantities $\bar{\kappa}$, κ_1 and κ_2 (defined at (2.1)) clearly tend to to 0 as $\pi(B^c) \longrightarrow 0$. In particular

$$\bar{\kappa} \equiv \bar{\kappa}^* = \sum_{j=1}^{d}\Big(\sum_{i:d_i \geq j} a_i\Big)\hat{\pi}^*(\ell - j) \sim \Big(\sum_{i:d_i=d} a_i\Big)\pi^*(\ell - d), \quad \text{as } \ell \longrightarrow \infty, \tag{3.1}$$

as will be seen in the proof of Lemma 3.1 below. Similarly

$$\|K^B\|_{\hat{\pi}}^2 = \sum_{j=1}^{d}\Big(\sum_{i:d_i \geq j} a_i\Big)^2 \hat{\pi}^*(\ell - j)$$

and this gives a bound on $(\kappa_1)^2$. Also

$$\begin{aligned}
\|R^B\|_{\hat{\pi}}^2 &= \sum_{y \in B} \hat{\pi}(y)\left(\frac{\sum_{x \in B^c} \pi(x)J(x,y)}{\pi(y)}\right)^2 \\
&= \sum_{y:\ell - d \leq \sum_{i=1}^{n} d_i y_i < \ell} \frac{1}{\pi(B)}\frac{1}{\pi(y)}\Big(\sum_{i=1}^{n} \pi(y + \delta_i)J(y + \delta_i, y)\Big)^2 \\
&= \sum_{y:\ell - d \leq \sum_{i=1}^{n} d_i y_i < \ell} \frac{1}{\pi(B)}\frac{1}{\pi(y)}\Big(\sum_{i=1}^{n} \pi(y)J(y, y + \delta_i)\Big)^2, \quad \text{by reversibility} \\
&= \sum_{y:\ell - d \leq \sum_{i=1}^{n} d_i y_i < \ell} \frac{\pi(y)}{\pi(B)}\Big(\sum_{i=1}^{n} a_i\Big)^2 \\
&= \Big(\sum_{i=1}^{n} a_i\Big)^2 \sum_{k=\ell-d}^{\ell-1} \hat{\pi}^*(k).
\end{aligned}$$

This gives the rate at which $(\kappa_2)^2$ tends to 0. Lastly, it is clear from the variational characterization of the Gap that $\mathrm{Gap}(L^*) \geq \mathrm{Gap}(L)$. However by Theorem 2.6 in Liggett (1989) $\mathrm{Gap}(L) \geq \min_i \mathrm{Gap}(L_i)$ where L_i is the generator of the i-th source; that is the generator of a $M/M/\infty$-queue with birth rate a_i and death rate b_i. The Gap of such a birth and death process was calculated to be b_i in Proposition 3.1 in Iscoe, McDonald and Qian (1993). We conclude that the hypotheses of Theorems 2.2 and 2.7 and Corollary 2.5 are verified for the ATM switch. Practical application of these theorems requires, however, good estimates on $\Lambda(B)$ and on $\hat{\pi}$. This is the main thrust of the rest of this section. In particular in Proposition 3.8 we show that $\Lambda(B_\ell)/\Lambda^*(\ell) \longrightarrow 1$ as $\ell \longrightarrow \infty$. In doing so, we show we may replace $\hat{\pi}$ by $(\rho^* \circ f) \cdot \hat{\pi}$ in Theorem 2.7. It turns out that this measure in turn is asymptotically equivalent to $\hat{\pi}$.

The Dirichlet (zero)-form associated with N^* was calculated in Iscoe, McDonald and Qian (1993) as follows,

$$(u, L^*u)_{\pi^*} = \sum_{r=0}^{\infty} u(r)L^*u(r)\pi^*(r) = \sum_{r=0}^{\infty}\sum_{i=1}^{n}[u(r + d_i) - u(r)]^2 a_i \pi^*(r) \tag{3.2}$$

We recall that the killed induced generator is denoted by $[L^B]^* \equiv [L^*]^{B^*}$. Also, we denote by $K^*(x) := \mathbf{1}_{B^*}(x)J^*(x, [B^*]^c)$, the killing rate of the induced process. Then $[L^B]^*$, being self-adjoint, has a positive, isolated eigenvalue $\Lambda^*(\ell)$ and associated right (and left) nonnegative eigenfunction ρ^* belonging to $L^2(\hat{\pi}^*)$. It will be convenient to renormalize ρ^* so that $\|\rho^*\|_{\hat{\pi}^*} = 1$.

LEMMA 3.1.

$$\Lambda^*(\ell) = O\big(\pi^*(\ell - d)\big), \quad \text{as } \ell \to \infty.$$

PROOF. By Lemma 2.12 in Iscoe and McDonald (1994), with $B = [0, \ell - 1]$,

$$\begin{aligned}
\Lambda^*(\ell) &\le \bar{\kappa}^* \\
&= \sum_{k=0}^{\ell-1} J^*(k, [\ell, \infty])\hat{\pi}^*(k) \\
&= \sum_{k=\ell-d}^{\ell-1} \sum_{i:d_i \ge \ell-k} a_i \hat{\pi}^*(k) \\
&= \sum_{j=1}^{d} \Big(\sum_{i:d_i \ge j} a_i\Big)\hat{\pi}^*(\ell - j) \\
&\sim \Big(\sum_{i:d_i=d} a_i\Big)\pi^*(\ell - d)
\end{aligned}$$

by Lemma A.2 in Iscoe, McDonald, and Qian (1993). ∎

LEMMA 3.2. *Let ρ^* be normalized such that $\|\rho^*\|_{\hat{\pi}^*} = 1$. Then*

