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Fibers of generic projections

Roya Beheshti and David Eisenbud

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Abstract

Let X be a smooth projective variety of dimension n in $\mathbf{P}^r$, and let $\pi : X \to \mathbf{P}^{n+c}$ be a general linear projection, with $c > 0$. In this paper we bound the scheme-theoretic complexity of the fibers of π. In his famous work on stable mappings, Mather extended the classical results by showing that the number of distinct points in the fiber is bounded by $B := n/c + 1$, and that, when n is not too large, the degree of the fiber (taking the scheme structure into account) is also bounded by B. A result of Lazarsfeld shows that this fails dramatically for $n \gg 0$. We describe a new invariant of the scheme-theoretic fiber that agrees with the degree in many cases and is always bounded by B. We deduce, for example, that if we write a fiber as the disjoint union of schemes Y' and Y'' such that Y' is the union of the locally complete intersection components of Y, then $\deg Y' + \deg Y''_{\mathrm{red}} \leqslant B$. Our method also gives a sharp bound on the subvariety of $\mathbf{P}^r$ swept out by the l-secant lines of X for any positive integer l, and we discuss a corresponding bound for highly secant linear spaces of higher dimension. These results extend Ran's 'dimension +2 secant lemma'.

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic zero. We denote by $X \subset \mathbf{P}^r = \mathbf{P}^r_k$ a smooth projective variety, and by $\pi : X \to \mathbf{P}^{n+c}$ a general linear projection with $c > 0$.

We are interested in how large and complex the fibers $\pi^{-1}(q)$ can be for $q \in \mathbf{P}^{n+c}$. Classical computations, greatly extended by the work of Mather, show that when n is small (for example, $c = 1$ and $n \leqslant 14$) the degree of any fiber is bounded by $n/c + 1$. For n large, however, Lazarsfeld has shown that the fibers can have exponentially greater degree (for example, with $c = 1$ and $n = 56$ there will sometimes be fibers of degree at least 70). In fact, when n is large compared with c we know no bound on the degrees of the fibers that depends on n and c alone. Nevertheless, the degree of a fiber is bounded by $n/c + 1$ in so many cases that it is tempting to repair the situation by looking for ways to replace the degree by some other locally defined invariant, one that 'often' agrees with the degree and always takes values $\leqslant n/c + 1$. In this paper we introduce an invariant that provides just such a replacement.

Any fiber of $\pi : X \to \mathbf{P}^{n+c}$ can be expressed as the scheme-theoretic intersection of X with a linear space Λ. Our invariant is defined more generally for the intersection of two schemes X, Y in an ambient scheme P.

We first define a coherent sheaf $Q(X, Y)$, supported on $Z := X \cap Y$, as the cokernel of the restriction map

$$Q(X, Y) := \mathrm{coker}(\mathrm{Hom}(\mathcal{I}_{Z/X}/\mathcal{I}^2_{Z/X}, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_{Y/P}/\mathcal{I}^2_{Y/P}, \mathcal{O}_Z)).$$

When the intersection is 'too small' in the sense that $\operatorname{codim} Y - \dim X > \dim Z$, then we define

$$q(X, Y) := \frac{\deg Q(X, Y)}{\operatorname{codim} Y - \dim X - \dim Z}.$$

The invariant $q(X, Y)$ is often equal to the degree of Z. We will show that this is the case, for example, when $X \cap Y$ is locally a complete intersection or, more generally, is a smooth point of the 'smoothing component' of its Hilbert scheme in Y or X. (Curiously, although much attention has been paid to computing intersection numbers in the case of excess intersection, where $\operatorname{codim} Y - \dim X < \dim Z$, we are not aware of non-trivial invariants other than $\deg Z$ for the case $\operatorname{codim} Y - \dim X > \dim Z$.)

Under good circumstances $Q(X, Y)$ is the module of obstructions to an infinitesimal flat deformation of Y inducing a flat deformation of $X \cap Y$. In general, it measures the 'excess intersection' of X and Y: when these are both locally complete intersections, $Q(X, Y)$ vanishes if and only if the intersection is dimensionally transverse. If X, Y and P are smooth and $Z = X \cap Y$ is finite, then $Q(X, Y)$ is equal to $Q(Y, X)$, and up to a direct sum with a free $\mathcal{O}_Z$ module, these depend only on Z (see Theorem 7.1 and Proposition 7.3). From this one can show that $q(X, Y)$ depends only on Z and the number $\operatorname{codim} Y - \dim X - \dim Z$.

We now return to the case of a generic projection $\pi : X \to \mathbf{P}^{n+c}$. A fiber Z can be written as $X \cap \Lambda$, where

$$\operatorname{codim} \Lambda - \dim X = c > 0 = \dim Z,$$

so $q(X, \Lambda)$ is defined and, as we have noted, it is equal to the degree of Z in many cases. Our main result, the following theorem, says that $q(X, \Lambda)$ behaves as we would hope.

THEOREM 1.1. *If X is a smooth projective variety of dimension n in $\mathbf{P}^r$, and if $\pi : X \to \mathbf{P}^{n+c}$ is a general projection, then every fiber $X \cap \Lambda$, where Λ is a linear subspace containing the projection center in codimension one, satisfies*

$$q(X, \Lambda) \leqslant \frac{n}{c} + 1.$$

The invariant $q(X, \Lambda)$ can be greater than or less than the degree of the scheme $X \cap \Lambda$. Locally at a point of $X \cap \Lambda$ that is not a complete intersection, we can show in the situation above that $q(X, \Lambda)$ is at least $2/c + 1$ (see also Conjecture 1.4).

A corollary of the work of Mather [Mat73] is that a fiber of π as in Theorem 1.1 can never have more than $n/c + 1$ distinct points. Mather's work, and also that of Ran [Ran91], implies that if a fiber is curvilinear, then it has degree at most $n/c + 1$. Both of these statements are improved by the following consequence of Theorems 1.1 and 4.4. The class of 'licci' schemes (schemes in the linkage class of complete intersections), which is explained in § 3, contains, for example, the class of complete intersections, but also all zero-dimensional schemes of embedding dimension at most two.

COROLLARY 1.2. *Suppose that X is a smooth n-dimensional variety in $\mathbf{P}^r$ and that π is a generic projection of X into $\mathbf{P}^{n+c}$, with $c \geqslant 1$. Write $\pi^{-1}(p) = X \cap \Lambda$, where Λ is a linear space containing the projection center in codimension one. If we decompose $\pi^{-1}(p)$ as $Y \cup Y'$ where Y is the union of all of the licci components, then*

$$\deg Y + \left(1 + \frac{3}{c}\right) \deg Y'_{\mathrm{red}} \leqslant q(X, \Lambda) \leqslant n/c + 1.$$

In the case $c = 1$ this may be improved to

$$\deg Y + 5 \deg Y'_{\mathrm{red}} \leqslant q(X, \Lambda) \leqslant n + 1.$$

1.1 Castelnuovo–Mumford regularity

It would be nice to have bounds on the complexity of the fiber in terms of more familiar invariants. One attractive possibility not contradicted by Lazarsfeld's examples is given by the following conjecture.

CONJECTURE 1.3. Let $X \subset \mathbf{P}^r$ be a smooth projective variety of dimension n, and let $\pi : X \to \mathbf{P}^{n+c}$ be a general linear projection. If $Z \subset X \subset \mathbf{P}^r$ is any fiber, then the Castelnuovo–Mumford regularity of Z, as a subscheme of $\mathbf{P}^r$ is $\leqslant n/c + 1$.

If true, this conjecture is sharp: an argument of Lazarsfeld, reproduced in Proposition 6.3 shows that the generic projections to $\mathbf{P}^{n+1}$ of codimension two, arithmetically Cohen–Macaulay varieties of dimension n have fibers with degree $n+1$ as long as the varieties do not lie on hypersurfaces of degree at most n; since such fibers are automatically colinear, their regularity is $n+1$.

The conjecture is also sharp for non-degenerate surfaces in $\mathbf{P}^5$. The Fano embedding of the Reye congruence (Example 6.2) is a non-degenerate smooth Enriques surface in $\mathbf{P}^5$ whose generic projection to $\mathbf{P}^3$ has fibers with three collinear points, and thus regularity three.

Conjecture 1.3 would follow from Theorem 1.1 and the following conjectural comparison between the invariant q and the regularity of the intersection.

CONJECTURE 1.4. Let $X \subset \mathbf{P}^r$ be a smooth projective variety of dimension n, and let $\pi : X \to \mathbf{P}^{n+c}$ be a general linear projection. If $Z \subset X \subset \mathbf{P}^r$ is any fiber, written as $Z = X \cap \Lambda$, where Λ is a linear subspace containing the projection center in codimension one, satisfies

$$\operatorname{reg} Z \leqslant q(X, \Lambda).$$

If Conjecture 1.3 is true, it could be used to give a new bound on the regularity of smooth varieties. The Eisenbud–Goto conjecture [EG84] predicts that for any integral non-degenerate variety X in $\mathbf{P}^r$, the Castelnuovo–Mumford regularity of X is less than or equal to $\deg(X) - \operatorname{codim}(X) + 1$. This conjecture has been verified for curves and smooth surfaces (see [GLP83, Laz87]). The best known bound on the regularity of a smooth variety X of dimension n is given by Bertram–Ein–Lazarsfeld [BEL91, Corollary 2.1], extending a theorem of Mumford:

$$\operatorname{reg}(X) \leqslant \min\{\operatorname{codim} X, 1 + \dim X\}(\deg X - 1) - 1.$$

Kwak [Kwa00] has improved this for smooth varieties of dimension at most six to a bound of the form $\deg X - \operatorname{codim} X + \text{constant}$. Using projection methods of [Kwa00, Laz87], Conjecture 1.3 would imply that for any smooth non-degenerate projective variety of dimension n in $\mathbf{P}^r$,

$$\operatorname{reg}(X) \leqslant \deg(X) - \operatorname{codim}(X) + 1 + c_{n,r}$$

where

$$c_{n,r} = \sum_{i=3}^{n+1} (i-2) \binom{r-n-2+i}{i}.$$

1.2 Secant lines and planes

We can also use $Q(X, \Lambda)$ to give bounds on the dimension of the subvariety of $\mathbf{P}^r$ swept out by l-secant lines of X. An l-secant line of X is a line in $\mathbf{P}^r$ whose intersection with X has degree at least l or is contained in X. With X as above, let S_l be the subvariety of $\mathbf{P}^r$ swept out by the l-secant lines of X. Ran's celebrated 'dimension+2 secant lemma' [Ran91] says that the dimension of S_{n+2} is at most $n+1$. We give a more conceptual proof, which allows us to go

further and get a sharp bound on the dimension of the subvariety swept out by the l-secant lines of X for any l.

