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ON HURWITZ CONSTANTS FOR FUCHSIAN GROUPS

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ABSTRACT. Explicit bounds for the Hurwitz constants for general cofinite Fuchsian groups have been found. It is shown that the bounds obtained are exact for the Hecke groups and triangular groups with signature $(0:2,p,q)$.

1**. Introduction.** It is known that $\mathrm{PSL}(2,\mathbf{R})$ can be identified with the group of all orientation-preserving isometries of the upper half- plane model for hyperbolic plane $H^2 = \{x+yi \in \mathbf{C}, y > 0\}$ endowed with metric $y^{-2}|dz|^2$ (see *e.g.* [1]). Transformation $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbf{R})$ acts on H^2 by the rule $T(z) = (az+b)/(cz+d)$. A geodesic in H^2 is a semicircle or a ray orthogonal to the real axis. Let $\Gamma \in \mathrm{PSL}(2,\mathbf{R})$ be a finitely generated Fuchsian group of the first kind. We assume that Γ is *zonal*, that is, Γ has a parabolic fixed point at ∞. Then there is a least positive w, the *width* of the cusp at ∞, such that $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Let α be a real irrational number. In 1891 A. Hurwitz [5] showed that the inequality

$$|\alpha - \frac{a}{c}| < \frac{1}{hc^2} \tag{1}$$

has infinitely many solutions in coprime integers a and c when $h = \sqrt{5}$, and $\sqrt{5}$ is the best constant possible. The first geometric proof of this result was obtained by L. Ford in [3] where he makes use of properties of the modular group.

Let Λ be the limit set of Γ and P the set of cusps (parabolic vertices). Let $\alpha \in \Lambda - P$. J. Lehner [6] showed that there is a positive constant h depending only on Γ such that the inequality

$$|\alpha - T\infty| < \frac{1}{h|c|^2} \tag{2}$$

holds for infinitely many left cosets of $\Gamma_\infty = \mathrm{Stab}(\infty, \Gamma)$ in Γ. When Γ is the modular group, $P = \mathbf{Q}$ and (2) is reduced to (1).

For a fixed $\alpha \in \Lambda - P$ we denote by $h(\alpha)$ the supremum of all such h in (2). The set of numbers

$$L(\Gamma) = \{1/h(\alpha), \alpha \in L - P\}$$

is the *Lagrange spectrum* for Γ and $C(\Gamma) = \sup L(\Gamma)$ the *Hurwitz constant* for Γ.

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For any real x_o, the region $P_\infty = \{(x, y) \in H^2 : x_o < x < x_o + w\}$, where w is the width of the cusp at ∞, is a fundamental domain of Γ_∞. The region

$$D = P_\infty \cap \{z \in H^2 : |T'(z)| < 1, T \in \Gamma\}$$

is an *isometric* fundamental domain for Γ in H^2. Here $T'(z) = |cz + d|^{-2}$. (The circle $|cz + d| = 1$ is called the *isometric* circle of T) (see *e.g.* [1]).

Assume that a side σ of D, which is not a vertical ray, lies on some isometric circle λ. The point of λ farthest from the real axis Λ is called the *summit* of σ. We shall call the distance from the farthest from the real axis point of σ to the real axis the *height* of σ and denote it by $\mathrm{ht}(\sigma)$. Suppose that $v = v_1$ belongs to the cycle of vertices $C = \{v_1, \ldots, v_n\}$ of D. It is known that C lies on some horocycle $\mathrm{Im}\, z = \mathrm{const}$ (see [1], p. 229 and p. 288).

Let σ_i and σ_i', $\mathrm{ht}(\sigma_i) \le \mathrm{ht}(\sigma_i')$, be the sides of D which meet at v_i, $i = 1, \ldots, n$. Denote

$$K(v) = K(C) = 2 \min\{\mathrm{ht}(\sigma_1), \ldots, \mathrm{ht}(\sigma_n)\}. \tag{3}$$

We shall say that a vertex of D is *odd* (*even*) if it is an elliptic fixed point of Γ of an odd (even) order. Let v be an endpoint of a side σ of D. Define $\lambda(v) = 1$ unless v is an odd vertex of D of order q and the summit of σ belongs to σ when

$$\lambda(v) = \left(1 + \left(1 - \cos\frac{\pi}{q}\right)^2\right)^{1/2}, \tag{4}$$

and denote

$$h_\Gamma = \inf \lambda(v)K(v), \tag{5}$$

where $K(v)$ is defined by (3) and the infimum being taken over all the vertices and cusps $v \neq \infty$ of D.