$$\lim_{\ell\to\infty} \max_{0\le k\le \ell-1} |\rho^*(k) - 1| = 0.$$

PROOF. By Lemma 3.1 there is a constant C_1 such that $\Lambda^*(\ell) \le C_1\pi^*(\ell - d)$. Set $a = \sum_{i:d_i=d} a_i$. Then for all large ℓ

$$\begin{aligned}
C_1 &\ge \Lambda^*(\ell)/\pi^*(\ell - d) \\
&= ([L^B]^*\rho^*, \rho^*)_{\hat{\pi}^*}/\pi^*(\ell - d) \\
&= \sum_{i=1}^{n}\sum_{k=0}^{\ell-1} a_i[\rho^*(k) - \rho^*(k + d_i)]^2 \frac{\hat{\pi}^*(k)}{\hat{\pi}^*(\ell - d)} \qquad \text{by (3.2)}, \\
&\ge \sum_{k=0}^{\ell-d-1} [\rho^*(k) - \rho^*(k + d)]^2 a \frac{\pi^*(k)}{\pi^*(\ell - d)} \\
&\ge \Big\{\sum_{k=0}^{\ell-d-1} [\rho^*(k) - \rho^*(k + d)]^2\Big\} a \frac{\pi^*(\ell - d - 1)}{\pi^*(\ell - d)}
\end{aligned}$$

by Lemma A3 in Iscoe, McDonald and Qian (1993). By Lemma A.2 of Iscoe, McDonald and Qian (1993), $\lim_{\ell\to\infty} \pi^*(\ell - d - 1)/\pi^*(\ell - d) = +\infty$. We conclude that

$$\lim_{\ell\to\infty} \sum_{k=0}^{\ell-d-1} [\rho^*(k) - \rho^*(k + d)]^2 = 0. \tag{3.3}$$

Given $\ell - d \le k \le \ell - 1$, write $k = md + r$ with $0 \le r < d$. Then

$$(3.4)\quad |1-\rho^*(k)| \le |1-\rho^*(r)| + \sum_{j=1}^{m-1} |\rho^*(k-[j+1]d) - \rho^*(k-jd)| + |\rho^*(k-d)-\rho^*(k)|.$$

Now, by Theorem 2.1, $\lim_{\ell\to\infty} |1-\rho^*(r)| = 0$ for each $0 \le r \le d-1$; and by (3.3)

$$\lim_{\ell\to\infty} \max_{d\le k\le \ell-1} |\rho^*(k-d)-\rho^*(k)| = 0.$$

Finally, with C denoting a generic constant (depending on d, λ, *etc.*) whose value varies from line to line, we have, by the Cauchy-Schwarz inequality:

$$\begin{aligned}\Big[\sum_{j=1}^{m-1} |\rho^*(k-[j+1]d) - \rho^*(k-jd)|\Big]^2 &\le \frac{\ell}{d}\sum_{i=0}^{\ell-2d-1} |\rho^*(i)-\rho^*(i+d)|^2 \\ &\le C\ell \sum_{i=0}^{\ell-2d-1} |\rho^*(i)-\rho^*(i+d)|^2 a\frac{\pi^*(i)}{\pi^*(\ell-2d-1)} \\ &\le C\ell\Lambda^*(\ell)/\pi^*(\ell-2d-1) \quad \text{by (3.2)}, \\ &\le C\ell\pi^*(\ell-d)/\pi^*(\ell-2d-1) \\ &\le C\ell\ell^{-(d+1)/d} = C\ell^{-1/d} \to 0\end{aligned}$$

as $\ell \to \infty$; the last inequality following from Lemma A.2 of Iscoe, McDonald and Qian (1993).

Note that for $k < \ell - d$, we can include the first term on the right-hand side of (3.4) in the summation following it. ∎

We now show that the estimate obtained in the proof of Lemma 3.1 is actually asymptotically sharp.

PROPOSITION 3.3. *Let* $a = \sum_{i:d_i=d} a_i$. *Then* $\Lambda^*(\ell) \sim a\pi^*(\ell-d)$, *as* $\ell \to \infty$.

PROOF. By Theorem 2.7, with $A = B^c$, and Corollary 2.14 of Iscoe and McDonald (1994) with $B = [0, \ell - 1]$

$$\begin{aligned}\Lambda^*(\ell) &\sim \sum_{k=0}^{\ell-1} \rho^*(k)K^*(k)\pi^*(k) \\ &\sim \sum_{k=0}^{\ell-1} K^*(k)\pi^*(k), \quad \text{by Lemma 3.2} \\ &= \sum_{j=1}^{d}\Big(\sum_{i:d_i\ge j} a_i\Big)\pi^*(\ell-j) \\ &\sim a\pi^*(\ell-d)\end{aligned}$$

by Lemma A.2 of Iscoe, McDonald and Qian (1993). ∎

Using Proposition 3.3, we can sharpen the analysis in the beginning of the proof of Lemma 3.1 to obtain the following estimate, which will be used in the proof of Proposition 3.8.