THEOREM 1.5. *Let X be a smooth projective variety of dimension n in $\mathbf{P}^r$, and let S_l be the subvariety of $\mathbf{P}^r$ swept out by all of the l-secant lines of X. If $l \geqslant 2$, then*

$$\dim S_l \leqslant \frac{nl}{l-1} + 1.$$

An argument of Lazarsfeld (Proposition 6.3) shows that this bound is achieved in the case $l = n + 1$ for arithmetically Cohen–Macaulay varieties of codimension two. In Theorem 5.1 we give a corresponding bound for secant planes as well.

Remark 1.6 (Length and degree). We generally use the word *degree* when speaking of schemes of dimension zero, and *length* when speaking of modules of dimension zero. Since we are working over an algebraically closed field, the degree $\deg Z$ of a scheme of dimension zero is the same as the *length* of the module $\mathcal{O}_Z$.

2. Mather's upper bounds and Lazarsfeld's examples

Mather's trasversality theorem holds for multi-jet spaces and a class of their subvarieties that are called modular. Mather showed that if X is an n-dimensional smooth projective variety, then the multi-jet of a general linear projection $f : X \to \mathbf{P}^{n+c}$ is transverse to all of the modular subvarieties of the multi-jet spaces $_sJ^k(X, \mathbf{P}^{n+c})$, and that all of the Thom–Boardman strata in multi-jet spaces are modular (see the last sentence of [Mat73]). He used these results in [Mat71] to conclude that in the nice range of dimensions ($n < \frac{6}{7}(n + c) + \frac{8}{7}$, or $n < \frac{6}{7}(n + c) + \frac{9}{7}$ and $c \leqslant 3$), a general projection is C^∞ stable, and this enabled him to describe local normal forms for general projections in the nice range of dimensions. In particular he proved that when $c = 1$ and $n \leqslant 14$, the degrees of fibers of a general projection is bounded by $n + 1$, just as in the classical cases.

The following result is a consequence of Mather's transversality theorems [Mat73] and the formula proved by Boardman [Boa67] for the codimensions of the Thom–Boardman strata. We say that a map $\pi : X \to Y$ between smooth spaces has *tangential corank* d at a point $p \in X$ if the induced map on tangent spaces has rank $= \dim X - d$ at p.

THEOREM 2.1 (Mather). *Let $X \subset \mathbf{P}^r$ be a smooth projective variety of dimension n and let $\pi : X \to \mathbf{P}^{n+c}$ be a general linear projection, with $c > 0$. If the fiber $\pi^{-1}(q)$ contains points $p_1, \ldots, p_d$ and π has corank d_i at p_i, then*

$$\sum_{i=1}^{d} \left(\frac{d_i^2}{c} + d_i + 1 \right) \leqslant \frac{n}{c} + 1.$$

For example, the number of distinct points in the fiber of a generic projection is at most $n/c + 1$. Mather's theory also implies that any curvilinear fiber (that is, one that can be embedded in a smooth curve) has degree at most $n/c + 1$. (Roberts [Rob71] proves these results by a more elementary method, but only for embeddings of varieties that have been composed with sufficiently high Veronese maps.) However, it is easy to see that knowing the Thom–Boardman symbol of the projection at every point is not enough to bound the degree of the fiber except in the curvilinear case.

In fact, Lazarsfeld shows in [Laz04, Volume 2, Proposition 7.2.17] that for if $X \subset \mathbf{P}^r$ is a sufficiently ample embedding of a smooth variety of dimension n, then a general projection of X

to $\mathbf{P}^{n+1}$ will have points of corank d whenever $d(d+1) \leqslant n$. He used this to show that the image of the projection would have points of multiplicity at least of the order of $2^{\lfloor \sqrt{n} \rfloor}$. The corank d condition also gives a lower bound on the lengths of the fibers.

PROPOSITION 2.2. *If X is a smooth projective variety of dimension n and $p \in X$ is a point of corank d for the projection map $\pi : X \to \mathbf{P}^{n+1}$, then the degree of the fiber $\pi^{-1}\pi(p)$ is at least $\binom{d+1}{\lceil d/2 \rceil}$.*

Proof. Let $\hat{\mathcal{O}}_{X,p}$ be the completion of the local ring of X at p, and let $\mathfrak{m}$ be its maximal ideal. Let $f_1, \ldots, f_{n+1} \in \hat{\mathcal{O}}_{X,p}$ be the functions locally defining π at p. Since p is a point of corank d, we can assume that $n - d$ of the f_i, say $f_{d+2}, \ldots, f_{n+1}$, form a regular system of parameters, so that $R := \hat{\mathcal{O}}_{X,p}/(f_{d+2}, \ldots, f_{n+1})$ is a power series ring in d variables, while $f_1, \ldots, f_{d+1} \in \mathfrak{m}^2$. Set $I = (f_1, \ldots, f_{d+1})$.

Setting $S = R[[y]]$, we can write $R/I = S/J$ where $J = (y) + IS$. Denote by $\mathfrak{p}$ the maximal ideal of S, and let $f_i' = f_i + q_i$, where the q_i are general quadratic forms of S. Let $I' = (f_1', \ldots, f_{d+1}')S$, and set $J' = (y) + I'$. Note that the leading terms of the f_i' form a regular sequence.

By the semicontinuity of fiber dimension, the length of $S/J = R/I$ is at least the length of S/J', so it suffices to show that the latter is at least $\binom{d+1}{\lceil d/2 \rceil}$.

Since the leading terms of the f_i' form a regular sequence, the f_i are a standard basis in the sense of Grauert and Hironaka. It follows that the Hilbert function of S/I' is the same as that of a complete intersection of quadrics,

$$\text{length}((I' + \mathfrak{p}^m)/(I' + \mathfrak{p}^{m+1})) = \binom{d+1}{m}.$$

Further, the length of $S/J' = S/((y) + I')$ is at least the length of the cokernel of the map given by multiplication

$$y : S/(I', \mathfrak{p}^m) \to S/(I', \mathfrak{p}^{m+1})$$

for any m. Thus,

$$\text{length } S/J' \geqslant \text{length } S/(I', \mathfrak{p}^{m+1}) - \text{length } S/(I', \mathfrak{p}^m) = \text{length}((I', \mathfrak{p}^m)/(I', \mathfrak{p}^{m+1})).$$

If we let $m = \lceil d/2 \rceil$, we obtain length $S/J \geqslant \binom{d+1}{\lceil d/2 \rceil}$ as required. $\square$

Applying the Stirling approximation for factorials we deduce an asymptotic formula.

COROLLARY 2.3. *If X is a smooth projective variety of dimension n embedded by a sufficiently positive line bundle, then a general projection X into $\mathbf{P}^{n+1}$ will have some fibers whose degrees are of the order of $\sqrt{(2/\pi)}\,(2^{\sqrt{n}}/n^{1/4})$.* $\square$

3. Proof of the main theorem

Proof of Theorem 1.1. Let $\Sigma \subset \mathbf{P}^r$ be a general linear subvariety of codimension $n + c + 1$ in $\mathbf{P}^r$, and let $\pi_\Sigma : X \to \mathbf{P}^{n+c}$ be projection from Σ, so that in particular π_Σ is generically injective and is a finite map. Thus, if Λ is any codimension $n + c$ linear subvariety of $\mathbf{P}^r$ that contains Σ, the intersection of Λ and X is a scheme of dimension zero. Fix a (general) Σ, and a $\Lambda \supset \Sigma$ that makes the degree of $X \cap \Lambda$ maximal.