Rankin [10] found explicit upper and lower bounds for the Hurwitz constant $C(\Gamma)$ for a general zonal cofinite Fuchsian group Γ. In [12], a modification of the Ford geometric approach to the problem of approximation of irrational real numbers by rational fractions is developed. This method is applied to find an upper bound for the Hurwitz constant for a geometrically finite discrete group acting in an n-dimensional hyperbolic space. When $n = 1$, that is, when Γ is a general zonal Fuchsian group, this bound is better than the one obtained by Rankin. In [11], this approach is used to find the approximation constants for the imaginary quadratic fields of discriminant -20 and -24. The main purpose of this paper is to obtain a further improvement for the upper bound for $C(\Gamma)$. The following result is stronger than Theorem 1 from [12] for $n = 1$.

THEOREM 1. *Let Γ be a zonal cofinite Fuchsian group with the limit set Λ and set of cusps P. Let $\alpha \in \Lambda - P$. Then there are infinitely many left cosets of* $\mathrm{Stab}(\infty, \Gamma)$ *in Γ whose members T satisfy*

$$\left|\alpha - \frac{a}{c}\right| = |\alpha - T\infty| < \frac{1}{h_\Gamma |c|^2} \tag{6}$$

($c \neq 0$). *Thus,* $C(\Gamma) \le 1/h_\Gamma$.

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We shall say that a side σ of D with an endpoint v is *critical* if $h_\Gamma = \lambda(v)K(v) = 2\lambda(v)\mathrm{ht}(\sigma)$. It follows from the results obtained in [12], where $\lambda(v) = 1$ for all the vertices of D, that $C(\Gamma) = 1/h_\Gamma$ if each of the endpoints of a critical side σ is an elliptic fixed point of an even order. In the following theorem, some other cases are enumerated when the equality holds for the Hurwitz constant in Theorem 1.

THEOREM 2. *Let Γ be a zonal cofinite Fuchsian group. Let D be an isometric fundamental domain of Γ and let σ be a critical side of D. Assume that an endpoint v of σ is a fixed point of Γ and that the summit s of σ is an elliptic fixed point of order two.*

1. If v is an elliptic fixed point of an odd order q, then

$$C(\Gamma) = \frac{1}{2\mathrm{ht}(\sigma)}\left(1 + \left(1 - \cos\frac{\pi}{q}\right)^2\right)^{-1/2}.$$

2. If v is an elliptic fixed point of an even order or a cusp of D, then

$$C(\Gamma) = \frac{1}{2\mathrm{ht}(\sigma)}.$$

Moreover, if v is a cusp, then $C(\Gamma)$ is an accumulation point in the Lagrange spectrum for Γ.

In Section 2 we introduce h-neighborhoods of vertices and cusps of D, study their properties, and use them to prove Theorem 1. In Section 3, Theorem 5, an analogue of Theorems 1 and 2 for the disc model of the hyperbolic plane, is given. In Section 4 we first prove Theorem 2 and the second part of Theorem 5 and then apply them to some triangular groups, including Hecke groups G_q. In Example 1, for even q, we find also the second minimum in the Lagrange spectrum of G_q. It was first found by Haas and Series [4] (see also [8]).

The author thanks the referee for his useful remarks which led to an improvement of this work.

2. ***h*-neighborhoods of vertices and cusps.** In this section we prove Theorem 1 using a modification of the notion of an h-neighborhood of a vertex or a cusp of D introduced in [12].

Let $\alpha \in \Lambda - P$. Denote by $L = L(\alpha)$ the vertical ray through α in H^2. Let $T \in \Gamma$. For any $h > 0$, let $R(T, h)$ be the open Euclidean disk in H^2 tangent to the real axis Λ at $T\infty = a/c$ having radius $1/(hc^2)$. We have $R(T, h) = TR_h$ where $R_h = R(\mathrm{id}, h) = \{z : \mathrm{Im}\, z \geq h/2\}$. Denote the boundaries of the horocyclic regions $R(T, h)$ and R_k by $Q(T, h)$ and Q_h respectively. Thus, the inequality (2) holds if and only if L cuts $Q(T, h)$.

Since α is not a parabolic fixed point, the line L passes through infinitely many fundamental regions $T(D)$, $T \in \Gamma$. Let the fundamental region through which L passes be $T_n(D)$, $n = 1, 2, \ldots$, taken in order as a point z moves along L from ∞ to α. Let z_n be the point of intersection of L with the common boundary of $T_{n-1}(D)$ and $T_n(D)$. Denote by L_n the part of L between z_n and α. Define $\mathcal{N}(h)$ to be the region in $H^2 \cup \mathcal{P}$ which is exterior to all $\mathcal{R}(T, h)$, $T \in \Gamma$.