COROLLARY 3.4. *Let ρ^* be normalized such that $\|\rho^*\|_{\hat{\pi}^*} = 1$. Then*

$$\sum_{i=1}^{n} \sum_{k=0}^{\ell-d_i-1} a_i[\rho^*(k) - \rho^*(k+d_i)]^2 \pi^*(k) = o\big(\Lambda^*(\ell)\big), \quad \text{as } \ell \to \infty.$$

PROOF. With a as in Proposition 3.3,

$$\begin{aligned}\Lambda^*(\ell) &= \sum_{i=1}^{n} \sum_{k=0}^{\ell-1} a_i[\rho^*(k) - \rho^*(k+d_i)]^2 \hat{\pi}^*(k) \quad \text{by (3.2)}, \\ &\geq \sum_{i=1}^{n} \sum_{k=0}^{\ell-d_i-1} a_i[\rho^*(k) - \rho^*(k+d_i)]^2 \pi^*(k) + a[\rho^*(\ell-d) - \rho^*(\ell)]^2 \pi^*(\ell-d).\end{aligned}$$

Therefore

$$\begin{aligned}\Lambda^*(\ell)^{-1} \sum_{i=1}^{n} \sum_{k=0}^{\ell-d_i-1} a_i[\rho^*(k) - \rho^*(k+d_i)]^2 \pi^*(k) &\leq 1 - [\rho^*(\ell-d)]^2 a\pi^*(\ell-d)\Lambda^*(\ell)^{-1} \\ &\longrightarrow 0, \quad \text{as } \ell \to \infty,\end{aligned}$$

by Lemma 3.1 and Proposition 3.3. ∎

The next two lemmas will also be used in the proof of Proposition 3.8.

LEMMA 3.5. *Let $X = (X_1, \ldots, X_n)$ be a random vector with law π. Set $\lambda_i = a_i/b_i$ and for $k \in S^*$, set*

$$E^{(k)} X_i = E\Big[X_i \mid \sum_{j=1}^{n} d_j X_j = k\Big] \quad \text{and} \quad \mathrm{Var}^{(k)} X_i = \mathrm{Var}\Big[X_i \mid \sum_{j=1}^{n} d_j X_j = k\Big].$$

Then

$$\begin{aligned}E^{(k)} X_i &= \lambda_i \frac{\pi^*(k-d_i)}{\pi^*(k)} \\ \mathrm{Var}^{(k)} X_i &= \lambda_i^2 \frac{\pi^*(k-2d_i)}{\pi^*(k)} + \lambda_i \frac{\pi^*(k-d_i)}{\pi^*(k)} - \lambda_i^2 \Big[\frac{\pi^*(k-d_i)}{\pi^*(k)}\Big]^2\end{aligned}$$

PROOF.

$$\begin{aligned}E^{(k)} X_i &= \sum_{x: f(x)=k} x_i \frac{\pi(x)}{\pi^*(k)} \\ &= \frac{1}{\pi^*(k)} \sum_{x: f(x)=k} b_i x_i \pi(x)/b_i \\ &= \frac{1}{\pi^*(k)} \sum_{x: f(x)=k} a_i \pi(x-\delta_i)/b_i \qquad (3.5) \\ &= \frac{1}{\pi^*(k)} a_i \pi^*(k-d_i)/b_i \\ &= \lambda_i \frac{\pi^*(k-d_i)}{\pi^*(k)}\end{aligned}$$

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and

$$
\begin{aligned}
E^{(k)}X_i^2 &= \sum_{x:f(x)=k} x_i^2 \frac{\pi(x)}{\pi^*(k)} \\
&= \frac{1}{\pi^*(k)} \sum_{x:f(x)=k} x_i\lambda_i\pi(x-\delta_i), \quad \text{as in (3.5)} \\
&= \frac{\lambda_i}{\pi^*(k)} \sum_{x:f(x)=k-d_i} (x_i+1)\pi(x) \\
&= \frac{\lambda_i}{\pi^*(k)} \sum_{x:f(x)=k-d_i} x_i\pi(x) + \frac{\lambda_i}{\pi^*(k)} \sum_{x:f(x)=k-d_i} \pi(x) \\
&= \lambda_i^2 \frac{\pi^*(k-2d_i)}{\pi^*(k)} + \lambda_i \frac{\pi^*(k-d_i)}{\pi^*(k)}, \quad \text{as in (3.5).}
\end{aligned}
$$

■

LEMMA 3.6. *For sufficiently large k,* $\mathrm{Var}^{(k)}(X_i)/E^{(k)}(X_i)$ *is well-defined and uniformly bounded in k.*

PROOF. By Lemma 3.5, the second half of the present lemma will follow from the factorization:

$$\frac{\pi^*(k-2d_i)}{\pi^*(k)} - \left[\frac{\pi^*(k-d_i)}{\pi^*(k)}\right]^2 = \left[\frac{\pi^*(k-2d_i)}{\pi^*(k-d_i)} - \frac{\pi^*(k-d_i)}{\pi^*(k)}\right]\frac{\pi^*(k-d_i)}{\pi^*(k)}$$

provided we show that

$$\frac{\pi^*(k-2d_i)}{\pi^*(k-d_i)} - \frac{\pi^*(k-d_i)}{\pi^*(k)} = O(1), \quad \text{as } k \longrightarrow \infty. \tag{3.6}$$

In the case that all the d_is coincide (and equal 1), $S^* = \mathcal{N}$ and (3.6) follows from explicit computation. Indeed, π^* is then a Poisson density and the ratio $\pi^*(k-1)/\pi^*(k)$ is a linear function of k.

Assume for the remainder of the proof, that the d_is are aperiodic. Then, for sufficiently large k, $k \in S^*$; so that $\pi^*(k) > 0$ and hence the ratios $g(k) := \pi^*(k-d_i)/\pi^*(k)$ are well-defined. In order to verify (3.6), it suffices to derive the asymptotic expansion

$$g(k) := \frac{\pi^*(k-d_i)}{\pi^*(k)} = c_0k^{pd_i} + c_1k^{p[d_i-1]} + \cdots + c_{d_i}k^p + O(1), \quad \text{as } k \longrightarrow \infty. \tag{3.7}$$

Indeed, (3.6) follows from (3.7) because for any (positive or negative) exponent $q \leq 1$

$$(k-d_i)^q - k^q = k^q\left[\left(1-\frac{d_i}{k}\right)^q - 1\right] = k^q[-d_iqk^{-1} + O(k^{-2})] = O(1), \quad \text{as } k \longrightarrow \infty; \tag{3.8}$$

and the left-hand side of (3.6) is $g(k-d_i) - g(k)$, which is a linear combination of differences of the type in (3.8) (plus the O-term from (3.7)).