Consider the natural surjections of sheaves

$$N_{\Lambda/\mathbf{P}^r} = \text{Hom}(\mathcal{I}_\Lambda, \mathcal{O}_\Lambda) \twoheadrightarrow \text{Hom}(\mathcal{I}_\Lambda, \mathcal{O}_Z) \twoheadrightarrow Q(X, \Lambda),$$

and let F be the kernel of the composite map $\gamma : N_{\Lambda/\mathbf{P}^r} \to Q(X, \Lambda)$. Identifying the local sections of $N_{\Lambda/\mathbf{P}^r}$ with the embedded deformations of Λ in $\mathbf{P}^r$ (the tangent space to the Grassmannian), we see from the definition of $Q(X, \Lambda)$ that the local sections of F are the deformations of Λ that induce flat deformations of Z. Since $Q(X, \Lambda)$ is supported on $Z \subset X$, which is disjoint from Σ, we have $F|_\Sigma = N_\Lambda|_\Sigma$. (See Theorem 7.1 for another description of F.)
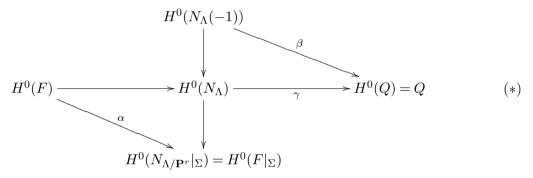
As Σ is a hyperplane section of Λ there is an exact sequence

$$0 \to N_{\Lambda/\mathbf{P}^r}(-1) \to N_{\Lambda/\mathbf{P}^r} \to N_{\Lambda/\mathbf{P}^r}|_\Sigma \to 0.$$

Putting this together with the exact sequence

$$0 \to F \to N_{\Lambda/\mathbf{P}^r} \to Q(X, \Lambda) \to 0,$$

coming from the definition of F, and taking global sections, we obtain a commutative diagram with exact row and column.

$$
\begin{array}{ccccc}
& & H^0(N_\Lambda(-1)) & & \\
& & \downarrow & \searrow^{\beta} & \\
H^0(F) & \xrightarrow{} & H^0(N_\Lambda) & \xrightarrow{\gamma} & H^0(Q) = Q \\
& \searrow_{\alpha} & \downarrow & & \\
& & H^0(N_{\Lambda/\mathbf{P}^r}|_\Sigma) = H^0(F|_\Sigma) & &
\end{array}
\qquad (*)
$$

Here α, β and γ are defined by applying H^0 to the evident maps, and $Q = Q(X, \Lambda)$.

There is an exact sequence of normal bundles

$$0 \longrightarrow N_{\Sigma/\Lambda} \longrightarrow N_{\Sigma/\mathbf{P}^r} \xrightarrow{\psi} N_{\Lambda/\mathbf{P}^r}|_\Sigma \longrightarrow 0.$$

Since $N_{\Sigma/\Lambda} = \mathcal{O}_\Sigma(1)$, every global section of $N_{\Lambda/\mathbf{P}^r}|_\Sigma$ lifts to a global section of $N_{\Sigma/\mathbf{P}^r}$, that is, a deformation of Σ. The length of $X \cap \Lambda$ is semicontinuous as we move Λ, and we have supposed that it is maximal among those lengths attained by a Λ containing a general plane Σ; so if we move Σ in a flat family $\Sigma_t \subset \mathbf{P}^r$, then (for small, or first-order t) there is a flat deformation $\Lambda_t \subset \mathbf{P}^r$ of Λ such that $\Sigma_t \subset \Lambda_t$ and the degree of $Z = X \cap \Lambda_t$ is constant; that is, the family $Z_t := X \cap \Lambda_t$ is flat. Thus, any first-order deformation of Σ can be lifted to a first-order deformation of Λ fixing the length of $X \cap \Lambda$.

Using the identification of sections of F with deformations of Λ fixing the length of Z, we see that

$$H^0(F) \xrightarrow{\alpha} H^0(F|_\Sigma) = H^0(N_{\Lambda/\mathbf{P}^r}|_\Sigma)$$

contains the image of $H^0(N_{\Sigma/\mathbf{P}^r})$ under ψ. Since $H^1(N_{\Sigma/\Lambda}) = H^1(\mathcal{O}_\Sigma(-1)) = 0$, ψ is surjective on global sections, and thus α is surjective.

A diagram chase using the surjectivity of α and the exactness of the row and column in diagram $(*)$ shows that the image of β is equal to the image of γ. Moreover, $N_\Lambda(-1) \cong \mathcal{O}_\Lambda^{n+c}$ is generated by global sections. Thus, we can apply the following result with $A = N_\Lambda(-1)$ and $B = Q(X, \Lambda)$.

PROPOSITION 3.1. *Suppose that $\delta : A \to B$ is an epimorphism of coherent sheaves on $\mathbf{P}^r$, and suppose that A is generated by global sections. If $\delta(H^0(A)) \subset H^0(B)$ has the same dimension as $\delta(H^0(A(1))) \subset H^0(B(1))$, then $\dim B = 0$ and $\delta(H^0(A(m))) = H^0(B(m)) \cong H^0(B)$ for all $m \geqslant 0$.*

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Proof. We may harmlessly assume that the ground field is infinite, so we may choose a linear form x on $\mathbf{P}^r$ that does not vanish on any associated subvariety of B. It follows that multiplication by x is a monomorphism on global sections, so, under our hypothesis,

$$x \cdot \delta(H^0(A)) = H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \cdot \delta(H^0(A)) = \delta(H^0(A(1))).$$

It follows that $x^m \cdot \delta(H^0(A)) = H^0(\mathcal{O}_{\mathbf{P}^r}(m)) \cdot \delta(H^0(A))$ for all $m \geqslant 0$. Since δ is a surjection of sheaves and A is globally generated, this space is equal to $H^0(B(m))$ for large m. It follows that the Hilbert polynomial of B is constant, so B is zero-dimensional, and $H^0(B(m)) = \delta(H^0(A(m)))$ for all $m \geqslant 0$ as required. $\qquad\square$

Returning to the proof of Theorem 1.1 we apply Proposition 3.1 and deduce that β is surjective. Thus,

$$\text{length } Q(X, \Lambda) \leqslant \dim_k H^0(N_\Lambda(-1)) = n + c,$$

as required. $\qquad\square$

4. Bounds on the invariant q

We begin by explaining why the 'expected' value of $q(X, Y)$ is $\deg(X \cap Y)$. We write Hilb_X for the Hilbert scheme of X, and $T_{\text{Hilb}_{X,[Z]}}$ for its tangent space at the point corresponding to the scheme Z. If $[Z]$ is a smooth point of the Hilbert scheme, and if Z deforms in X to a reduced set of smooth points, then $\dim T_{\text{Hilb}_{X,[Z]}} = (\deg Z)(\dim X)$, which we may thus consider to be the 'expected value'. We denote by $T^1(Z)$ the Zariski tangent space to the deformation space of Z. We denote by $\text{Der } \mathcal{O}_Z$ the module of k-linear derivations from $\mathcal{O}_Z$ to itself.

THEOREM 4.1. *Suppose that $X, Y \subset P$ are k-schemes of finite type with X smooth and Y locally a complete intersection in P. If $Z := X \cap Y$ is finite, then*

$$\dim_k Q(X, Y) = (\deg Z)(\text{codim } Y - \dim X) - (\dim_k T_{\text{Hilb}_{X,[Z]}} - (\deg Z)(\dim X))$$

$$= (\deg Z)(\text{codim } Y - \dim X) - \dim_k T^1(Z) + \dim_k \text{Der } \mathcal{O}_Z.$$

Proof. To compute the dimension of $Q(X, Y)$, we note first that since $I_Y := I_{Y/P}$ is locally a complete intersection the module I_Y/I_Y^2 is locally free over $\mathcal{O}_Y$ of rank equal to the codimension of Y. It follows that $\text{Hom}(I_Y/I_Y^2, \mathcal{O}_Z) \cong (\mathcal{O}_Z)^{\text{codim } Y}$. Using this and the definition of Q we obtain

$$\dim_k Q(X, Y) = (\deg Z)(\text{codim } Y) - \dim_k \text{Hom}((I_X + I_Y)/(I_X + I_Y^2), \mathcal{O}_Z).$$

On the other hand, we may identify $\text{Hom}((I_X + I_Y)/(I_X + I_Y^2), \mathcal{O}_Z)$ with the tangent space to the functor of embedded deformations of Z in X, that is, with the tangent space to the Hilbert scheme Hilb_X at $[Z]$. There is an exact sequence

$$0 \to \text{Hom}(\Omega_{Z/k}, \mathcal{O}_Z) \to \text{Hom}(\Omega_{X/k}|_Z, \mathcal{O}_Z)$$
$$\to \text{Hom}((I_X + I_Y)/(I_X + I_Y^2), \mathcal{O}_Z) \to T^1(Z) \to 0$$

(see Eisenbud [Eis95b, Exercise 16.8]). Since X is smooth we have

$$\dim_k \text{Hom}(\Omega_{X/k}|_Z, \mathcal{O}_Z) = (\deg Z)^{\dim X}$$

and the desired formula follows. $\qquad\square$

COROLLARY 4.2. *Let $X, Y \subset P$ be schemes of finite type with X smooth and Y locally a complete intersection in P. Suppose that $Z = X \cap Y$ is finite, and consider the corresponding point $[Z]$ in the Hilbert scheme Hilb_X. If $[Z]$ is a smooth point and lies in the closure of the*

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locus of reduced subschemes, then

$$q(X, Y) = \frac{\dim_k Q(X, Y)}{\operatorname{codim} Y - \dim X} = \deg \mathcal{O}_Z.$$

In particular, this is the case when Z is a complete intersection.

Proof. The dimension of the closure of the locus of sets of $\deg Z$ reduced points has dimension $(\deg Z)(\dim X)$. □

The case of complete intersections extends to that of schemes in the 'linkage class of a complete intersection' described below.

Remark 4.3. The tangent space to Z in the Hilbert scheme of X is often larger than $\deg \mathcal{O}_Z^{\dim X}$, for example when $[Z]$ is at a point where the 'smoothing component' meets another component; a result of Iarrobino [Iar72] shows that there are always such points when $\dim X > 2$. However, it can also be smaller. The first such example was discovered by Iarrobino and Emsalem [EI78] (see [CEVV08] for an exposition): the Hilbert scheme of finite subschemes of degree eight in $\mathbf{C}^4$ contains a reduced component whose generic point is the scheme defined by seven general quadrics. This component is isomorphic to the product of $\mathbf{C}^4$ and the Grassmanian $Gr(7, 10)$, which has dimension 25, whereas the locus of reduced 8-tuples of points has dimension 32. Such a point can appear as the intersection of a 4-plane X with a scheme Y of codimension seven. In this case we have $\dim_k Q(X, Y) = 31 > \deg Z(\operatorname{codim} Y - \dim X) = 24$.