Let σ and σ' be the sides of D that meet at v. Assume that their summits s and s' belong to the closure of D. Let $2\,\mathrm{ht}(\sigma) = h_o < h$. We denote by $\mathcal{T}(v)$ the component of the closure of D which contains v and is bounded by the sides σ and σ', two vertical lines passing through the summits s and s', and $y = h/2$. (If $h < h_o$, then $\mathcal{T}(v)$ is a triangular region). We shall call the union of all $T\mathcal{T}(v)$, $T \in \Gamma$, which contain v the *h-neighborhood* of v and denote it by $N(v, h)$. There are two kinds of sides of $N(v, h)$: parts of horocycles $Q(T, h)$, $T \in \Gamma$, which will be called the *horocyclic* sides of $N(v, h)$, and separating them the *geodesic* sides which are the images of the vertical segments, sides of $\mathcal{T}(v)$.

For every vertex or cusp v of D, define $k(v)$ to be the largest h such that any geodesic passing through $N(v, h)$ cuts a horocyclic side of $N(v, h)$. Suppose that $v = v_1$ belongs to the cycle of vertices $C = \{v_1, \dots, v_n\}$. Let σ_i and σ_i', $\mathrm{ht}(\sigma_i) \le \mathrm{ht}(\sigma_i')$, be the sides of D which meet at v_i, $i = 1, \dots, n$. If $h \le K(v)$ (see (3)) then $N(v, h)$ does not have geodesic sides. Hence

$$k(v) \ge K(v). \tag{7}$$

It will be shown (see Lemma 3) that the equality holds in (7) when v is an even vertex or cusp of D but when v is an odd vertex this bound can be improved.

Let vertex v of D be an elliptic fixed point of order q. Assume that some geodesic L intersects $\mathcal{T}(v)$ but does not cut the horocyclic sides of $N(v, h)$. Then L crosses two geodesic sides of $N(v, h)$ one of which is a vertical side adjacent to the horocyclic side that lies on $y = h/2$. Since the order of v is q there are q horocyclic and q geodesic sides of $N(v, h)$. Figure 1 shows the sets $\mathcal{T}(v)$ and $N(v, h)$, $h > h_\Gamma$, when Γ is the Hecke group G_4. (Figure 1 in [12] shows these sets when $h = h_\Gamma$).

Assume that the cyclic group $\mathrm{Stab}(v, \Gamma)$ is generated by W, $W^q = \mathrm{id}$, and $W(\sigma) = \sigma'$. Denote by Q_k the side of $N(v, h)$ that lies on the horocycle $Q(W^k, h)$ and by B_k the geodesic side of $N(v, h)$ that lies on the geodesic with endpoints $W^k\infty$ and $W^{k+1}\infty$. Thus, B_0 and B_{q-1} are the vertical sides of $N(v, h)$ (see Figure 1 where $q = 4$). The summit s of σ is the hyperbolic midpoint of B_0 and $s_k = W^k s$ is the midpoint of B_k.

The improvement of Theorem 1 from [12] for $n = 1$ is based on the following.

LEMMA 3. *Let D be an isometric fundamental domain for a zonal cofinite Fuchsian group Γ. Assume that the sides σ and σ' of D meet at a vertex v which is a fixed point of Γ of order q. If v is an odd vertex of order q, then*

$$k(v) = 2\left(1 + \left(1 - \cos\frac{\pi}{q}\right)^2\right)^{1/2} \mathrm{ht}(\sigma).$$

If v is an even vertex or a cusp, then $k(v) = 2\,\mathrm{ht}(\sigma)$.

PROOF. Let $c_k = c(W^k)$. The radius of Q_k equals $1/(hc_k^2)$. In particular, the radius of $Q(W, h_o)$ $1/(h_o c_1^2) = h_o/4$ since it is tangent to $y = h_o/2$ at the summit of σ. Hence