In what follows, we let (c_i), (c_i'), etc. denote generic sequences of constants which may vary in value from line to line in any calculation—they are coefficients in various asymptotic expansions. It is only the existence of the expansions which is important; the precise values of the coefficients are not needed. Also we set $p = 1/d$ and $c = (d\lambda)^{-p}$.

A somewhat weaker form of (3.7) was established (for another purpose) in Lemma A.2 of Iscoe, McDonald and Qian (1993) in the special case $d = 1$; namely that $g(k) \sim ck^p$. The analysis here is similar, so we shall be brief. The starting point was the asymptotic expansion

$$\pi^*(k) = \frac{1}{\sqrt{2\pi}} \frac{s(k)^{-k} \exp\left[\sum_{j=1}^n d_j^2 \lambda_j \left(s(k)^{d_j} - 1\right)\right]}{[\sum_{j=1}^n d_j \lambda_j s(k)^{d_j}]^{1/2}} \left\{1 + \frac{c_1}{k} + \frac{c_2}{k^2} + \cdots\right\} \tag{3.9}$$

where $s \equiv s(k)$ is the positive solution of $\sum_{j=1}^n d_j \lambda_j s^{d_j} = k$ so that $s(k) \sim (k/\lambda d)^{1/d}$, as $k \longrightarrow \infty$. Therefore

$$g(k) = \left[\frac{s(k)}{s(k-d_i)}\right]^k \left[\exp\left(\sum_{j=1}^n \lambda_j [s(k-d_i)^{d_j} - s(k)^{d_j}]\right)\right] \left[\frac{\sum_{j=1}^n d_j^2 \lambda_j s(k)^{d_j}}{\sum_{j=1}^n d_j^2 \lambda_j s(k-d_i)^{d_j}}\right]^{\frac{1}{2}}$$
$$\times\, s(k-d_i)_i^d [1 + O(k^{-2})], \text{ as } k \longrightarrow \infty, \tag{3.10}$$

since $[1 + c_1/(k-1) + O(k^{-2}])/[1 + c_1/k + O(k^{-2})] = 1 + O(k^{-2})$ as $k \longrightarrow \infty$.

The method of reversion (see Chapter 1 of Olver (1974)) applied to the defining equation for $s(k)$, *viz.*

$$\sum_{j=1}^n d_j \lambda_j s(k)^{d_j} = k,$$

yields the asymptotic expansion

$$s(k) = ck^p + c_0 + c_1 k^{-p} + c_2 k^{-2p} + \cdots + c_d k^{-1}[1 + o(1)]. \tag{3.11}$$

From (3.11) we can easily derive the expansions

$$s(k)^{d_j} = c^{d_j} k^{pd_j} + c_1 k^{p[d_j-1]} + c_2 k^{p[d_j-2]} + \cdots + c_{d_j}[1 + o(1)] \tag{3.12}$$

and

$$s(k)^{-1} = c^{-1} k^{-p} \{1 + c_1 k^{-p} + c_2 k^{-2p} + \cdots + c_d k^{-p-1}[1 + o(1)]\}. \tag{3.13}$$

Now, (3.11) also implies that

$$\begin{aligned} s(k-d_i) &= s(k-d_i) - s(k) + s(k) \\ &= c[(k-d_i)^p - k^p] + c_1[(k-d_i)^{-p} - k^{-p}] + O(k^{-2p-1}) + s(k) \\ &= -pcd_i k^{p-1} + O(k^{p-2}) + c_1\left[\frac{1 - (1 - \frac{d_i}{k})^p}{k^p (1 - \frac{d_i}{k})^p}\right] + O(k^{-2p-1}) + s(k) \\ &= s(k) - pcd_i k^{p-1} + O(k^{-p-1}). \end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\begin{aligned} \frac{s(k-d_i)}{s(k)} &= 1 - pcd_i k^{p-1} s(k)^{-1} + O(k^{-2p-1}) \\ &= 1 - pd_i k^{-1}\{1 + c_1 k^{-p} + c_2 k^{-2p} + \cdots + c_d k^{-p-1}[1 + o(1)]\}, \end{aligned} \tag{3.15}$$

since $-2p-1 > -p-2$. In particular

$$s(k-d_i) = s(k)[1+O(k^{-1})] = ck^p + c_0 + c_1k^{-p} + \cdots + c_{d-1}k^{p-1}[1+O(1)]. \tag{3.16}$$

Also

$$\frac{s(k)}{s(k-d_i)} = 1 + pd_ik^{-1}\{1 + c_1k^{-p} + c_2k^{-2p} + \cdots + c_dk^{-1}[1+o(1)]\}$$

and

$$\begin{aligned}\left[\frac{s(k)}{s(k-d_i)}\right]^k &= \exp\Big(k\log\big[1+pd_ik^{-1}\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+o(1)]\}\big]\Big)\\ &= \exp\big(pd_i\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+o(1)]\}\big)\\ &= e^{pd_i}\cdot\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+o(1)]\}. \end{aligned} \tag{3.17}$$