4.1 Lower bounds on the invariant q

Since the computation of Q is local, it suffices to treat the local case. If $\mathcal{O}$ is a regular local ring, then ideals $J, J' \subset \mathcal{O}$ (or the subschemes they define) are said to be linked by a complete intersection K if $J' = (K : J)$ and $J = (K : J')$, where $(K : J)$ denotes the ideal $\{f \in \mathcal{O} \mid fJ \subset K\}$.

The ideal J is *in the linkage class of a complete intersection*, written *licci*, if it can be linked to a complete intersection in finitely many steps. See Peskine–Szpiro [PS74] for general information about this notion.

We write $\mu(Q)$ for the minimal number of generators of an $\mathcal{O}$-module Q, and it is obvious that length $Q \geqslant \mu(Q)$. Example 4.7 suggests that the length of $Q(X, Y)$ can be equal to $\mu(Q(X, Y))$ even in large cases.

THEOREM 4.4. *Let $\mathcal{O}$ be an equicharacteristic zero regular local ring of dimension r. Suppose that $X \subset \operatorname{Spec} \mathcal{O}$ is smooth and $Y \subset \operatorname{Spec} \mathcal{O}$ is a complete intersection. Set $c = \operatorname{codim} Y - \dim X$ and $Z = X \cap Y$. Suppose that $c \geqslant 1$, and that $\dim Z = 0$.*

(1) *If I_Z is licci then*

$$q(X, Y) = \deg(X \cap Y).$$

(2) *If I_Z is not licci, then*

$$q(X, Y) \geqslant \frac{1}{c}\mu(Q(X, Y)) \geqslant \max\left(1 + \frac{3}{c}, \frac{5}{c}\right).$$

In particular, $q(X, Y) \geqslant 1$.

Here are some facts about the licci property that will be important to us.

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PROPOSITION 4.5. *Suppose that A is a local Gorenstein ring with maximal ideal $\mathfrak{m}$, and $L \subset A$ is an ideal of finite projective dimension such that A/L is Cohen–Macaulay.*

(1) *If $f_1, \dots, f_m \in L$ is a regular sequence, and $L/(f_1, \dots, f_m)$ is licci, then L is licci.*

(2) *If codim $L \leqslant 2$, then L is licci.*

(3) *If $\mu(L) \leqslant 4$, then L is licci.*

(4) *Suppose that A is regular, and A/L is of finite length. Let $T := \mathfrak{m}/(\mathfrak{m}^2 + L)$ be the Zariski tangent space of A/L. If $\dim T \leqslant 2$, or if L is generated by $\operatorname{codim} L + 1$ elements and $\dim T \leqslant 3$, then L is licci.*

Proof sketch. The first assertion is immediate from the definition. The second was proved by Apéry and Gaeta [Gae52], and is reproved in modern language in Peskine–Szpiro [PS74].

When L is generated by at most two elements, then it has codimension at most two and is thus covered by the theorem of Apéry and Gaeta. If $\mu(L) = \operatorname{codim} L$, then L is itself a complete intersection. The only remaining case with $\mu(L) \leqslant 4$ is the case of a 4-generator ideal of codimension three. It follows from the paper of Peskine–Szpiro that L is then linked to an ideal L' of finite projective dimension that has a symmetric resolution, and the main theorem of Buchsbaum–Eisenbud [BE77] shows that L' is generated by the $2n \times 2n$ pfaffians of a $2n + 1 \times 2n + 1$ matrix. Ideas similar to those of Gaeta show that such ideals are licci; see also Watanabe [Wat73] that implicitly contains a different (and prior) proof of the slightly restricted case where A is regular. This proves part (3).

Part (4) follows from the previous parts: if $\dim T \leqslant 2$, then there is a regular sequence $g_1, \dots, g_s$ in L, with $s = \dim A - \dim T$, such that $A/(g_2, \dots, g_s)$ is again regular. If $\dim T = 2$ we may apply parts (1) and (2) to conclude that L is licci. If $\dim T = 3$, then

$$\mu(L/(g_1, \dots, g_s)) = \mu(L) - s = \operatorname{codim} L + 1 - s = \dim A + 1 - (\dim A - 3) = 4$$

so we may apply parts (1) and (3) to conclude that L is licci. $\square$

Proof of Theorem 4.4. First suppose that I_Z is licci. We may harmlessly complete the ring $\mathcal{O}$ and thus can apply the result of Buchweitz [Buc, Theorem 6.4.4, p. 235], which shows that a licci scheme represents a smooth point on its Hilbert scheme (this result is proven in the analytic category). Ulrich [Ulr86, Theorem 2.1] implies that a licci scheme of dimension at most three is smoothable. Part (1) thus follows from Theorem 4.1.

To prove part (2), set $\mu = \mu(Q(X, Y))$ and $n = \dim Y$. Let $J = (I_X + I_Y)/I_X$ be the image of I_Z in $\mathcal{O}_X = \mathcal{O}/I_X$, and consider the defining exact sequence

$$0 \to \operatorname{Hom}(J/J^2, \mathcal{O}_Z) \to \operatorname{Hom}(I_Y/I_Y^2, \mathcal{O}_Z) \to Q(X, Y) \to 0.$$

Since $\operatorname{Hom}(I_Y/I_Y^2, \mathcal{O}_Z)$ is a free $\mathcal{O}_Z$-module of rank $r - n$, $\operatorname{Hom}(J/J^2, \mathcal{O}_Z)$ must have an $\mathcal{O}_Z$-free summand of rank $m := r - n - \mu$. Since $\mathcal{O}_Z$ is artinian, this implies that J/J^2 has also an $\mathcal{O}_Z$-free summand of rank m.

We may write this free summand in the form J/K for some ideal K of $\mathcal{O}_X$ such that $J \supset K \supset J^2$. Since $\mathcal{O}_X$ is regular, J has finite projective dimension. The proof of Theorem 1.1 of Vasconcelos [Vas67] shows that there is a regular sequence $f_1, \dots, f_m$ in J such that

$$J = (f_1, \dots, f_m) + K$$

and such that $J/(f_1, \dots, f_m)$ is an ideal of finite projective dimension in the ring $\mathcal{O}_X/(f_1, \dots, f_m)$. By Proposition 4.5(1) it suffices to show that when $\mu \leqslant c + 2$ or $\mu \leqslant 4$ the ideal $J/(f_1, \dots, f_m)$ is licci.

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If $\mu \leqslant c + 2$, then

$$\dim \mathcal{O}_X/(f_1, \ldots, f_m) = r - n - c - m = (r - n - c) - (r - n - \mu) = \mu - c \leqslant 2$$

so Proposition 4.5(2) shows that J is licci. On the other hand, if $\mu \leqslant 4$, then

$$\mu(J/(f_1, \ldots, f_m)) = \mu(J) - m \leqslant \mu(I_Y) - m = r - n - m = (r - n) - (r - n - \mu) \leqslant 4$$

so Proposition 4.5(2) shows that J is licci. □

If $\pi : X \to P$ is a map of smooth varieties, and $x \in X$ is a point, then the tangential corank of π at x is the dimension of the Zariski tangent space of the fiber to π through x. Theorem 4.4 allows us to analyze the invariant q at points of small tangential corank.

COROLLARY 4.6. *Let $x \in X \subset \mathbf{P}^r$ be a point on a smooth projective variety of dimension n, and let $\pi : X \to \mathbf{P}^{n+c}$ be a linear projection from a center that does not meet X, so that the fiber Z of π through x is the intersection of X with a linear subspace Λ of dimension $r - n - c$. If the tangential corank of π at x is at most two, or $c = 1$ and the tangential corank of π at x is at most three, then $q(X, \Lambda) = \deg(X \cap \Lambda)$.*

Proof. From Proposition 4.5(4) we see that the fiber is licci at x, and the assertion then follows from Theorem 4.4. □

Example 4.7. Let $A = k[a_1, \ldots, a_n, x_1, \ldots, x_{n+1}]$ and let

$$I = (x_1 - f_1(a), \ldots, x_{n+1} - f_{n+1}(a)); \quad X = V(I) \subset \operatorname{Spec} A$$

where the $f_i(a)$ are generic quadrics in the variables a_i. Let $\Lambda = V(x_1, \ldots, x_{n+1})$. The variety X is smooth (it is the graph of the map $f := (f_i)$) and meets the plane Λ, its tangent plane, in a scheme Z supported at the origin.

For $2 \leqslant n \leqslant 8$ and random examples over a large finite field, the values of $\deg(Z)$, $q(\Lambda, X)$, which is equal to $q(X, \Lambda)$ by Theorem 7.1, and the minimal number of generators of the module $Q(\Lambda, X)$, written $\mu(Q(\Lambda, X))$, are given in the following table, computed with Macaulay2 [GS]. Note that for even n we have $\mu(Q(\Lambda, X)) = q(\Lambda, X) = \operatorname{codim} I$. It follows immediately that in these cases

$$Q(\Lambda, X) = I/(a_1, \ldots, a_n, x_1, \ldots, x_{n+1})I,$$

a vector space concentrated in degree two.

n	$\deg Z$	$q(\Lambda, X)$	$\mu(Q(\Lambda, X))$
2	3	3	3
3	6	6	3
4	10	5	5
5	20	20	6
6	35	7	7
7	70	57	8
8	126	9	9

In the notation of Theorem 4.4 we have $c = 1$. In these examples we have $\deg(\Lambda \cap X) = q(\Lambda, X)$ if and only if $n = 2, 3$, and these are exactly the cases where $\Lambda \cap X$ is licci.