$$c_0 = 0, \quad c_1 = 2/h_o. \tag{8}$$

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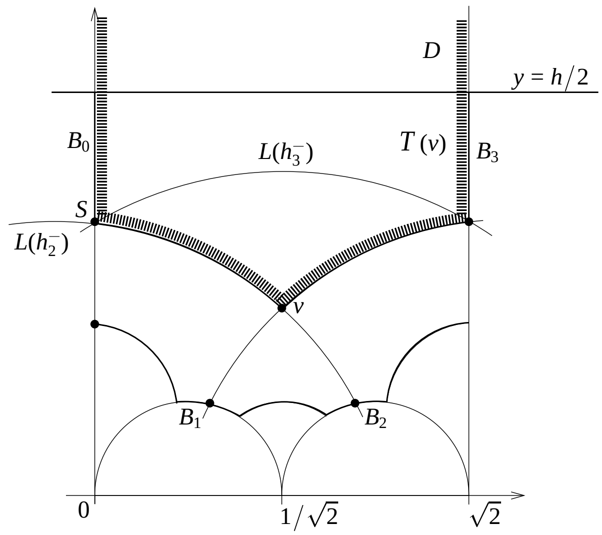

FIGURE 1

Since the trace of W is $2\cos\theta$, where $\theta = \pi/q$, W satisfies the equation $W^2 - 2\cos\theta W + I = 0$, where I is the identity matrix. Thus, c_k satisfy the finite difference equation

$$c_{k+1} - 2(\cos\theta)c_k + c_{k-1} = 0.$$

Solving this equation subject to the initial conditions (8) we get

$$c_k = \frac{2}{h_o}\frac{\sin(k\theta)}{\sin\theta}. \tag{9}$$

Notice that when v is a cusp of D, then, taking the limit in (9) as $\theta \longrightarrow 0$, we obtain

$$c_k = \frac{2}{h_o}k. \tag{10}$$

Let h_k^- (h_k^+) be the smallest value of h for which there is a geodesic which cuts both B_0 (B_{q-1}) and B_k without cutting a horocyclic side of $N(v,h)$. It is clear from the geometry that the radius of a geodesic $L(h)$ which is internally tangent to one of the horocycles $Q(W^m,h)$ and $Q(W^{m+1},h)$ and externally to the other is a decreasing function of h. Hence $L(h_k^-)$ and $L(h_k^+)$ are tangent to Q_0, Q_k, and Q_{k+1}.

Let L be the geodesic which passes through s and s_k and is tangent to Q_0. Since s is the midpoint of B_0, L is also tangent to Q_1. Let R be the reflection in H^2 with respect to the geodesic which passes through v and perpendicular to L. Then $R(Q_{k+1}) = Q_0$, $R(Q_k) = Q_1$, and $R(s) = s_k$. Hence L is tangent to Q_k, and Q_{k+1}. Thus, $L = L(h_k^-)$. Similarly, $L(h_k^+)$ passes through s' and s_k and is tangent to Q_0, Q_{q-1}, Q_k, and Q_{k+1} (see Figure 1 [12] where Γ is the Hecke group G_4). Notice that $L(h_0^+) = L(h_{q-1}^-)$.

The center of Q_k is $x_k + i/(hc_k)$ where $x_k = W^k\infty$. The centers ζ and the radii $h/2$ of the geodesics $L(h_k^-)$ and $L(h_k^+)$ satisfy the system

$$\begin{aligned} \left|\zeta - x_k + \frac{i}{hc_k^2}\right| &= \frac{h}{2} \pm \frac{1}{hc_k^2} \\ \left|\zeta - x_{k+1} + \frac{i}{hc_k^2}\right| &= \frac{h}{2} \mp \frac{1}{hc_k^2} \end{aligned} \tag{11}$$

where

$$|x_{k+1} - x_k| = \frac{2}{|h_o c_k c_{k+1}|}$$

since horocycles Q_k and Q_{k+1} are tangent to each other when $h = h_o$. Solving system (11) we get

$$h = h_o\left(1 + \left(\frac{c_k^2 - c_{k+1}^2 \pm 4h_o^{-2}}{2c_k c_{k+1}}\right)^2\right)^{1/2}.$$

from which, by (9), we obtain

$$h_k^- = h_o\big(1 + \cot^2(k\theta)\sin^2\theta\big)^{1/2}, \tag{12}$$

and

$$h_k^+ = h_{k+1}^-. \tag{13}$$

($k = 1, \ldots, q-1$). When $q = 2m+1$ is odd, the smallest value of h_k^+ (and h_k^-) is $k(v) = h_m^+ = h_m^- = \lambda(v)h_o$ where $\lambda(v)$ is defined by (4), and the centers of $L(h_m^-)$ and $L(h_m^+)$ are $\operatorname{Re} v \mp h_o/2$. When $q = 2m$, $k(v) = h_m^- = h_o$. If v is a cusp of D, then, taking the limit in (12) as $\theta \longrightarrow 0$, we obtain

$$h_k^- = h_o(1 + k^{-2})^{1/2}, \tag{14}$$

and (13) also holds for $k = 1, 2, \ldots$. Thus, $k(v) = 2\operatorname{ht}(\sigma)$ in that case. The lemma is proved.