Next, (3.12) and (3.13) yield

$$\begin{aligned}&\exp\Big(\sum_{j=1}^n \lambda_j[s(k-d_i)^{d_j} - s(k)^{d_j}]\Big)\\ &\quad= \exp\Big(\sum_{j=1}^n \lambda_j s(k)^{d_j}\Big[\Big(\frac{s(k-d_i)}{s(k)}\Big)^{d_j} - 1\Big]\Big)\\ &\quad= \exp\Big(\sum_{j=1}^n \lambda_j s(k)^{d_j}\big[-d_jpd_ik^{-1}\{1+c_1k^{-p}+\cdots+c_dk^{-1}[1+o(1)]\}+O(k^{-2})\big]\Big)\\ &\quad= \exp\Big(-pd_i\big[\lambda ds(k)^dk^{-1}\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+o(1)]\}+O(k^{-1})\big]\Big)\\ &\qquad\times\exp\big(c_1'k^{-p}+c_2'k^{-2p}+\cdots+c_d'k^{-1}+O(k^{-p-1})+O(k^{-p-1})\big)\\ &\quad= e^{-pd_i}\cdot\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+O(1)]\}\\ &\qquad\times\{1+c_1'k^{-p}+c_2'k^{-2p}+\cdots+c_d'k^{-1}[1+o(1)]\}\\ &\quad= e^{-pd_i}\cdot\{1+c_1k^{-p}+c_2k^{-2p}+\cdots+c_dk^{-1}[1+O(1)]\}\end{aligned} \tag{3.18}$$

Finally, (3.12) and (3.16) yield

$$\begin{aligned}&\left[\frac{\sum_{j=1}^n d_j^2\lambda_j s(k)^{d_j}}{\sum_{j=1}^n d_j^2\lambda_j s(k-d_i)^{d_j}}\right]^{\frac12}\\ &\qquad= \left[\frac{d^2\lambda k + c_1k^{1-p} + c_2k^{1-2p} + \cdots + c_d[1+o(1)]}{\{d^2\lambda k + c_1'k^{1-p} + c_2'k^{1-2p}+\cdots+c_d'[1+o(1)]\}\{1+O(k^{-1})\}}\right]^{\frac12}\\ &\qquad= 1 + c_1k^{-p} + c_2k^{-2p} + \cdots + c_dk^{-1}[1+o(1)].\end{aligned} \tag{3.19}$$

Combining (3.11), (3.16), (3.17), (3.18), (3.19) and noting that $pd_i - 1 \le 0$, we obtain the desired conclusion (3.7) from (3.10) as follows:

$$g(k) = e^{pd_i}\cdot\{1 + c_1'k^{-p} + \cdots + c_d'k^{-1}[1+o(1)]\}$$

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$$
\begin{aligned}
&\times e^{-pd_i} \cdot \{1 + c_1'' k^{-p} + \cdots + c_d'' k^{-1}[1 + O(1)\} \\
&\times \{1 + c_1''' k^{-p} + \cdots + c_d''' k^{-1}[1 + o(1)]\} \\
&\times \{c^{d_i} k^{pd_i} + c_1 k^{p[d_i - 1]} + \cdots + c_{d_i - 1} k^{pd_i - 1}[1 + o(1)]\}\{1 + O(k^{-2})\} \\
= &\{1 + \bar{c}_1 k^{-p} + \bar{c}_2 k^{-2} + \cdots + \bar{c}_{d-1} k^{p-1} + O(k^{-1})\} \\
&\times \{c^{d_i} k^{pd_i} + c_1 k^{p[d_i - 1]} + \cdots + c_{d_i - 1} k^{pd_i - 1}[1 + o(1)]\} \\
= &c^{d_i} k^{pd_i} + c_1 k^{p[d_i - 1]} + \cdots + c_{d_i - 1} k^{pd_i - 1}[1 + o(1)] + O(1).
\end{aligned}
$$

■

The main tool used in the proof of Proposition 3.8 is the following variational estimate due to Temple and Kato [*cf.* Kato (1949) or Theorem XIII.5 in Reed and Simon (1978)].

THEOREM 3.7. *Let $\mathcal{L}$ be a self-adjoint operator, on a Hilbert space $\mathcal{H}$, whose spectrum is bounded below by an isolated eigenvalue Λ: $\Lambda < \Lambda_1 \equiv \inf \sigma(\mathcal{L}) \setminus \{\Lambda\}$. Fix $0 \neq v \in \mathcal{H}$ and let r denote the Rayleigh quotient:*

$$r = (\mathcal{L}v, v)/\|v\|^2.$$

If $r < \bar{\Lambda}$ for some $\bar{\Lambda} < \Lambda_1$, then

$$r - \frac{\epsilon^2}{\bar{\Lambda} - r} \leq \Lambda \leq r$$

where $\epsilon^2 \equiv \epsilon(v)^2 = \|\mathcal{L}v - rv\|^2/\|v\|^2 = \|\mathcal{L}v\|^2/\|v\|^2 - r^2$.

PROPOSITION 3.8. *Let $B \equiv B_\ell = \{x : f(x) \leq \ell - 1\}$, $B^* \equiv B^*_\ell = [0, \ell - 1]$, and let $\Lambda \equiv \Lambda(\ell)$, $\Lambda^* \equiv \Lambda^*(\ell)$ be the principal eigenvalues of L^B and $[L^B]^* \equiv L^{*B^*}$. Then $\Lambda(\ell) \sim \Lambda^*(\ell)$, as $\ell \longrightarrow \infty$.*

PROOF. We apply the Temple-Kato result, Theorem 3.7, to $\mathcal{L} \equiv L^B$ and $\Lambda \equiv \Lambda(\ell)$, with test function $v \equiv \tilde{\rho} = \rho^* \circ f$, where ρ^* is the principal (non-negative) eigenfunction associated with Λ^*, normalized such that $\|\rho^*\|_{\hat{\pi}^*} = 1$; and recall $f(x) = \sum_{i=1}^n d_i x_i$. Thus

$$\tilde{\rho}(x) = \rho^*(k), \quad \text{if } \sum_{i=1}^n d_i x_i = k.$$

As such, $\|\tilde{\rho}\|_{\hat{\pi}} = 1$ and the Rayleigh-Ritz quotient, r, is given by

$$(L^B \tilde{\rho}, \tilde{\rho})_{\hat{\pi}} = ([L^B]^* \rho^*, \rho^*)_{\hat{\pi}^*} = \Lambda^*(\ell).$$