The case $n = 4$ shows that the bound $q(\Lambda, X) \geqslant 5$ in part (2) of Theorem 4.4 can be sharp. For an example where the other option is sharp consider the smallest non-licci scheme, which is $Z = \operatorname{Spec} k[x, y, z]/(x, y, z)^2$. This scheme can be written as the intersection of a smooth

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complete intersection X of six general quadrics in some $\mathbf{P}^r$ ($r \geqslant 7$) with a 3-plane Λ, and thus $c = 3$. It is easy to compute the dimension of the tangent space to the Hilbert scheme of Λ at $[Z]$ has dimension 18, so by Theorem 4.1, or direct computation,

$$q(\Lambda, X) = \mu(Q(\Lambda, X)) = 6 = c + 3.$$

5. Secant lines

Let $X \subset \mathbf{P}^r$ be a smooth subvariety of dimension n, and denote by S_l, $l \geqslant 2$, the subvariety of $\mathbf{P}^r$ swept out by all the l-secant lines of X. Let $c = \lfloor n/(l-1) \rfloor + 1$, and let Σ be a linear subspace of codimension $n + c + 1$ in $\mathbf{P}^r$. If S_l intersects Σ at a point q, then there is an l-secant line Λ_1 of X which passes through q, and this line together with Σ span a linear subspace Λ of codimension $n + c$ in $\mathbf{P}^r$ that intersects X in a scheme of degree at least $l > (n + c)/c$. If we knew that $q(X, \Lambda)$ was bounded below by $\deg(X \cap \Lambda_1)$, then Theorem 1.1 would show that a general such Σ does not intersect S_l, and so $\dim S_l \leqslant n + c$. Although we do not know how to prove such a comparison theorem for $q(X, \Lambda)$, Theorem 1.5 shows that this upper bound on the dimension of S_l is satisfied. The examples in § 6 show that it is sharp.

Proof of Theorem 1.5. Let G_l be the subvariety of the Grassmannian of lines in $\mathbf{P}^r$ parametrizing the lines whose intersection with X has degree at least l, and let G^0 be an irreducible component of G_l. We must show that the lines parametrized by G^0 sweep out a subvariety of dimension at most $n + c$ in $\mathbf{P}^r$.

Let $[\Lambda]$ be a general point in G^0. We may assume that $\Lambda \not\subset X$, since otherwise this conclusion is obvious. Set $Z = X \cap \Lambda$, and let $F = \ker(N_{\Lambda/\mathbf{P}^r} \to Q(X, \Lambda))$. Since F is a torsion-free sheaf of rank $r - 1$ on $\mathbf{P}^1$, it splits as

$$F = \mathcal{O}_\Lambda(a_1) \oplus \cdots \oplus \mathcal{O}_\Lambda(a_{r-1}), a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{r-1}.$$

We see from the definition that the Euler characteristic of F is

$$r - 1 + \sum_i a_i = 2(r - 1) - \text{length}(Q(X, \Lambda)).$$

Since any subscheme of $\mathbf{P}^1$ is a local complete intersection, part (1) of Theorem 4.4 yields

$$\sum_i a_i = r - 1 - \text{length}(Q(X, \Lambda)) = r - 1 - (r - n - 1)l. \tag{1}$$

We next show that $a_i \geqslant -l + 1$ for every i by showing that $H^1(F(l - 2)) = 0$.

Consider the short exact sequence

$$0 \to F(l - 2) \to N_{\Lambda/\mathbf{P}^r}(l - 2) \to Q(X, \Lambda) \to 0.$$

Since $H^1(N_{\Lambda/\mathbf{P}^r}(l - 2)) = 0$, to prove $H^1(F(l - 2)) = 0$ we need to show that the map

$$H^0(N_{\Lambda/\mathbf{P}^r}(l - 2)) \to H^0(Q)$$

is surjective. This map factors through

$$H^0(N_{\Lambda/\mathbf{P}^r}(l - 2)) \to H^0(N_{\Lambda/\mathbf{P}^r}(l - 2)|_Z) \to H^0(Q).$$

The first map is surjective since the Castelnuovo–Mumford regularity of a finite scheme is bounded by the degree of the scheme, and the second map is surjective since $N_{\Lambda/\mathbf{P}^r}|_Z \to Q(X, \Lambda)$ is surjective and Z is zero-dimensional. This shows that $a_i \geqslant -l + 1$ for every i.

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Now let k be the largest index such that $a_k \geqslant 0$, then

$$\sum_i a_i \geqslant (r - 1 - k)(-l + 1). \tag{2}$$

Combining (1) and (2), we obtain

$$r - 1 - (r - n - 1)l \geqslant (r - 1 - k)(-l + 1),$$

and so $k \leqslant nl/(l - 1)$.

Let $\mathcal{I} \subset \mathbf{P}^r \times G^0$ be the incidence correspondence, and denote by p_1 and p_2 the two projections from $\mathcal{I}$ to $\mathbf{P}^r$ and G^0. We obtain a commutative diagram.

$$
\begin{array}{ccc}
T_{\mathcal{I},([\Lambda],p)} & \xrightarrow{\ dp_2\ } & T_{G^0,[\Lambda]} = H^0(F) \\
\downarrow{\scriptstyle dp_1} & & \downarrow \\
T_{\mathbf{P}^r,p} & \xrightarrow{\hspace{3cm}} & N_{\Lambda/\mathbf{P}^r}|_p = F|_p
\end{array}
$$

If $([\Lambda], p)$ is a general point of $\mathcal{I}$, then since $k \leqslant nl/(l - 1)$, the image of the restriction map $H^0(F) \to F|_p$ has dimension at most $nl/(l - 1)$. Therefore, the dimension of the image of dp_1, which is equal to the dimension of the subvariety swept out by all of the lines parametrized by G^0, is at most $(nl/(l - 1)) + 1$. $\qquad\square$

A similar argument proves an analogous result for the two-dimensional linear subvarieties intersecting X in a scheme of dimension zero.

THEOREM 5.1. *If $S_{l,t}$ is the closure of the subvariety of $\mathbf{P}^r$ swept out by the 2-planes Λ such that the intersection of Λ and X is a scheme of degree at least l and regularity at most t, then*

$$\dim S_{l,t} \leqslant \frac{\binom{t+1}{2}(r - 2) - l(r - 2 - n)}{\binom{t}{2}} + 2.$$

Proof. Let G^0 be an irreducible component of the space of 2-planes in $\mathbf{P}^r$ which intersects X in a scheme of degree at least l and regularity at most t, and let $[\Lambda]$ be a general point in G^0. Consider the exact sequence

$$0 \to F \to N_{\Lambda/\mathbf{P}^r} \to Q(X, \Lambda) \to 0.$$

Since $Z = \Lambda \cap X$ is t-regular, an argument parallel to that given in the proof of Theorem 1.5 shows that the map

$$H^0(N_{\Lambda/\mathbf{P}^r}(t - 2)) \to H^0(Q(X, \Lambda))$$

is surjective. Thus,

$$h^0(F(t - 2)) = \binom{t + 1}{2}(r - 2) - l(r - 2 - n).$$

On the other hand, if the planes parametrized by G^0 cover a subvariety of dimension at least k, then for a general point $p \in \Lambda$, the image of the restriction map $H^0(F) \to F|_p$ is at least $(k - 2)$-dimensional, so

$$h^0(F(t - 2)) \geqslant H^0(\mathcal{O}(t - 2)^{k-2}) = \binom{t}{2}(k - 2),$$

and we obtain the desired bound. $\qquad\square$

6. Examples

Here we give examples showing that the bounds in Theorem 1.5 are sharp, and that Conjecture 1.3 has the best possible bound on the regularity of the fibers.

Example 6.1. Let $r = \lfloor (nl/(l-1)) + 1 \rfloor$, and let X be the complete intersection of $r - n$ general hypersurfaces $Y_1, \ldots, Y_{r-n}$ of degree l in $\mathbf{P}^r$. We show that $\mathbf{P}^r$ is swept out by the l-secant lines of X and so

$$\dim S_l = r = \left\lfloor \frac{nl}{l-1} + 1 \right\rfloor.$$

Consider the intersection of any $r - n - 1$ of the Y_i, say $Y = Y_1 \cap \cdots \cap Y_{r-n-1}$. Under these circumstances, it is known that Y is covered by lines; we give a proof for the reader's convenience.

Let p be a point in Y, let V be the cone of lines in Y through p, and let V_i be the cone of lines in Y_i through p. We can compute the equations of V_i as follows. We may assume that $p = (1 : 0 : \cdots : 0)$ and write the equation of Y_i in the form

$$x_0^{l-1} F_1^i + \cdots + x_0 F_{l-1}^i + F_l^i,$$

where F_d^i is a form of degree d in $x_1, \ldots, x_n$. Consider the line W through p and another point, which we may take to be $p' = (0 : 1 : 0 : \cdots : 0)$. Substituting the parametrization $(1 : t : 0 : \cdots : 0)$ of W into the equation of Y_i, we see that $\{F_1^i(p') = \cdots = F_l^i(p') = 0\}$ if and only if W lies in Y_i. Thus, $\operatorname{codim} V_i \leqslant l$. Since

$$\operatorname{codim} V \leqslant \sum_{1 \leqslant i \leqslant r-n-1} \operatorname{codim} V_i = (r - n - 1)l \leqslant r - 1,$$

the cone V is at least one-dimensional; that is, there is at least one line through p contained in Y.