Now, from (5), (7), and Lemma 3 we get

$$h_\Gamma \leq \inf k(v), \tag{15}$$

the infimum being taken over all the vertices and cusps $v \neq \infty$ of D.

PROOF OF THEOREM 1. Let $h < h_\Gamma$. Assume that Theorem 1 is false. Then there exists an integer n such that $L_n \subset \mathcal{N}(h_\Gamma)$. Assume that L_n passes through $N(v, h)$ in $\mathcal{N}(h_\Gamma)$. Since, by (15), $h < h_\Gamma \leq k(v)$, L_n cuts a horocyclic face of $N(v, h)$ in contradiction with the assumption.

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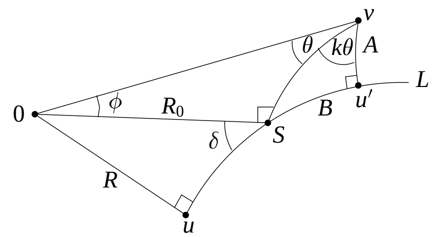

FIGURE 2

3**. The disc model.** In this section, the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ with a metric derived from the differential $ds = 2(1 - |z|^2)^{-1}|dz|$ is used as a model for the hyperbolic plane. An orientation-preserving isometry of Δ can be identified with $T = \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{C})$. Let Γ be a cofinite Fuchsian group acting in Δ. The unit circle Λ is the limit set of Γ. Let $\alpha \in \Lambda - P$. (Now P can be empty). We consider the approximation of α by the elements of the orbit $\Gamma 0$. It is shown in [12] how the general case of approximation of α by the orbit Γw where $w \in \Delta$ can be reduced to the case of $w = 0$. It is also shown that the Hurwitz constant for the orbit $\Gamma 0$ coincides with that for $\Gamma\infty$.

Let D be the Dirichlet polygon for Γ with center 0. For a geodesic L in Δ, the point of L which is closest to the origin is the *summit* of L. Let η and η' be the endpoints of L. Denote by r and R the Euclidean and hyperbolic distances from the origin to L respectively. Let $h = |\eta - \eta'|$. Then we have

$$h = 2\frac{1-r^2}{1+r^2} = \frac{2}{\cosh R}.$$

Let v be a vertex or cusp of D. Denote $K = \{z \in \mathbf{C} : |z| = r\}$. We define h-neighborhoods $N(v, h)$ as in Section 2, replacing the horocycles $Q(T, h)$ and vertical rays through the summits in H^2 by hyperbolic circles $T(K)$, $T \in \Gamma$, and radii through the summits in Δ. We intend to derive an analogue of formula (12) and then to obtain an analogue of Theorem 1 for the disc model. Let σ be a side of D whose summit s belongs to the closure of D. Assume that an endpoint v of σ is an elliptic fixed point of order q. Denote by W a generator of $\mathrm{Stab}(v, \Gamma)$. Let $s_k = W^k s$, $k = 1, \ldots, q-1$. Denote by $L(h_k^-)$ and $L(h_k^+)$ the geodesics passing through s and s_k and s_{q-1} and s_k respectively. Denote by R_o and R_k^- the hyperbolic distances from the origin to s and to $L(h_k^-)$ respectively. Let u and u' be the feet of perpendiculars from the origin and v to $L(h_k^-)$ respectively (see Figure 2).
Denote $A = \rho(v, u')$, $B = \rho(s, u')$, and $C = \rho(s, v)$ where $\rho(z, z')$ is the hyperbolic distance between two points $z, z' \in \Delta$. Let δ be the angle at s in the triangle with vertices O, s, and u. In the triangle Osv, the angles at v and s are $\theta = \pi/q$ and $\pi/2$, and we denote by ϕ the angle at the origin. In the triangle vsu', the angles at v, s, and u' are $k\theta$, $\pi/2 - \delta$, and $\pi/2$ respectively. Then we have (see [1], pp. 146–147)

$$\sinh R_k^- = \sinh R_o \sin\delta, \tag{16}$$

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and

$$\cosh C = \cosh A \cosh B, \quad \sinh A = \sinh C \cos \delta, \quad \sinh B = \sinh C \sin(k\theta).$$

Eliminate A and B, substitute $\cosh C = (\cos\phi)/(\sin\theta)$, and solve this system for $\sin\delta$, to obtain, by (16),

$$\sinh R_k^- = \sinh R_o \left(1 + \frac{\sin^2\theta \cot^2(k\theta)}{\cos^2\phi}\right)^{-1/2}, \tag{17}$$
$$R_k^+ = R_{k+1}^-.$$

If v is a cusp of D, then, as $\theta \longrightarrow 0$ in (17), we obtain

$$\sinh R_k^- = \sinh R_o \left(1 + \frac{1}{k^2\cos^2\phi}\right)^{-1/2}. \tag{18}$$

We define $k'(v)$ as in Section 2. The following statement is an analogue of Lemma 3.

LEMMA 4. *Let D be an isometric fundamental domain for a cofinite Fuchsian group Γ which acts in Δ. Assume that the sides σ and σ' of D meet at vertex v which is a fixed point of Γ of order q. If v is an odd vertex of order q, then*

$$\sinh R(v) = \sinh R_o \left(1 + \left(\frac{1-\cos\frac{\pi}{q}}{\cos\phi}\right)^2\right)^{1/2}$$

where $k'(v) = 2/\cosh R(v)$. If v is an even vertex or a cusp, then $k'(v) = 2/\cosh R_o$.

Let v be an endpoint of a side σ of D. Define $\lambda'(v) = 1$ unless v is an odd vertex of D of order q and the summit of σ belongs σ when

$$\lambda'(v) = \left(1 + \left(\frac{1-\cos\theta}{\cos\phi}\right)^2\right)^{1/2}, \quad \theta = \frac{\pi}{q}. \tag{19}$$

Let $w \in D$. Suppose that $v = v_1$ belongs to the cycle of vertices $C = \{v_1, \ldots, v_n\}$ of D. Let σ_i and σ_i', $\rho(w, \sigma_i) \geq \rho(w, \sigma_i')$, be the sides of D which meet at v_i, $i = 1, \ldots, n$. Denote

$$K(w, v) = \max\{\rho(w, \sigma_1), \ldots, \rho(w, \sigma_n)\}. \tag{20}$$

Assume that the orbit Γw is used to approximate $\alpha \in \Lambda - P$. Let

$$\sinh R_\Gamma = \sup \lambda'(v) K(w, v), \tag{21}$$

where $K(w, v)$ is defined by (20) and the supremum being taken over all the vertices and cusps $v \neq \infty$ of D. Denote $V = (1 - |w|^2)^{-1/2} \begin{pmatrix} 1 & w \\ \overline{w} & 1 \end{pmatrix}$ and

$$TV = \begin{pmatrix} a'' & \overline{c''} \\ c'' & \overline{a''} \end{pmatrix} \tag{22}$$

For a fixed $w \in \Delta$, we define the approximation constant $h(\alpha)$, the Lagrange spectrum, and the Hurwitz constant for the group Γ as in Section 1. The first part of the following theorem can now be proved as in [12].

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THEOREM 5. *Suppose that* Γ *is a cofinite Fuchsian group acting on the unit disc* Δ*. Let* $w \in \Delta$ *and* $D(w)$ *be the Dirichlet polygon with center* w*. Let* Λ *be the limit set and* P *the set of parabolic fixed points of* Γ*. Let* $\alpha \in \Lambda - P$*. Then there are infinitely many* $T \in \Gamma$ *satisfying*

$$|\alpha - Tw| < \frac{\cosh R_\Gamma}{2|c''|^2}$$

where c'' *is defined by (22) and* R_Γ *by (19) and (21).*

Assume that the endpoints v *and* v' *of a critical side of* D *are fixed points of* Γ*. Let the summit* s *of* σ *be an elliptic fixed point of order two. Then the Hurwitz constant*

$$C(\Gamma) = \frac{1}{2}\cosh R_\Gamma \lambda'(v),$$

where $\lambda'(v)$ *is defined by (19). Moreover, if* v *is a cusp, then* $C(\Gamma)$ *is an accumulation point in the Lagrange spectrum for* Γ*.*

The second part of Theorem 5 is an analogue of Theorem 2 for the disc model.

4. **Applications.** In this section we first prove Theorem 2 and then apply it to Hecke groups and some other groups. We shall say that a geodesic L with endpoints η and η' is *extremal* if

$$h(\eta) = |\eta' - \eta|.$$

Suppose that L is the axis of a hyperbolic element $S \in \Gamma$. Then η and η' are the fixed points of S. It is known (see *e.g.* [4]) that

$$h(\eta) = \sup |T(\eta') - T(\eta)|$$

where the supremum is taken over all $T \in \Gamma$. It follows that the Hurwitz constant $C(\Gamma) \geq 1/h(\eta)$.

LEMMA 6. *Let* L *be an axis of a hyperbolic element* $S \in \Gamma$*. Let* η *and* η' *be the fixed points of* S*. Suppose that* L *passes through* $N(v, h)$*, where* $h = |\eta' - \eta|$*, but does not cut a horocyclic side of* $N(v, h)$*. If* $L \cap N(v, h)$ *contains a fundamental domain of* $\mathrm{Stab}(L, \Gamma)$ *on* L *then* L *is extremal.*

PROOF. Assume that L is not extremal. Then, for some $T \in \Gamma$, there is $L' = T(L)$ such that $\mathrm{ht}(L') > \mathrm{ht}(L)$. Since $L \cap N(v, h)$ contains a fundamental domain of $\mathrm{Stab}(L, \Gamma)$ on L and $N(v, h)$ is covered by images of the regions $T(v)$, $v \in C$, for some cycle C of vertices of D, $\mathrm{Im}\, z \leq h/2$ for $z \in V(L)$ for any $V \in \Gamma$ which contradicts the assumption that $\mathrm{ht}(L') > \mathrm{ht}(L)$.

COROLLARY 7. *Let a vertex* v *of* D *be a fixed point of* Γ*. Let* σ *and* σ' *be the sides of* D *that meet at* v*. Assume that the summit* s *of* σ *is an elliptic fixed point of order two. Let* W *be a generator of* $\mathrm{Stab}(v, \Gamma)$*. Then the geodesic passing through the points* s *and* $W^k s$ *is extremal.*

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PROOF. Assume that $As = s$, $A^2 = \mathrm{id}$. Then $S = AW^kAW^{-k} \in \Gamma$ is a hyperbolic element, and L is the axis of S. The arc $[s, s_k)$, $s_k = W^k s$, of L is a fundamental region of $\mathrm{Stab}(L, \Gamma)$ on L. It was shown above that $L = L(h_k^-)$ does not intersect a horocyclic side of $N(v, h_k^-)$. Hence the arc $[s, s_k)$ belongs to $N(v, h_k^-)$ and by Lemma 6 L is extremal.

It is clear that Lemma 6 and Corollary 7 hold for the disc model too. Corollary 7 implies that all the geodesics $L(h_k^-)$ and $L(h_k^+)$ are extremal. When v is an elliptic vertex of order q, by (12), (13),

$$\left(1 + \cot^2(k\theta)\sin^2\theta\right)^{-1/2}/h_o \in \mathcal{L}(\Gamma), \quad k = 1, \ldots, q-1,$$

where $\theta = \pi/q$. In particular, $1/h_\Gamma \in \mathcal{L}(\Gamma)$ and therefore $C(\Gamma) = 1/h_\Gamma$. This proves case a) and the first part of case b) of Theorem 2. Similarly the second part of Theorem 5 can be proved.

If v is a cusp of D, then, by (14),

$$(1 + k^{-2})^{-1/2}/h_o \in \mathcal{L}(\Gamma), \quad k = 1, 2, \ldots$$

which implies that $C(\Gamma) = 1/h_\Gamma$ is an accumulation point of $\mathcal{L}(\Gamma)$. This completes the proof of case c) of Theorem 2.

EXAMPLE 1. Let $q > 2$ be an integer. Let $\Gamma = G_q = \langle A, B \rangle$ where

$$A = \begin{pmatrix} 1 & 2\cos(\pi/q) \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is a triangular group with signature $(0 : 2, q, \infty)$ known as a Hecke group. A fundamental domain of this group is bounded by the unit circle $|z| = 1$ and two vertical lines $x = \pm\cos(\pi/q)$. Thus $h_o = 2$ and, $h_\Gamma = 2\lambda(v) = 2\left(1 + (1 - \cos\frac{\pi}{q})^2\right)^{1/2}$ if q is odd, and $h_\Gamma = 2$ if q is even. Theorem 2, where σ lies on $|z| = 1$, $v = e^{i\pi/q}$, and $s = i$, is applicable since $Bs = s$. Hence $C(\Gamma) = 1/h_\Gamma$.

Now let v be an even vertex of D of order $q = 2m$. We shall find the second point in the Lagrange spectrum of G_q. The notation established in Section 2 is maintained in this example. Assume that $h > 2$. The vertex v is the fixed point of involution $R = W^m$ and it lies on the geodesic $L = L(h_m^-)$. Hence L is the axis of the hyperbolic element $T = RB$, $T(B_o) = B_m$, and the geodesic interval $[s, s_m)$ is a fundamental domain on L of the cyclic group generated by T. Thus L cuts only geodesic faces $T^i(B_o)$, $i = 0, \pm 1, \pm 2, \ldots$.

Assume that an extremal geodesic L' passes through $N(v, h)$. If L' cuts only the same geodesic faces as L, then $L' = L$. Assume that L' passes through $T^iN(v, h)$ and that it cuts $T^i(B_o)$ and $T^i(B_k)$, $k \neq m$. Then geodesic $T^{-i}(L')$ passes through $N(v, h)$ and cuts B_o and B_k, $k \neq m$. Hence $2\mathrm{ht}(L') \geq 2\mathrm{ht}\left(T^{-i}(L')\right) \geq h_k^-$, $k \neq m$. By (12), $2\mathrm{ht}(L') \geq h_m^+ = h_{m+1}^- = 2(1 + \sin^2\frac{\pi}{q}\tan^2\frac{\pi}{q})^{1/2}$. The case when the Γ-orbit of geodesic L' does not contain an extremal geodesic can be dealt with as it is done in [2], Chapter II. In that case one has to use an analogue of the isolation theorem (see [2], p. 25) for a zonal cofinite Fuchsian group. Thus $1/h_{m+1}^-$ is the second minimum in the $\mathcal{L}(G_q)$. It is attained at the endpoints of the geodesic $L(h_{m+1}^-)$ (In Figure 1, $q = 4$). These results were first obtained by Haas and Series [4] (see also [8]).

EXAMPLE 2. Let Γ be the limit of the Hecke groups as $q \longrightarrow \infty$. The points ± 1 are the cusps of D. By Theorem 2, the Hurwitz constant $C(\Gamma) = 1/2$ is an accumulation point in the Lagrange spectrum for Γ.

EXAMPLE 3. Let $\Gamma = \Gamma(2)$, the principal congruence group of level 2 consisting of matrices $T \equiv \pm I \pmod 2$ (*cf.* [10]). We choose the fundamental domain D as follows. It is bounded by $x = -1/2$, $x = 3/2$, $|2z \pm 1| = 1$, and $|2z - 3| = 1$. Then Theorem 2, where σ lies on $|2z - 1| = 1$, $v = 0$ and $v' = 1$ are cusps of D, and $s = (1 + i)/2$ is the summit of σ, is applicable since D is symmetrical with respect to $x = 1/2$. It implies that $C(\Gamma) = 1$ is an accumulation point in $L(\Gamma)$.

EXAMPLE 4. Let now $\phi = \pi/p$ and $\theta = \pi/q$ where p and q are positive integers such that $1/p + 1/q < 1/2$. Let $\rho = (\cos^2\theta - \sin^2\phi)^{1/2}$. Then (see [9], p. 87–88) the group

$$\Gamma = \left\langle \frac{i}{\sin\phi}\begin{pmatrix} \cos\theta & \rho \\ -\rho & -\cos\theta \end{pmatrix}, \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \right\rangle$$

is a triangular group with signature $(0 : 2, p, q)$ which maps the unit disc Δ onto itself. Let $w = 0$. A Dirichlet polygon $D(0)$ of this group is bounded by the straight Euclidean lines joining the origin 0 with $P = \cos(\phi + \theta)e^{i\phi}/\rho$ and $\bar{P}$, and the circle $|z - (\cos\theta)/\rho| = (\sin\phi)/\rho$.

If q is even then, by Theorem 5, the Hurwitz constant for Γ equals $(\cos\theta)/(2\sin\phi)$ (*cf.* [12]). Similar results can be obtained for an arbitrary triangular groups using their matrix representation given in [9], p. 105.

When q is odd, by (19), we have

$$\sinh R_\Gamma = \frac{\rho}{\sin\phi}\left(1 + \left(\frac{1 - \cos\theta}{\cos\phi}\right)^2\right)^{1/2},$$

since $\sinh R_o = \rho/\sin\phi$, and the Hurwitz constant for Γ equals $(\cosh R_\Gamma)/2$.

EXAMPLE 5. Let $q = \infty$ in Example 4. By Theorem 5, $C(\Gamma) = 1/(2\sin\phi)$ and it is an accumulation point in the Lagrange spectrum for Γ. When $p = 3$, Γ is conjugate in $SL(2, \mathbf{C})$ to the modular group and $C(\Gamma) = 3^{-1/2}$.

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