Taking $v = \tilde{\rho}$ in Theorem 3.7 it suffices to show $\epsilon^2 = o(\Lambda^*)$; *i.e.*,

$$\frac{\|L^B \tilde{\rho} - \Lambda^* \tilde{\rho}\|^2_{\hat{\pi}}}{\Lambda^*} \longrightarrow 0, \text{ as } \ell \longrightarrow \infty.$$

Now, if $f(x) = k$,

$$
\begin{aligned}
L^B \tilde{\rho}(x) &= \sum_{i=1}^n a_i[\tilde{\rho}(x) - \tilde{\rho}(x + \delta_i)] + \sum_{i=1}^n b_i x_i[\tilde{\rho}(x) - \tilde{\rho}(x - \delta_i)] \\
&= \sum_{i=1}^n a_i[\rho^*(k) - \rho^*(k + d_i)] + \sum_{i=1}^n b_i x_i[\rho^*(k) - \rho^*(k - d_i)]
\end{aligned}
$$

and

$$\Lambda^*\tilde{\rho}(x) = \Lambda^*\rho^*(k) = [L^B]^*\rho^*(k)$$
$$= \sum_{i=1}^n a_i[\rho^*(k) - \rho^*(k+d_i)] + \sum_{i=1}^n \Big[\sum_{x:f(x)=k} b_i x_i \frac{\pi(x)}{\pi^*(k)}\Big][\rho^*(k) - \rho^*(k-d_i)].$$

Therefore

$$\|L^B\tilde{\rho} - \Lambda^*\tilde{\rho}\|_{\hat{\pi}}^2 = \sum_{x\in B}[L^B\tilde{\rho}(x) - \Lambda^*\tilde{\rho}(x)]^2\hat{\pi}(x) = \sum_{k=0}^{\ell-1}\sum_{x:f(x)=k}[L^B\tilde{\rho}(x) - \Lambda^*\tilde{\rho}(x)]^2\hat{\pi}(x)$$
$$= \sum_{k=0}^{\ell-1}\sum_{x:f(x)=k}\Big(\sum_{i=1}^n\Big[b_i x_i - \sum_{x:f(x)=k} b_i x_i\frac{\pi(x)}{\pi^*(k)}\Big][\rho^*(k) - \rho^*(k-d_i)]\Big)^2\hat{\pi}(x)$$
$$\leq \sum_{k=0}^{\ell-1}\sum_{x:f(x)=k} n\sum_{i=1}^n[b_i x_i - \sum_{x:f(x)=k} b_i x_i\frac{\pi(x)}{\pi^*(k)}]^2[\rho^*(k) - \rho^*(k-d_i)]^2\hat{\pi}(x)$$
$$= n\sum_{i=1}^n\sum_{k=0}^{\ell-1}\sum_{x:f(x)=k}\Big[b_i x_i - \sum_{x:f(x)=k} b_i x_i\frac{\pi(x)}{\pi^*(k)}\Big]^2[\rho^*(k) - \rho^*(k-d_i)]^2\hat{\pi}^*(k)$$
$$= n\sum_{i=1}^n\sum_{k=0}^{\ell-1}\mathrm{Var}^{(k)}(b_i X_i)[\rho^*(k) - \rho^*(k-d_i)]^2\hat{\pi}^*(k)$$

where $(X_i; 1 \leq i \leq n)$ are random variables as in Lemma 3.5. For the remainder of the calculation, C denotes a generic constant which may vary from line to line. By Lemma 3.6, the last step may be estimated by

$$\|L^B\tilde{\rho} - \Lambda^*\tilde{\rho}\|_{\hat{\pi}}^2 \leq C\sum_{i=1}^n\sum_{k=d_i}^{\ell-1} b_i^2 E^{(k)}(X_i)[\rho^*(k) - \rho^*(k-d_i)]^2\pi^*(k)$$
$$= C\sum_{i=1}^n\sum_{k=d_i}^{\ell-1} a_i b_i[\rho^*(k) - \rho^*(k-d_i)]^2\pi^*(k-d_i)$$
$$= C\sum_{i=1}^n\sum_{k=0}^{\ell-d_i-1} a_i[\rho^*(k) - \rho^*(k+d_i)]^2\pi^*(k)$$
$$= o\big(\Lambda^*(\ell)\big), \quad \text{as } \ell \longrightarrow \infty$$

by Corollary 3.4. ∎

COROLLARY 3.9. *$E_{\hat{\pi}}\tau \sim \Lambda(\ell)^{-1} \sim \bar{\kappa}^{-1} \sim [a\pi^*(\ell - d)]^{-1}$ as $\ell \longrightarrow \infty$.*

PROOF. This is an immediate consequence of Corollary 2.5, Proposition 3.8, and Proposition 3.3. Note that $\bar{\kappa} \equiv \bar{\kappa}^* \sim a\pi^*(\ell - d)$ by (3.1). ∎

The next lemma is the final preliminary to the proof of the main result of this section; it is also of independent interest.

LEMMA 3.10. *(Maximum Principle) Let $\ell_1 < \ell$ and $\mu < \Lambda(\ell_1)$. Suppose that $v: B_\ell \longrightarrow \mathcal{R}^1$ satisfies*

$$L^B v = \mu v \quad on\ B_{\ell_1},$$
$$v \leq 0 \quad on\ B_\ell \setminus B_{\ell_1}.$$

Then $v \leq 0$ on B_ℓ.