Any line in Y intersects X in l points, so any point of Y is contained in S_l. Since any point of $\mathbf{P}^r$ is contained in the intersection of $r - n - 1$ independent hypersurfaces in the linear system spanned by $Y_1, \ldots, Y_{r-n}$, there is an l-secant line of X passing through every point of $\mathbf{P}^r$.

In particular, in the case $l = n + 1$, $r = n + 2$, this argument shows that if $X^n \subset \mathbf{P}^{n+2}$ is a complete intersection of two surfaces of degree $n + 1$, then any projection of X to $\mathbf{P}^{n+1}$ has fibers of length $n + 1$. This consequence is greatly generalized by the argument of Lazarsfeld given in Proposition 6.3 below.

6.1 A non-degenerate surface in $\mathbf{P}^5$ with many trisecants

Example 6.2. The Fano model of the classical Reye congruence is a non-degenerate Enriques surface in $\mathbf{P}^5$ whose 3-secant lines sweep out a four-dimensional subvariety in $\mathbf{P}^5$ (see Conte and Verra [CV93, Propositions 3.10 and 3.14], and Cossec [Cos83, § 3.3]). It can be described as follows. Let $A = [f_{ij}]_{1 \leqslant i,j \leqslant 4}$ be a symmetric 4×4 matrix whose entries are general linear forms in $\mathbf{P}^5$, and let S be the subvariety of $\mathbf{P}^5$ defined by all of the 3×3 minors of A. Then S is a smooth non-degenerate surface in $\mathbf{P}^5$. We show that the 3-secant lines of X sweep out the degree-four hypersurface in $\mathbf{P}^5$ defined by the determinant of A. Since A is general, for a general vector $V = [g_1, g_2, g_3, g_4]$ in the row space of A, the g_i are independent linear forms and their intersection defines a line l_V in $\mathbf{P}^5$. Then l_V intersects X in a scheme of degree three: without loss of generality, we can assume that V forms the first row of A; since A is symmetric, the first column of A vanishes on l_V too. So the intersection of l_V and X is the same as the intersection of l_V and the 3×3 minor $[f_{ij}]_{2 \leqslant i,j \leqslant 4}$ which is a scheme of degree three. If p is a general point in

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$\{\det A = 0\}$, then there is a general vector in the row space of A which vanishes on p, and so p is contained in a 3-secant line of X.

This is the only example we know of a non-degenerate smooth n-fold in $\mathbf{P}^{2n+1}$ with $n \geqslant 2$ such that some fibers of a general projection to $\mathbf{P}^{n+1}$ have regularity $n + 1$—that is, the fiber consists of points contained in a line. Are there other such examples?

6.2 Codimension two

If $X \subset \mathbf{P}^{n+2}$, then every fiber of a projection to $\mathbf{P}^{n+1}$ is contained in a line, so the regularity of each fiber is equal to its degree. Thus, one can check the degrees of fibers by checking their regularity. The following result was shown us by Lazarsfeld; with his generous permission we include a proof along the lines he suggested.

PROPOSITION 6.3 (Lazarsfeld). *Suppose that $X \subset \mathbf{P}^{n+2}$ is arithmetically Cohen–Macaulay of dimension n. If X does not lie on any hypersurface of degree less than $n + 1$, then any projection of X to $\mathbf{P}^{n+1}$ from a point off X must have fibers of degree (and regularity) at least $n + 1$; that is, the closure of the union of $n + 1$-secant lines to X fills $\mathbf{P}^r$.*

We note that Zak's famous theorem on linear normality [Zak93, ch. 2] can be phrased in a similar way: it says that if X^n is a smooth subvariety of codimension at most $1 + \lceil n/2 \rceil$ in $\mathbf{P}^r$, not contained in a hypersurface of degree less than two, then the closure of the union of 2-secant lines to X fills $\mathbf{P}^r$. Is there a nice statement about k-secant lines, $2 < k < n + 1$, that interpolates between these two results?

According to Hartshorne's conjecture [Har74, Introduction], smooth projective varieties of codimension two and dimension greater than six are complete intersections, so Proposition 6.3 may include all codimension-two varieties of dimension greater than four. For dimension two, we examined 48 examples of surfaces in $\mathbf{P}^4$ catalogued by Decker *et al.* [DES93] and Popescu [Pop98] to see whether their trisecants fill $\mathbf{P}^4$. These examples were produced using code originally written in the program Macaulay classic of Bayer and Stillman, and translated into singular by Oleksandr Motsak. We used the program Magma to make the computations using algorithms based on the paper of Eisenbud–Harris [EH] and unpublished work of Eisenbud–Ulrich. Of the 48 examples, 45 lie on no quadrics. Of these 45, there is just one whose trisecants do not fill $\mathbf{P}^4$: the elliptic scroll of degree five, whose ideal is generated by five cubics.
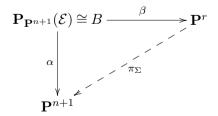
We now turn to the proof of Proposition 6.3. To compute things about a linear projection π_Σ from a linear space $\Sigma \subset \mathbf{P}^r$ to $\mathbf{P}^{n+1}$, we resolve it by blowing up. The general setup is as follows. Let Σ be a plane of dimension $\lambda - 1$, where $\lambda := r - n - 1$. Let $\beta : B \to \mathbf{P}^r$ be the blowup of Σ, and let E be the exceptional fiber. Then

$$B \cong \mathbf{P}_{\mathbf{P}^{n+1}}(\mathcal{E}),$$

where

$$\mathcal{E} \cong \mathcal{O}_{\mathbf{P}^{n+1}} \oplus \mathcal{O}_{\mathbf{P}^{n+1}}(-1)^\lambda \quad \text{and} \quad \mathcal{O}_{\mathbf{P}_{\mathbf{P}^{n+1}}(\mathcal{E})}(1) = \mathcal{O}_B(E).$$

With this notation the projection fits into the diagram

where $\alpha : \mathbf{P}_{\mathbf{P}^{n+1}}(\mathcal{E}) \to \mathbf{P}^{n+1}$ is the structure map of the projective bundle. In these terms we can describe the functor $\alpha_* \beta^*$, and more generally the derived functors $\mathbf{R}^i \alpha_* \beta^*$, quite explicitly, at least for their action on sums of line bundles.

LEMMA 6.4 [Har77, II, 17.11 and III, Exercise 8.4]. *Let α, β and $\mathcal{E}$ be as above.*

(1) *There are canonical isomorphisms*

$$\alpha_* \beta^*(\mathcal{O}_{\mathbf{P}^r}(d)) \cong \mathcal{O}_{\mathbf{P}^{n+1}}(d) \otimes \mathrm{Sym}_d(\mathcal{E}) \cong \bigoplus_{j=0}^{d}(\mathcal{O}_{\mathbf{P}^{n+1}}(d-j) \otimes \mathrm{Sym}_j(\mathcal{O}_{\mathbf{P}^{n+1}}^{\lambda})).$$

These induce isomorphisms

$$H^0(\alpha_* \beta^*(\mathcal{O}_{\mathbf{P}^r}(d))) \cong H^0(\mathcal{O}_{\mathbf{P}^{n+1}}(d) \otimes \mathrm{Sym}_d(\mathcal{E})) \cong H^0(\mathcal{O}_{\mathbf{P}^r}(d)).$$

In particular, $\alpha_* \beta^*(\mathcal{O}_{\mathbf{P}^r}(d)) = 0$ when $d < 0$.

(2) *There are canonical isomorphisms*

$$\mathbf{R}^{\lambda}(\alpha_*)\beta^*(\mathcal{O}_{\mathbf{P}^r}(-d)) \cong \mathcal{O}_{\mathbf{P}^{n+1}}(-d) \otimes (\mathrm{Sym}_{d-\lambda-1}(\mathcal{E}))^* \otimes \bigwedge^{\lambda+1} \mathcal{E}^*. \qquad \square$$

Proof of Proposition 6.3. We adopt the notation of Lemma 6.4, with $\lambda = 1$, and we write $\pi : X \to \mathbf{P}^{n+1}$ for the projection restricted to X. To show that the regularity of some fiber is at least $n + 1$ it suffices to show that for some point $y \in \mathbf{P}^{n+1}$ we have

$$H^1(\mathcal{I}_{\pi^{-1}(y)}(n-1)) \neq 0.$$

Since $\Sigma \cap X = \emptyset$ we have $\pi^{-1}(y) = \alpha^{-1}(y) \cap \beta^{-1}(X)$ and $\mathcal{I}_{\beta^{-1}X} = \beta^* \mathcal{I}_X$. Thus, it suffices to show that $H^1(\beta^*(\mathcal{I}_X(n-1))|_{\alpha^{-1}(y)}) \neq 0$ for some y. By the theorem on cohomology and base-change (see for, example, Hartshorne [Har77, Theorem III.12.11]) if suffices finally to show that $\mathbf{R}^1 \alpha_*(\beta^*(\mathcal{I}_X(n-1))) \neq 0$.

Because we have assumed that $X \subset \mathbf{P}^{n+2}$ is arithmetically Cohen–Macaulay, $\mathcal{I}_X$ has a resolution

$$0 \to F_1 \to F_0 \to \mathcal{I}_X \to 0$$

where F_0 and F_1 are sums of line bundles. The map β is locally an isomorphism on X, and outside X the sheaf $\mathcal{I}_X$ is locally free, so we may pull the resolution back by β (after tensoring with $\mathcal{O}_{\mathbf{P}^{n+2}}(n-1)$) to obtain a short exact sequence of sheaves on B of the form

$$0 \to \beta^*(F_1(n-1)) \to \beta^*(F_0(n-1)) \to \beta^*(\mathcal{I}_X(n-1)) \to 0.$$

Using the fact that the fibers of α are one-dimensional, from this a right exact sequence we obtain

$$\mathbf{R}^1 \alpha_*(\beta^*(F_1(n-1))) \xrightarrow{\phi} \mathbf{R}^1 \alpha_*(\beta^*(F_0(n-1))) \to \mathbf{R}^1 \alpha_*(\beta^*(\mathcal{I}_X(n-1))) \to 0$$

and it suffices to show that the map labelled ϕ is not surjective.