PROOF. Extend v to $S \equiv \mathcal{R}^n$ by setting $v = 0$ on B_ℓ^c; and set $\tau = \inf\{t \geq 0 : X_t \in B_{\ell_1}^c\}$. Then since

$$P_x(\tau > t) \leq C_x e^{-\Lambda(\ell_1)t}$$

for each $x \in B_{\ell_1}$, where C_x is a constant depending on x, we conclude that $E_x[e^{\mu\tau}] < +\infty$ for each $x \in B_{\ell_1}$. By an optional stopping argument, applied to the martingale: $e^{\mu t}v(X_t)$, we obtain the representation

$$v(x) = E_x[e^{\mu\tau}v(X_\tau)]$$

(equating expectations at times $t = 0$ and $t = \tau$). The nonpositivity of v follows. ∎

REMARK. If $\mu < 0$ then a direct (deterministic) analysis of the equation $L^B v = \mu v$ shows that v cannot attain a positive maximum value (occuring necessarily in B_{ℓ_1}). This yields an alternative proof in this case.

LEMMA 3.11. *Let $\rho \equiv \rho_\ell$ denote a non-negative eigenfunction associated with the principal eigenvalue $\Lambda(B_\ell)$. Then ρ is decreasing in each coordinate. In particular $max_{x\in B_\ell}\rho(x) = \rho(0)$.*

PROOF. For any fixed j, $1 \leq j \leq n$, set $v(x) = \rho(x + \delta_j) - \rho(x)$, where ρ is extended to be 0 off B_ℓ. Note that $x + \delta_j \in B_\ell^c$ when $x \in B_{\ell-d_j}^c$; so that $v(x) = 0 - \rho(x) \leq 0$ for $x \in B_\ell \setminus B_{\ell_1}$, where $\ell_1 := \ell - d_j$. It will be shown below that $L^B v = \mu v$ on B_{ℓ_1}, where $\mu := \Lambda(\ell) - b_j < \Lambda(\ell) \leq \Lambda(\ell_1)$; the latter following from the Rayleigh-Ritz characterization of Λ. Granted this result, we then conclude immediately from Lemma 3.10 that $v \leq 0$ on B_ℓ, which is equivalent to the present lemma.

It remains to verify that v satisfies $L^B v = \mu v$ on B_{ℓ_1}. For $x \in B_{\ell_1}$, $x + \delta_j \in B_\ell$; so that $L^B\rho(x + \delta_j) = \Lambda(\ell)\rho(x + \delta_j)$. Now

$$L^B v(x) = L^B[\rho(\cdot + \delta_j)](x) - L^B\rho(x) = L^B[\rho(\cdot + \delta_j)](x) - \Lambda(\ell)\rho(x);$$

and

$$\begin{aligned} &L^B[\rho(\cdot + \delta_j)](x) \\ &\quad = \sum_{i=1}^n \{a_i[\rho(x+\delta_j) - \rho(x+\delta_j+\delta_i)] + b_i x_i[\rho(x+\delta_j) - \rho(x+\delta_j-\delta_i)]\} \\ &\quad = \sum_{i=1}^n \{a_i[\rho(x+\delta_j) - \rho(x+\delta_j+\delta_i)] + b_i(x_i+\delta_j)[\rho(x+\delta_j) - \rho(x+\delta_j-\delta_i)]\} \\ &\qquad\quad - b_j[\rho(x+\delta_j) - \rho(x)] \\ &\quad = L^B\rho(x+\delta_j) - b_j v(x) = \Lambda(\ell)\rho(x+\delta_j) - b_j v(x). \end{aligned}$$

Therefore $L^B v(x) = \Lambda(\ell)\rho(x+\delta_j) - b_j v(x) - \Lambda(\ell)\rho(x) = \big(\Lambda(\ell) - b_j\big)v(x)$ as claimed. ∎

In the next result, we use the notation $\| \ \|_\infty$ to denote the total-variation norm of a (signed) measure. Also $H(x, \cdot) := J(x, \cdot \cap B^c)$; and recall that for a fixed $A \subset B^c$, $H(x) \equiv H(x, A)$ and $\bar{H} = \sum_{x\in B} H(x)\rho(x)^B\hat{\pi}(x)$.

THEOREM 3.12. *Let* $a = \sum_{i:d_i=d} a_i$ *and* $B_\ell = \{x \in S : f(x) \leq \ell - 1\}$. *Then*

$$\Big\| P_{\hat\pi}(X_\tau \in \cdot) - E_{\hat\pi}\tau \sum_{x \in B_\ell} H(x,\cdot)\hat\pi(x) \Big\|_\infty \longrightarrow 0, \quad \text{as } \ell \longrightarrow \infty,$$

where $E_{\hat\pi}\tau \sim [a\pi^*(\ell - d)]^{-1}$, *as* $\ell \longrightarrow \infty$. *(See (3.9) for the asymptotic behaviour of* π^*, *itself.) In particular* $\lim_{\ell\to\infty} P_{\hat\pi}(X_\tau \in \partial B_\ell) = 1$ *where* $\partial B_\ell := \{x \in S : f(x) = \ell\}$.