We may write

$$F_1 = \bigoplus_{i=1}^{t-1} \mathcal{O}_{\mathbf{P}^{n+2}}(-e_i), \quad F_0 = \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbf{P}^{n+2}}(-d_i)$$

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for some integers t, d_i, e_i, and considering first Chern classes on $\mathbf{P}^{n+2}$ we see that $\sum_{i=1}^{t-1} e_i = \sum_{i=1}^{t} d_i$. Applying Lemma 6.4 we see that

$$\mathbf{R}^1\alpha_*(\beta^*(F_1(n-1))) = \bigoplus_{i=1}^{t-1} \mathcal{O}_{\mathbf{P}^{n+1}}(-e_i+n-1) \otimes \mathrm{Sym}_{e_i-n-1}(\mathcal{E})^* \otimes \wedge^2(\mathcal{E})^*,$$

and similarly

$$\mathbf{R}^1\alpha_*(\beta^*(F_0(n-1))) = \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbf{P}^{n+1}}(-d_i+n-1) \otimes \mathrm{Sym}_{d_i-n-1}(\mathcal{E})^* \otimes \wedge^2(\mathcal{E})^*.$$

Because we have assumed that X does not lie on a hypersurface of degree less than $n+1$, all of the integers d_i are $\geqslant n+1$, so all of the summands are non-zero. Thus

$$\mathrm{rank}\,\mathbf{R}^1\alpha_*(\beta^*(F_0(n-1))) = \sum_{i=1}^{t} d_i - tn = \mathrm{rank}\,\mathbf{R}^1\alpha_*(\beta^*(F_1(n-1))) - n.$$

Set $p := \sum_i d_i - tn$. Taking modules of twisted global sections, we may represent ϕ by a map from a graded free module of rank $p+n$ over the homogeneous coordinate ring of $\mathbf{P}^{n+1}$ to another such module, of rank p. It follows by Macaulay's generalized principal ideal theorem (see, for example, [Eis95a, Exercise 10.9]) that if such a map is not surjective, then its cokernel has codimension at most $n+1$. Thus to prove that the cokernel is non-zero as a sheaf on $\mathbf{P}^{n+1}$, it suffices to show that the map ϕ does not induce a surjection on twisted global sections. This is immediate from the direct sum decompositions given above: after twisting by $\mathcal{O}_{\mathbf{P}^{n+1}}(1)$ we obtain

$$h^0 \mathbf{R}^1\alpha_*(\beta^*(F_0(n-1))) \otimes \mathcal{O}_{\mathbf{P}^{n+1}}(1) = t$$

and

$$h^0 \mathbf{R}^1\alpha_*(\beta^*(F_1(n-1))) \otimes \mathcal{O}_{\mathbf{P}^{n+1}}(1) = t - 1. \qquad \square$$

7. Symmetry and decomposition of Q

The sheaf $Q(X, Y)$ is defined in terms that can be understood as the comparison of the deformations of X in the ambient space to the deformations of $X \cap Y$ in Y. It turns out that under favorable circumstances $Q(X, Y) = Q(Y, X)$, and this module is determined up to a free summand by the scheme $X \cap Y$ itself.

THEOREM 7.1. *Suppose that $X, Y \subset \mathbf{P}^r$ are smooth subvarieties with ideal sheaves $\mathcal{I}_X, \mathcal{I}_Y$, and set $Z = X \cap Y$. Let $F \subset N_{Y/\mathbf{P}^r} = \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Y)$ be the subsheaf consisting of those local sections that map $\mathcal{I}_X \cap \mathcal{I}_Y$ into $\mathcal{I}_{Z/Y} \subset \mathcal{O}_Y$.*
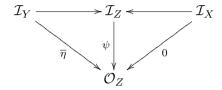
(1) *If $\eta \in H^0(N_{Y/\mathbf{P}^r})$ is a flat first-order deformation of Y, then η induces a flat first-order deformation of $X \cap Y$ if and only if $\eta \in H^0(F)$.*

(2) *There is a natural short exact sequence*

$$0 \to F \to N_{Y/\mathbf{P}^r} \to Q(X, Y) \to 0.$$

(3) *We have $Q(X, Y) = Q(Y, X)$.*

Proof. (1) We must show that $\eta \in H^0(F)$ if and only if there is a map $\psi : \mathcal{I}_Z \to \mathcal{O}_Z$ making the diagram

$$
\begin{array}{ccccc}
\mathcal{I}_Y & \longrightarrow & \mathcal{I}_Z & \longleftarrow & \mathcal{I}_X \\
{\scriptstyle \eta}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle 0}\downarrow \\
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_Z & \longleftarrow & \mathcal{O}_X
\end{array}
$$

commute. Writing $\overline{\eta}$ for the composition of η with the projection map $\mathcal{O}_Y \to \mathcal{O}_Z$, this commutativity is equivalent to the commutativity of the following diagram.

$$
\begin{array}{ccccc}
\mathcal{I}_Y & \longrightarrow & \mathcal{I}_Z & \longleftarrow & \mathcal{I}_X \\
 & {\scriptstyle \overline{\eta}}\searrow & {\scriptstyle \psi}\downarrow & {\scriptstyle 0}\swarrow & \\
 & & \mathcal{O}_Z & &
\end{array}
$$

From the short exact sequence

$$0 \to \mathcal{I}_X \cap \mathcal{I}_Y \to \mathcal{I}_X \oplus \mathcal{I}_Y \to \mathcal{I}_Z \to 0$$

we see that such a ψ exists if and only if $\eta : \mathcal{I}_Y \to \mathcal{O}_Y$ induces the zero map $\mathcal{I}_X \cap \mathcal{I}_Y \to \mathcal{O}_Y \to \mathcal{O}_Z$, that is, η maps $\mathcal{I}_X \cap \mathcal{I}_Y$ into $\mathcal{I}_{Z/Y}$, proving part (1).

(2) It follows from the definition that

$$Q(X, Y) = \frac{\mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Z)}{\{f \mid f(\mathcal{I}_X \cap \mathcal{I}_Y) = 0\}},$$

so the kernel of $N_{Y/\mathrm{Pr}} \to Q(X, Y)$ is equal to F.

(3) Because X is smooth, the derivation $d : \mathcal{I}_X \to \Omega_{\mathbf{P}^r}$ induces a locally split injection $d : \mathcal{I}_X/\mathcal{I}_X^2 \to \Omega_{\mathbf{P}^r}|_X$. It follows that the induced map of sheaves
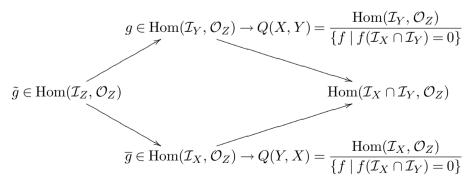
$$\mathrm{Hom}(\Omega_{\mathbf{P}^r}, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z)$$

is an epimorphism. Since $d : \mathcal{I}_X \to \Omega_{\mathbf{P}^r}$ factors as the inclusion of $\mathcal{I}_X \subset \mathcal{I}_Z$ and the map $d : \mathcal{I}_Z \to \Omega_{\mathbf{P}^r}$, we see that the restriction map

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z),$$

is also an epimorphism. The same considerations hold for X in place of Y.

Consider the commutative diagram

$$
\begin{array}{ccc}
 & g \in \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Z) \to Q(X, Y) = \dfrac{\mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Z)}{\{f \mid f(\mathcal{I}_X \cap \mathcal{I}_Y) = 0\}} & \\
\nearrow & & \searrow \\
\tilde{g} \in \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) & & \mathrm{Hom}(\mathcal{I}_X \cap \mathcal{I}_Y, \mathcal{O}_Z) \\
\searrow & & \nearrow \\
 & \overline{g} \in \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z) \to Q(Y, X) = \dfrac{\mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z)}{\{f \mid f(\mathcal{I}_X \cap \mathcal{I}_Y) = 0\}} &
\end{array}
$$

where the diagonal maps are restriction homomorphisms. By the argument above, the two maps coming from $\mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z)$ are epimorphisms. We define a map $\phi : \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Z) \to Q(Y, X)$

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as follows. Given a local section $g \in \mathrm{Hom}(\mathcal{I}_Y, \mathcal{O}_Z)$ then on a sufficiently small open set of $\mathbf{P}^r$ we may lift g back to a local section $\tilde{g} \in \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$. Let $\overline{g}$ be the image of $\tilde{g}$ in $\mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z)$, and let $\phi(g)$ be the image of $\overline{g}$ in $Q(Y, X)$.
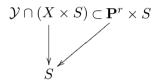
If $\tilde{g}'$ is a different lifting, then $\tilde{g} - \tilde{g}'$ goes to zero in $\mathrm{Hom}(\mathcal{I}_X \cap \mathcal{I}_Y, \mathcal{O}_Z)$, and thus $\phi(g)$ is well-defined. It follows at once that ϕ is a homomorphism, and since the map

$$\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_Z)$$

is an epimorphism, so is ϕ. Moreover, ϕ annihilates the maps f such that $f(\mathcal{I}_X \cap \mathcal{I}_Y) = 0$, so ϕ induces an epimorphism $Q(X, Y) \to Q(Y, X)$. The inverse map is constructed by a symmetrical procedure, proving part (3). $\qquad\square$