PROOF. Writing simply B for B_ℓ, for each $A \subset B^c$, the proof of Theorem 2.7 implies

$$\begin{aligned}\Big| P_{\hat\pi}(X_\tau \in A) - \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B} \rho(x)H(x,A)\hat\pi(x) \Big| &\leq \frac{\|1-\rho^B\|_{\hat\pi}\|H-\bar H\|_{\hat\pi}}{\mathrm{Gap}(L)} \\ &\equiv \eta(A) \\ &\leq \frac{2\sum_{i=1}^n a_i \|1-\rho^B\|_{\hat\pi}}{\mathrm{Gap}(L)}\end{aligned}$$

where $\rho := \rho^B / \rho^B(0)$, so that $\rho(0) = 1$ and $\rho^B = \rho / \sum_{x\in B}\rho(x)\hat\pi(x)$. In particular

$$\Big| 1 - \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B}\rho(x)H(x,B^c)\tilde\pi(x) \Big| \leq \eta(B^c). \tag{3.20}$$

By Lemma 3.11, $\rho(x) \leq \rho(0) = 1$. Therefore

$$\begin{aligned}&\Big| P_{\hat\pi}(X_\tau \in A) - \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B} H(x,A)\hat\pi(x) \Big| \\ &\qquad \leq \Big| P_{\hat\pi}(X_\tau \in A) - \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B}\rho(x)H(x,A)\hat\pi(x) \Big| \\ &\qquad\quad + \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B}[1-\rho(x)]H(x,A)\hat\pi(x) \\ &\qquad \leq \eta(A) + \frac{E_{\hat\pi}\tau}{\sum_{x\in B}\rho(x)\hat\pi(x)} \cdot \sum_{x\in B}[1-\rho(x)]H(x,A)\hat\pi(x). \end{aligned} \tag{3.21}$$

Also, since $\bar\kappa = \sum_{x\in B} H(x,B^c)\hat\pi(x)$,

$$\begin{aligned} &E_{\hat\pi}\tau \cdot \sum_{x\in B}[1-\rho(x)]H(x,A)\hat\pi(x) \qquad (3.22)\\ &\leq E_{\hat\pi}\tau \cdot \sum_{x\in B}[1-\rho(x)]H(x,B^c)\hat\pi(x) \\ &\leq \Big| \sum_{x\in B}\rho(x)\hat\pi(x) - E_{\hat\pi}\tau \cdot \sum_{x\in B}\rho(x)H(x,B^c)\hat\pi(x) \Big| \\ &\quad + |1 - E_{\hat\pi}\tau \cdot \bar\kappa| + 1 - \sum_{x\in B}\rho(x)\hat\pi(x) \\ &\leq \eta(B^c)\sum_{x\in B}\rho(x)\hat\pi(x) + |1 - E_{\hat\pi}\tau\cdot\bar\kappa| + 1 - 1/\rho^B(0) \\ &\longrightarrow 0 \qquad (3.23)\end{aligned}$$

as $B \longrightarrow S$, by (3.20) and Corollary 3.9.

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Combining (3.21) with (3.22), we conclude that

$$\left\| P_{\hat{\pi}}(X_\tau \in \cdot) - \frac{E_{\hat{\pi}}\tau}{\sum_{x\in B}\rho(x)\hat{\pi}(x)} \cdot \sum_{x\in B} H(x,\cdot)\hat{\pi}(x)\right\|_\infty \longrightarrow 0, \quad \text{as } \ell \longrightarrow \infty.$$

Finally,

$$\begin{aligned}
&\left\| P_{\hat{\pi}}(X_\tau \in \cdot) - E_{\hat{\pi}}\tau \sum_{x\in B} H(x,\cdot)\hat{\pi}(x)\right\|_\infty \\
&\quad \le \|P_{\hat{\pi}}(X_\tau \in \cdot)\|_\infty \cdot \Big[1 - \sum_{x\in B}\rho(x)\hat{\pi}(x)\Big] \\
&\quad\quad + \sum_{x\in B}\rho(x)\hat{\pi}(x)\left\| P_{\hat{\pi}}(X_\tau \in \cdot) - \frac{E_{\hat{\pi}}\tau}{\sum_{x\in B}\rho(x)\hat{\pi}(x)} \cdot \sum_{x\in B} H(x,\cdot)\hat{\pi}(x)\right\|_\infty \\
&\quad \longrightarrow 0, \quad \text{as } \ell \longrightarrow \infty,
\end{aligned}$$

which yields the first conclusion of the theorem. The second conclusion then follows immediately from Corollary 3.9. ∎

COROLLARY 3.13.

$$\|P_{\hat{\pi}}(X_\tau \in \cdot) - e(\cdot)\|_\infty \longrightarrow 0, \quad \text{as } \ell \longrightarrow \infty,$$

where e is a probability measure on the hyperplane $\partial B_\ell = \{x : \sum_{i=1}^n d_i x_i = \ell\}$, with probability mass function $\pi(x)\sum_{i=1}^n b_i x_i / \sum_{i=1}^n a_i \pi^*(\ell - d_i)$.

PROOF. From Theorem 3.12, the hitting distribution is concentrated, asymptotically, on the hyperplane $\{x : \sum_{i=1}^n d_i x_i = \ell\}$ with a probability mass function at x proportional to

$$\sum_{i=1}^n a_i \pi(x - \delta_i) = \sum_{i=1}^n b_i x_i \pi(x).$$

The normalizing constant is

$$\begin{aligned}
\sum_{x:\sum_{i=1}^n d_i x_i = \ell} \sum_{i=1}^n b_i x_i \pi(x) &= E\Big[\sum_{i=1}^n b_i X_i; \Big\{\sum_{i=1}^n d_i X_i = \ell\Big\}\Big] = \sum_{i=1}^n b_i E^{(\ell)} X_i \cdot \pi^*(\ell) \\
&= \sum_{i=1}^n b_i \lambda_i \frac{\pi^*(\ell - d)}{\pi^*(\ell)} \pi^*(\ell) \quad \text{from Lemma 3.5.}
\end{aligned}$$

The result follows. ∎

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