COROLLARY 7.2. *Suppose that $X, Y \subset \mathbf{P}^r$ are smooth varieties that meet in a finite scheme Z of degree l, and let F be the kernel of the surjection $N_Y \to Q(X, Y)$ defined above. Suppose that S is a reduced algebraic subset of the Hilbert scheme of subschemes of $\mathbf{P}^r$ containing the point $[Y]$ that corresponds to Y. If the points of S near p correspond to subschemes that meet X in schemes of degree at least l, then the tangent space to S at p is a subspace of $H^0(F)$. In particular, the dimension of the tangent space to S at p is at most $h^0(N_Y) - \dim_k Q(X, Y)$.*

Proof. Let $\pi : \mathcal{Y} \to S$ be the restriction of the universal family. By semicontinuity, the degree of the intersection $\pi^{-1}(q) \cap X$, for $q \in S$ near p, is equal to l. Since S is reduced, this guarantees that the family of intersections

$$\mathcal{Y} \cap (X \times S) \subset \mathbf{P}^r \times S$$
$$\downarrow \swarrow$$
$$S$$

is flat over a neighborhood of p. The tangent space to S at p thus consists of first-order deformations of Y that induce flat deformations of $X \cap Y$, and the first statement follows from part (1) of Theorem 7.1. The dimension statement is then immediate from the exact sequence in part (2) of the theorem. $\qquad\square$

To put part (3) of Theorem 7.1 into context, we note that, under somewhat more general circumstances, the structure of $Q(X, Y)$ depends mostly on the intersection $Z = X \cap Y$.

PROPOSITION 7.3. *Let Z be a finite scheme over k. There is a module $\overline{Q}(Z)$ depending only on Z such that if $X, Y \subset P$ are k-schemes of finite type with X smooth and Y locally a complete intersection in P such that $Z = X \cap Y$, then $Q(X, Y) \cong \overline{Q}(Z) \oplus \mathcal{O}_Z^m$ for some m.*

Proof. From the definition we see that $Q(X, Y)$ is the direct sum of local contributions, so we may harmlessly suppose that Z has only one closed point. Choose a minimal surjection $\phi : A := k[[x_1, \ldots, x_n]] \to \mathcal{O}_Z$ so that n is the dimension of the Zariski tangent space of $\mathcal{O}_Z$. Let I_Z be the kernel of this map, and let $f : F \to I_Z$ be a minimal surjection from a free $\mathcal{O}_Z$-module. Set

$$\overline{Q}(Z, \phi, f) = \mathrm{coker}\Big(\mathrm{Hom}_P(I_Z/I_Z^2, \mathcal{O}_Z) \xrightarrow{\mathrm{Hom}(f, \mathcal{O}_Z)} \mathrm{Hom}_P(F, \mathcal{O}_Z) \Big).$$

We will show that $\overline{Q}(Z) := \overline{Q}(Z, \phi, f)$ is independent of the choices of the minimal surjections ϕ and f. Note that $\overline{Q}(Z, \phi, f)$ has no free summand.

First, if $f' : F' \to I_Z$ is any surjection from a free $\mathcal{O}_Z$-module, then we may write $F' \cong F \oplus G$ in such a way that f' becomes the map $(f, 0)$ so

$$\overline{Q}(Z, \phi, f') = \overline{Q}(Z, \phi, f) \oplus (G \otimes_P \mathcal{O}_Z).$$

In particular, this shows that $\overline{Q}(Z, \phi, f)$ is independent of the choice of f so long as f is minimal.

Next, if $A' \to \mathcal{O}_Z$ is a surjection from a different power series ring (of any dimension), then we may choose a third power series ring A'' surjecting onto both A' and A. It thus suffices to show that, if $\psi : A'' \to A$ is a surjection of power series rings, then

$$\overline{Q}(Z, \phi\psi, f) = \overline{Q}(Z, \phi, f) \oplus \mathcal{O}_Z^m,$$

where m is the difference d of dimensions between A and A''. Let $(y_1, \ldots, y_d)$ be the kernel of ψ. Lifting generators of I_Z back to A'' as power series independent of the y_i, we see that the kernel I_Z'' of the surjection $A'' \to \mathcal{O}_Z$ may be written as $(y_1, \ldots, y_d) + I_Z'$, where $y_1, \ldots, y_d$ are a regular sequence modulo I_Z'. It follows that

$$I_Z''/I_Z''^2 = I_Z'/I_Z'^2 \oplus \mathcal{O}_Z^d \cong I_Z/I_Z^2 \oplus \mathcal{O}_Z^d.$$

This shows that $\overline{Q}(Z, \phi, f)$ is independent of the choices of ϕ and f.

If $X, Y \subset P$ are schemes, with X smooth, Y a complete intersection, and $Z = X \cap Y$, then $\mathcal{O}_Z$ is a homomorphic image of the completion $\hat{\mathcal{O}}_{X,Z}$ of the local ring of X at the closed point of Z, and

$$\hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}_P} I_Y/I_Y^2 \to (I_X + I_Y)/(I_X^2 + I_Y)$$

is a map of a free module onto $I_{Z/X}/I_{Z/X}^2$, so $Q(X, Y)$ is the direct sum of $\overline{Q}(Z)$ and a free module, as required. $\qquad\square$

7.1 Decomposing Q

The following result is sometimes useful in computing the length of $Q(X, Y)$. Since it reduces at once to the affine case, we work with ideals L, I in a Noetherian ring A. We define $Q(L, I)$ to be the cokernel of the map

$$\mathrm{Hom}\left(\frac{I + L}{I^2 + L}, \frac{A}{I + L}\right) \to \mathrm{Hom}\left(\frac{I}{I^2 + IL}, \frac{A}{I + L}\right).$$

To simplify the notation, if $K \subset A$, then we write $Q_{A/K}(L, I)$ for $Q((L + K)/K, (I + K)/K)$ computed in the ring A/K.

PROPOSITION 7.4.

(1) If $L' \subset L$ is an ideal such that $I \cap L' = IL'$, then $Q(L, I) \cong Q_{A/L'}(L, I)$. In particular, if $I \cap L = IL$, then $Q(L, I) = 0$.

(2) If $I' \subset I$, then $Q_{A/I'}(L, I) \subset Q(L, I)$.

(3) If $I = I' + I''$ with $I' \cap I'' \subset I'I''$ and $I' \cap (I'' + L) = I'(I'' + L)$, then $Q(L, I) = Q_{A/I'}(L, I) + Q_{A/I''}(L, I)$.

Proof. (1) It is clear that A/J and $(I + L)/(I^2 + L)$ remain the same modulo L'. The kernel of the surjection $I/I^2 \to (I + L')/(I^2 + L')$ is the image of $I \cap L'$. By hypothesis, $I \cap L' \subset IL' \subset IJ$. Since $\mathrm{Hom}(I/I^2, A/J) = \mathrm{Hom}(I/IJ, A/J)$, we are done.

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For parts (2) and (3) we refer to the diagram.

$$
\begin{array}{ccccccc}
0 \longrightarrow \operatorname{Hom}\!\left(\dfrac{I'+I^2+L}{I^2+L},\,A/J\right) & \xrightarrow{\ \beta\ } & \operatorname{Hom}\!\left(\dfrac{I'+I^2}{I^2},\,A/J\right) & & & & \\[2ex]
\big\uparrow{\scriptstyle\gamma} & & \big\uparrow & & & & \\[2ex]
0 \longrightarrow \operatorname{Hom}\!\left(\dfrac{I+L}{I^2+L},\,A/J\right) & \longrightarrow & \operatorname{Hom}\!\left(\dfrac{I}{I^2},\,A/J\right) & \longrightarrow & Q_A(L,I) & \longrightarrow & 0 \\[2ex]
\big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle\alpha} & & \\[2ex]
0 \longrightarrow \operatorname{Hom}\!\left(\dfrac{I+L}{I'+I^2+L},\,A/J\right) & \longrightarrow & \operatorname{Hom}\!\left(\dfrac{I}{I'+I^2},\,A/J\right) & \longrightarrow & Q_{A/I'}(L,I) & \longrightarrow & 0 \\[2ex]
\big\uparrow & & \big\uparrow & & \big\uparrow & & \\[2ex]
0 & & 0 & & 0 & &
\end{array}
$$

The first two rows, and the first two columns, are obviously exact. For part (2) we will show the injectivity of the map labelled α. We first identify

$$(I+L)/(I^2+L) \quad \text{with } I/(I^2+(I\cap L)),$$

and

$$(I+L)/(I'+I^2+L) \quad \text{with } I/(I'+I^2+(I\cap L)).$$

The injectivity of α is thus equivalent to the statement that any map $I/I^2 \to A/J$ that annihilates both of the images of I' and of L annihilates the image of $I'+L$, proving part (2).

If the hypotheses of part (3) are satisfied, then

$$\frac{I}{I^2} = \frac{I'+I^2}{I^2} \oplus \frac{I''+I^2}{I^2},$$

and similarly modulo L. Thus, both the left-hand vertical sequences in the diagram are split exact, and we deduce both that $Q_A(L,I) = Q_{A/I'}(L,I) \oplus \operatorname{coker}\beta$, and that $\operatorname{coker}\beta = Q_{A/I''}(L,I)$. $\qquad\square$

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Roya Beheshti   beheshti@math.wustl.edu

Department of Mathematics, Washington University, St Louis, MO 63130, USA

David Eisenbud   eisenbud@math.berkeley.edu

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA