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RESEARCH ARTICLE

From topological recursion to wave functions and PDEs quantizing hyperelliptic curves

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Abstract
Starting from loop equations, we prove that the wave functions constructed from topological recursion on families of degree 2 spectral curves with a global involution satisfy a system of partial differential equations, whose equations can be seen as quantizations of the original spectral curves. The families of spectral curves can be parametrized with the so-called times, defined as periods on second type cycles, and with the poles. These equations can be used to prove that the WKB solution of many isomonodromic systems coincides with the topological recursion wave function, which proves that the topological recursion wave function is annihilated by a quantum curve. This recovers many known quantum curves for genus zero spectral curves and generalizes this construction to hyperelliptic curves.

Contents

1. Introduction

Topological recursion is a powerful tool, which was first discovered in the context of large size asymptotic expansions in random matrix theory [10, 11, 12, 15] and established as an independent universal theory around 2007 [21]. Its most important role was to unveil a common structure in many different topics in mathematics and physics, which helped to build bridges among them and gain general context. For instance, it has been related to fundamental structures in enumerative geometry and integrable systems, such as intersection theory on moduli spaces of curves and cohomological field theories. A quantum curve is a Schrödinger operator-like noncommutative analogue of a plane curve that annihilates the so-called wave function, which can be seen as a Wentzel–Kramers–Brillouin (WKB) asymptotic solution of the corresponding differential equation. Inspired by the intuition coming from matrix models, it has been conjectured that there exists such a quantum curve associated to a spectral curve, which is the input of the topological recursion, and whose WKB asymptotic solution is reconstructed by the topological recursion output.

This claim was verified in [9] for a class of spectral curves called admissible, which are basically spectral curves whose Newton polygons have no interior point. Admissible spectral curves include a very large number of spectral curves of genus zero. Therefore, they recover many cases previously studied in the literature in various algebro-geometric contexts. In the present work, we go beyond admissible spectral curves and study the quantum curve problem for spectral curves with a global involution, given by algebraic curves whose defining polynomials are of the form $y^2 = R(x)$, with R a rational function on x. This setting includes the genus zero spectral curves with a global involution $y \mapsto -y$, such as the well-known Airy curve, which are many less than the set of admissible curves, but it also includes all genus one spectral curves, that is all elliptic curves, and all hyperelliptic curves, which are curves of genus $\hat{g} > 1$, where R is a polynomial in x. Quantum curves encode enumerative invariants in an interesting way, and help to build the bridge between the geometry of integrable systems and topological recursion. One of the most celebrated applications of quantum curves is in the context of knot theory, where the quantum curve of the A-polynomial of a knot provides a conjectural constructive generalization of the volume conjecture [5, 13, 14].

1.1. Quantum curves and topological recursion

We start by presenting the idea of quantum curves and their relation to topological recursion. Consider $P \in \mathbb{C}[x, y]$, and let

$$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}$$

be a plane curve.

Consider $\hbar > 0$ a formal parameter. A quantization of the plane curve $\mathcal{C}$ is a differential operator $\widehat{P}$ of the form

$$\widehat{P}(\widehat{x}, \widehat{y}; \hbar) = P_0(\widehat{x}, \widehat{y}) + O(\hbar),$$

where $\widehat{x} = x\cdot$, $\widehat{y} = \hbar\frac{\mathrm{d}}{\mathrm{d}x}$. In fact, $\widehat{P}$ is a normal-ordered operator valued (all the $\widehat{y}$ in a monomial are placed to the right of all the $\widehat{x}$) formal series (or transseries) whose leading order term $P_0(\widehat{x}, \widehat{y})$ recovers the polynomial equation of the original spectral curve. Actually, in general, $P_0(x, y)$ can be a reducible polynomial with $P(x, y)$ one of its factors. The operators $\widehat{x}$ and $\widehat{y}$ satisfy the following commutation relation which justifies the name "quantization":

$$[\widehat{y}, \widehat{x}] = \hbar.$$

One can consider a Schrödinger-type differential equation

$$\widehat{P}(\widehat{x}, \widehat{y})\psi(z, \hbar) = 0, \text{ with } z \in \mathcal{C}, \tag{1}$$

whose solution can be calculated via the WKB method, that is we require ψ to have a formal series (respectively, transseries) $\hbar$ expansion of the form

$$\psi(z, \hbar) = \exp\left(\sum_{m\geq 0} \hbar^{m-1} S_m(z)\right)$$

(respectively, of the form of a formal series in powers of exponentials of inverse powers of $\hbar$ whose coefficients are formal power series in $\hbar$). The coefficients $S_k(z)$ are determined recursively via (1). One fundamental question is if the formal solution ψ can be computed directly from the original plane curve $\mathcal{C}$. The conjectural answer is provided by the topological recursion and is already established in many cases.

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The input data of the topological recursion is called spectral curve. For the purpose of this paper, a *spectral curve* will be given as in [17], that is by the data $(\Sigma, x, y, \omega_{0,1}, \omega_{0,2})$, where Σ is a compact Riemann surface, and x and y are meromorphic functions on Σ, such that the zeroes of $\mathrm{d}x$ do not coincide with the zeroes of $\mathrm{d}y$. Then x and y must be algebraically dependent, that is $P(x, y) = 0$ with $P \in \mathbb{C}[x, y]$. We consider $\omega_{0,1} := y\mathrm{d}x$ and $\omega_{0,2}$ a symmetric bidifferential B on Σ^2 with double poles along the diagonal and vanishing residues. The output of the topological recursion are symmetric meromorphic multidifferentials $\omega_{g,n}(z_1, \ldots, z_n)$ on Σ^n. In particular, for $n = 0$, the $F_g := \omega_{g,0}$ are complex numbers or expressions depending on parameters of the spectral curves. In the present work, we assume that the poles of y but not of x cannot be ramification points of x.

The perturbative wave function $\psi(z)$ constructed from topological recursion is defined (see [21]) as

$$\frac{1}{x(z) - x(o)} \exp\Bigg(\sum_{g\geq 0, n\geq 1} \frac{\hbar^{2g-2+n}}{n!} \int_o^z \cdots \int_o^z \Big(\omega_{g,n}(z_1, \ldots, z_n) - \delta_{g,0}\delta_{n,2} \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2} \Big) \Bigg),$$

where $o \in \Sigma$ is a chosen base point for integration. In general, the quantum curve is obtained by choosing $o = \zeta$, such that $x(\zeta) = \infty$, and one may need to regularize the $(g, n) = (0, 1)$ term in the limit $o \longrightarrow \zeta$. We choose a regularization for $(g, n) = (0, 2)$ which slightly differs from some part of the literature and produces the first factor of the expression. We emphasize that our definition transforms as a spinor $\frac{1}{2}$-form under change of coordinates. Actually, we will generalize the definition of the wave function by allowing integration over any divisor, as in [17].

A further question is if the differential operator $\widehat{P}$ can be directly constructed from the topological recursion. This has also been answered affirmatively in many cases. In this article, we actually construct an operator that we believe is a more fundamental object, which appears more naturally for curves of genus $\hat{g} > 0$, and provides a partial differential equation (PDE) which also allows to reconstruct the wave function ψ. Our system of PDEs will also imply a quantum curve in the more classical sense considered in this section.

1.2. Generalized cycles

Let us now recall the concept of generalized cycles on a Riemann surface as in [17], which will help us introduce suitable local coordinates in the space of spectral curves. These local coordinates can be seen as deformation parameters giving rise to families of spectral curves and will play a key role when producing our system of PDEs for a large class of spectral curves.

The so-called times t_i, introduced in [17], can be viewed as local coordinates in the space of spectral curves. Time deformations ∂_{t_i} belong to the tangent space, which is isomorphic to the space of meromorphic differential forms on the spectral curve, and via form-cycle duality, it can be identified with the space of generalized cycles on the spectral curve. In [17], generalized cycles are defined as elements of the dual of the space of meromorphic forms on Σ, such that integrating $\omega_{0,2} = B$ on them gives meromorphic 1-forms.

In practice, a generating family is given by three kinds of cycles, dual to three kinds of forms:

- **First kind cycles:** This type of cycles are usual noncontractible cycles, that is elements $[\gamma] \in \mathrm{H}_1(\Sigma, \mathbb{C})$. If Σ is compact of gnus $\hat{g}$, then $\dim \mathrm{H}_1(\Sigma, \mathbb{C}) = 2\hat{g}$.
- **Second kind cycles:** A cycle of the second type $\gamma = \gamma_p.f$ consists of a small circle γ_p around a point $p \in \Sigma$ weighted by a function f holomorphic in a neighborhood of γ_p and meromorphic in a neighborhood of p, with a possible pole at p (of any degree), that is by definition $\int_\gamma \omega := 2\pi i \operatorname{Res}_p f\omega$.
- **Third kind cycles:** They are open chains $\gamma = \gamma_{q\to p}$ (paths up to homotopic deformation with fixed endpoints), whose boundaries $\partial\gamma = [p] - [q]$ are degree zero divisors.

Let $x : \Sigma \longrightarrow \mathbb{C}$ be the meromorphic function that makes the spectral curve a branched cover of the Riemann sphere. A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

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$$\{\xi_p^k\}_{k\in\mathbb{Z}},\ \text{with}\ \xi_p = (x - x(p))^{1/\mathrm{ord}_p(x)}.$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\mathrm{ord}_p(x)}$, with $\mathrm{ord}_p(x) < 0$. The following set of cycles generates an integer lattice in the space of second kind cycles:

$$\mathcal{A}_{p,k} = \gamma_p.\xi_p^k,\ \ p \in \Sigma, k \geq 0, \tag{2}$$

$$\mathcal{B}_{p,k} = \frac{1}{2\pi i}\gamma_p.\frac{\xi_p^{-k}}{k},\ \ p \in \Sigma, k \geq 1. \tag{3}$$

Given a meromorphic 1-form ω on Σ, for every pole p of ω, we define for every $j \geq 0$, the *KP times*:

$$t_{p,j} := \frac{1}{2\pi i}\int_{\mathcal{A}_{p,j}} \omega = \frac{1}{2\pi i}\int_{\gamma_p} \xi_p^j\,\omega = \operatorname*{Res}_p\,(\xi_p)^j\omega\,, \tag{4}$$

so that

$$\omega \sim \sum_{j=0}^{\deg_p(\omega)} t_{p,j}\xi_p^{-j-1}d\xi_p + \text{analytic at } p.$$

Since we assumed Σ to be compact, the number of poles is finite. Moreover, all the times with $j \geq \deg_p\omega$ are vanishing. Therefore, only a finite number of times are nonzero.

1.3. Context and outline

One of our motivations to study the problem of quantum curves for any spectral curve with a global involution was to be able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also aimed to give the first quantum curves for spectral curves of genus $\hat{g} > 1$, and we were especially interested to see how introducing deformations with respect to the times could give rise to systems of PDEs that we consider more natural in general.

1.3.1. Comparison to the literature

In [8], they generalize the techniques employed in [9] to find the quantum curves for admissible curves to apply them to the family of genus one spectral curves given by the Weierstrass equation. They find an order two differential operator that annihilates the perturbative wave function ψ. However, it is not a quantum curve, since it contains infinitely many $\hbar$ corrections which are not meromorphic functions of x. They also check the first orders of the conjectural quantum curve [4, 16, 20] for the nonperturbative wave function.

In [25], they focus on the Painlevé I spectral curve, which is a degenerate torus, and from topological recursion, they get a PDE that annihilates the wave function, which is compatible with the isomonodromic system and, together with another identity coming from integrable systems, provides a quantum curve that annihilates the wave function. In [23], the first author slightly generalizes the same results to the case of any elliptic curve, that is he considers not only the degenerate case of Painlevé I but tori where none of the two cycles are pinched. In both papers, they show that the $\hbar$ corrections from [8] can be controlled by a derivative with respect to a deformation parameter. The quantum curves still contain infinitely many $\hbar$-correction terms, but in this case, these corrections are given by the asymptotic expansion of the solution of Painlevé I around $\hbar \to 0$.

In [24], the approach is reversed: they prove that Lax pairs associated with $\hbar$-dependent Painlevé equations satisfy the topological type property of [1], which implies that one can reconstruct the $\hbar$-expansion of the isomonodromic τ-function from topological recursion. Finally, in [28], they generalize this result showing that it is always possible to deform a differential equation $\partial_x\Psi(x) = \mathcal{L}(x)\Psi(x)$,

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with $\mathcal{L}(x) \in \mathfrak{sl}_2(\mathbb{C})$ by introducing a formal parameter $\hbar$ in such a way that it satisfies the topological type property.

In the present work, we recover the PDE from [23, 25] from loop equations (which are necessary for topological recursion but not sufficient[1]) and as part of a system that we obtain because we consider a wave function where the integrals are over any divisor of degree zero. With our system, we are able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also give an additional meaning to the deformation parameter that appears naturally in [23, 25] for the case of elliptic curves, making use of the powerful idea of deforming with respect to the generalized cycles introduced in the previous section. The elliptic curve case is a very concrete case in which there is only one such deformation parameter, but we see that we need to consider several in the higher genus cases.

1.3.2. Outline

In Section 2, we introduce the type of curves we consider in this work and relate them to the concept of spectral curves as input of the topological recursion. We also give the link to the spectral curves in the setting of isomonodromy systems, which serves as a motivation to us. Moreover, we compute the deformation parameters of the family of elliptic curves, which recovers the Painlevé I isomonodromy system setting in the degenerate case; in particular, the so-called Kadomtsev–Petviashvili (KP) times.

In Section 3, we recall the loop equations for our specific setting and deduce some interesting consequences relating them to time deformations, which appear when considering spectral curves of genus $\hat{g} > 0$.

In Section 4, we prove our main result. From loop equations, we obtain a system of partial differential equations that annihilates our wave function defined from topological recursion integrating over a general divisor. We also give the shape of this system in the particular cases of genus zero and elliptic curves. Finally, we consider our system for the particular case of a two-point divisor, which then consists of only two PDEs, with differentials with respect to two spectral variables and the deformation parameters, called times. We are able to combine the two PDEs in such a way that we eliminate one of the spectral variables.

In Section 5, we argue that if the spectral curve comes from an isomonodromic system, then the topological recursion nonperturbative wave function has to coincide with the solution of the isomonodromic system, which implies an ordinary differential equation (ODE), which is the quantum curve we were looking for. As particular interesting cases, we recover the first Painlevé system and equation, and its higher analogues defined in terms of Gelfand–Dikii polynomials. In Appendix A, we prove that the integrable kernel of any isomonodromic system satisfies the same PDE that we found for the two-point wave function constructed from topological recursion.

2. Spectral curves with a global involution

In this article, we focus on algebraic plane curves of the form

$$y^2 = P(x), \tag{5}$$

with $P(x) \in \mathbb{C}[x]$ an arbitrary polynomial of x, and we will generalize to $y^2 = R(x)$ with $R(x) \in \mathbb{C}(x)$ an arbitrary rational function of x.

The degree of the polynomial is related to the genus of the curve. For example, in the case in which P is a polynomial of degree $2m + 1$ or $2m + 2$, the curve has genus $\hat{g} \leq m$, with equality if the plane curve is smooth. If the degree is odd, the curve has one point at infinity, and if the degree is even, the curve has two points at infinity. If $\hat{g} > 1$, the curve is called hyperelliptic; if $\hat{g} = 1$ (with a distinguished point), it is called elliptic, and if $\hat{g} = 0$, it is called rational.

[1]Topological recursion provides a specific solution of loop equations. General solutions of loop equations don't necessarily satisfy the so-called projection formula, which implies that the purely holomorphic part of $\omega_{g,n}$ vanishes (in the sense of [7, Section 2.2.2]), and they are governed by a generalization called blobbed topological recursion, which was introduced in [7].

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2.1. Spectral curves as input of the topological recursion

The method of topological recursion (TR) associates to a spectral curve $\mathcal{S}$, a doubly indexed family of meromorphic multidifferentials $\omega_{g,n}$ on Σ^n:

$$\text{TR:}\quad \text{Spectral curve } \mathcal{S} = (\Sigma, x, y\mathrm{d}x, B) \rightsquigarrow \text{Invariants } \omega_{g,n} (F_g = \omega_{g,0}).$$

A spectral curve is the data of a Riemann surface Σ, a holomorphic projection $x : \Sigma \to \mathbb{C}\mathrm{P}^1$ to the base $\mathbb{C}\mathrm{P}^1$ which turns Σ into a ramified cover of the sphere, a meromorphic 1-form $y\mathrm{d}x$ on Σ, and B, a second kind fundamental differential, that is a symmetric $1 \boxtimes 1$ form on $\Sigma \times \Sigma$ with normalized double pole on the diagonal and no other pole, behaving near the diagonal as:

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2} + \text{holomorphic at } z_1 = z_2.$$

In case the spectral curve is of genus zero, that is $\Sigma = \mathbb{C}\mathrm{P}^1$, it is known that such a B is unique and is worth

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

If the genus $\hat{g}$ of Σ is ≥ 1, B is not unique, since one can add any symmetric bilinear tensor product of holomorphic 1-forms. A way to find a unique one is to choose a Torelli marking, which is a choice of a symplectic basis $\{\{\mathcal{A}_i\}_{i=1}^{\hat{g}}, \{\mathcal{B}_i\}_{i=1}^{\hat{g}}\}$ of $H_1(\Sigma, \mathbb{Z})$. There exists a unique B normalized on the $\mathcal{A}$-cycles of $H_1(\Sigma, \mathbb{Z})$

$$\oint_{\mathcal{A}_i} B(z_1, \cdot) = 0.$$

Such a bidifferential has a natural construction in algebraic geometry and is called the normalized *fundamental differential of the second kind* on Σ. See [18] for constructing B for general algebraic plane curves.

We define the *filling fractions* as the $\mathcal{A}$-periods of $\omega_{0,1}$:

$$\epsilon_i := \oint_{\mathcal{A}_i} y\mathrm{d}x, \quad \text{for } i = 1, \ldots, \hat{g}. \tag{6}$$

Remark 2.1. The coordinate $x : \Sigma \to \mathbb{C}\mathrm{P}^1$ in the definition of spectral curve can be thought of as a ramified covering of the sphere. We call the *degree* of the spectral curve the number of sheets of the covering, that is the number of preimages of a generic point. In this article, we focus on spectral curves of degree 2 with a global involution $(x, y) \mapsto (x, -y)$.

2.2. Spectral curves from isomonodromic systems

Painlevé transcendents have their origin in the study of special functions and of isomonodromic deformations of linear differential equations. They are solutions to certain nonlinear second-order ordinary differential equations in the complex plane with the Painlevé property, that is the only movable singularities are poles.

An $\hbar$-dependent *Lax pair* is a pair $(\mathcal{L}(x, t; \hbar), \mathcal{R}(x, t; \hbar))$ of 2×2 matrices, whose entries are rational functions of x and holomorphic in t, such that the system of partial differential equations

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$$\begin{cases} \hbar \dfrac{\partial}{\partial x}\Psi(x,t) = \mathcal{L}(x,t;\hbar)\Psi(x,t), \\ \hbar \dfrac{\partial}{\partial t}\Psi(x,t) = \mathcal{R}(x,t;\hbar)\Psi(x,t) \end{cases}$$

is compatible. We call such a system an *isomonodromy system*.

The compatibility condition, that is $\frac{\partial}{\partial t\partial x}\Psi = \frac{\partial}{\partial x\partial t}\Psi$, is equivalent to the so-called zero-curvature equation:

$$\hbar\frac{\partial \mathcal{L}}{\partial t} - \hbar\frac{\partial \mathcal{R}}{\partial x} + [\mathcal{L},\mathcal{R}] = 0.$$

In [26], Jimbo and Miwa gave a list of the Lax pairs whose compatibility conditions are equivalent to the six Painlevé equations.

Let us consider the expansion around $\hbar = 0$ of the first equation of the system: $\mathcal{L}(x,t;\hbar) = \sum_{k\geq 0}\hbar^k \mathcal{L}_k(x,t)$. The associated *spectral curve* is given by

$$\det(y\,\mathrm{Id} - \mathcal{L}_0(x,t)) = 0, \tag{7}$$

which is actually a family of algebraic curves parametrized by t.

2.2.1. Motivational example: The first Painlevé equation

Let us consider the first Painlevé equation with a formal small parameter $\hbar$:

$$P_I: \ \frac{\hbar^2}{2}\frac{\partial^2}{\partial t^2}U - 3U^2 = t.$$

The leading term $u = u(t)$ of a formal power series solution $U(t,\hbar) = u(t) + \sum_{k\geq 1}\hbar^{2k}u_k(t)$ satisfies $t = -3u^2$ and determines the subleading terms recursively:

$$u_k = c_k u^{1-5k}, c_k \in \mathbb{Q}$$

by the recursion

$$c_0 = 1\,, \qquad 2c_{k+1} = \frac{25k^2-1}{6^3}c_k - \sum_{j=1}^{k} c_j c_{k+1-j}.$$

The coefficient $u_k(t)$ has a singularity at $u = 0$, that is $t = 0$. This special point is called a turning point of P_I. We shall assume that $t \neq 0$. We denote by $\dot{U}$ the derivative with respect to t of $U(t,\hbar)$.

The Tau-function $\mathcal{T}(t)$ is defined in such a way that

$$U(t) = -\hbar^2\,\frac{\partial^2}{\partial t^2}\log\mathcal{T}.$$

The Painlevé equation ensures that $\mathcal{T}$ is an entire function with simple zeros at the movable poles of U.

The Lax pair associated to the first Painlevé equation is given by

$$\mathcal{L}(x,t;\hbar) := \begin{pmatrix} \frac{\hbar}{2}\dot{U} & x-U \\ (x-U)(x+2U) + \frac{\hbar^2}{2}\ddot{U} & -\frac{\hbar}{2}\dot{U} \end{pmatrix} \text{ and } \mathcal{R}(x,t;\hbar) := \begin{pmatrix} 0 & 1 \\ x+2U & 0 \end{pmatrix}. \tag{8}$$

The leading term of $\mathcal{L}$ in its expansion around $\hbar = 0$ is given by

$$\mathcal{L}_0(x,t) = \begin{pmatrix} 0 & x-u \\ x^2+ux-2u^2 & 0 \end{pmatrix}.$$

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The spectral curve reads

$$\det(y\,\mathrm{Id} - \mathcal{L}_0(x,t)) = y^2 - (x-u)^2(x+2u) = 0, \tag{9}$$

which is actually a family of algebraic curves parametrized by t. Since we have assumed $t \neq 0$, the two roots $x = u$ and $x = -2u$ of $R(x) = (x-u)^2(x+2u)$ are distinct, but the root $x = u$ has multiplicity 2. These curves have genus zero or, more precisely, constitute a family of tori with one of the cycles pinched.

In general, we want to study the family of tori given by

$$y^2 = x^3 + tx + V, \tag{10}$$

where $R(x) = x^3 + tx + V$ has three different roots. The case $t = -3u^2$, $V = 2u^3$ recovers the particular degenerate case (9).

2.3. *Parametrizations and deformation parameters of the elliptic case*

In the elliptic case, we give the parametrizations and compute the coordinates because it has more structure than the genus zero case, since we need to introduce one deformation parameter, and it also illustrates how the general case works. The degenerate case corresponds to Painlevé I, which is our prototypical example when making the connection to isomonodromy systems.

2.3.1. Degenerate case

When $V = 2u^3$, consider the parametrization of the spectral curve given by

$$\begin{cases} x(z) = z^2 - 2u, \\ y(z) = z^3 - 3uz, \end{cases}$$

with $z \in \Sigma = \mathbb{C}\mathrm{P}^1$ and the fundamental form of the second kind on Σ:

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

It satisfies

$$y^2 = x^3 + tx + V,$$

with $t = -3u^2$, $V = 2u^3$.

This curve has one ramification point at $z = 0$ and one pole at $z = \infty$.

Near $z = \infty$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}}).$$

This implies that

$$t_{\infty,1} = \frac{1}{2\pi i} \int_{\mathcal{A}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{-\frac{1}{2}}\, y\mathrm{d}x = -t,$$

$$\int_{\mathcal{B}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{\frac{1}{2}}\, y\mathrm{d}x = -V,$$

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$$t_{\infty,5} = \frac{1}{2\pi i}\int_{\mathcal{A}_{\infty,5}} y\mathrm{d}x = \operatorname*{Res}_{z\to\infty} x^{-\frac{5}{2}}\, y\mathrm{d}x = -2,$$

$$\int_{\mathcal{B}_{\infty,5}} y\mathrm{d}x = \frac{1}{5}\operatorname*{Res}_{z\to\infty} x^{\frac{5}{2}}\, y\mathrm{d}x = \frac{tV}{10},$$

where we use the generalized cycles $\mathcal{A}_{p,k}$ and $\mathcal{B}_{p,k}$ defined in (2), and we have considered only the values $k = 1, 5$ for which $t_{\infty,k} \neq 0$.

The degenerate cycle $\mathcal{A}$ corresponds to a small simple closed curve encircling $z_0 = \sqrt{3u}$ and the cycle $\mathcal{B}$ corresponds to the chain $(-\sqrt{3u} \to \sqrt{3u})$. Therefore, we have

$$\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{A}} y\mathrm{d}x = 0, \qquad I = \oint_{\mathcal{B}} y\mathrm{d}x = \frac{-8}{15}\,(3u)^{\frac{5}{2}}.$$

The prepotential $F_0 = \omega_{0,0}$, defined in general in [17, 21], is worth

$$F_0 = \frac{1}{2}\left(\epsilon I + \sum_k t_{\infty,k}\int_{\mathcal{B}_{\infty,k}} y\mathrm{d}x\right) = \frac{1}{2}\left(tV - 2\frac{tV}{10}\right) = \frac{2}{5}tV = -\frac{12}{5}u^5$$

and satisfies

$$\frac{\partial F_0}{\partial t} = 2u^3 = V, \qquad \frac{\partial^2 F_0}{\partial t^2} = \frac{\partial V}{\partial t} = -u.$$

2.3.2. Nondegenerate case: elliptic curves

When $V \neq 2u^3$, we shall consider the Weierstrass parametrization of the torus of modulus τ, and with a scaling ν:

$$\begin{cases} x(z) = \nu^2\wp(z), \\ y(z) = \frac{\nu^3}{2}\wp'(z), \end{cases}$$

with $z \in \Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ the torus of modulus τ. The fundamental form of the second kind on Σ, normalized on the $\mathcal{A}$ cycle is:

$$B(z_1, z_2) = (\wp(z_1 - z_2) + G_2(\tau))\mathrm{d}z_1\mathrm{d}z_2,$$

with $G_k(\tau)$ the k^{th} Eisenstein series.

It satisfies

$$y^2 = x^3 + tx + V,$$

with

$$t = -15\nu^4 G_4(\tau), \qquad V = -35\nu^6 G_6(\tau).$$

Instead of parametrizing the spectral curve with t and V, we shall parametrize it with t and ϵ, where

$$\epsilon = \frac{1}{2\pi i}\oint_{\mathcal{A}} y\mathrm{d}x = 3\nu^5 G_4'(\tau).$$

We shall now write

$$V = V(t, \epsilon).$$

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We have

$$\mathrm{d}V = 2\pi i \nu \mathrm{d}\epsilon - \nu^2 G_2(\tau)\mathrm{d}t.$$

This curve has three ramification points at $z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, and one pole at $z = 0$. Near $z = 0$, we have

$$y \sim x^{\frac{3}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}}).$$

This implies that

$$t_{\infty,1} = \frac{1}{2\pi i}\int_{\mathcal{A}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to 0} x^{-\frac{1}{2}}\, y\mathrm{d}x = -t,$$

$$\int_{\mathcal{B}_{\infty,1}} y\mathrm{d}x = \operatorname*{Res}_{z\to 0} x^{\frac{1}{2}}\, y\mathrm{d}x = -V.$$

$$t_{\infty,5} = \frac{1}{2\pi i}\int_{\mathcal{A}_{\infty,5}} y\mathrm{d}x = \operatorname*{Res}_{z\to 0} x^{-\frac{5}{2}}\, y\mathrm{d}x = -2,$$

$$\int_{\mathcal{B}_{\infty,5}} y\mathrm{d}x = \frac{1}{5}\operatorname*{Res}_{z\to 0} x^{\frac{5}{2}}\, y\mathrm{d}x = \frac{tV}{10}.$$

We also consider

$$I = \oint_{\mathcal{B}} y\mathrm{d}x.$$

The prepotential $F_0 = \omega_{0,0}$ is worth

$$F_0 = \frac{1}{2}\left(tV - 2\frac{tV}{10} + I\epsilon\right) = \frac{2}{5}tV + \frac{1}{2}I\epsilon,$$

and satisfies

$$\mathrm{d}F_0 = V\mathrm{d}t + I\mathrm{d}\epsilon.$$

Remark 2.2. In particular, for elliptic curves, we can express the independent term of the curve V as a deformation of the prepotential $\omega_{0,0}$ with respect to the coefficient of the linear term t:

$$\frac{\partial}{\partial t}\omega_{0,0} = V. \tag{11}$$

This is a consequence of $\frac{\partial}{\partial t_{\infty,1}}\omega_{0,0} = \int_{\mathcal{B}_{\infty,1}} \omega_{0,1}$ (which is already proved in [21, Theorem 5.1]).

In terms of the torus modulus τ and scaling ν, we have

$$t = -15\nu^4 G_4(\tau), \qquad V = -35\nu^6 G_6(\tau), \qquad \epsilon = 3\nu^5 G_4'(\tau), \qquad I = 2\pi i\tau\epsilon + \frac{4}{5}\nu t.$$

We also have

$$\omega_{1,0} = F_1 = \frac{1}{48}\log\left(4t^3 + 27V^2\right) + \frac{1}{4}\log\frac{2}{\nu}.$$

3. Loop equations and deformation parameters

We start by recalling the loop equations for the topological recursion applied to any spectral curve of degree 2 with a global involution. Let $y^2 = R(x)$, with $R \in \mathbb{C}(x)$. The family of curves that we consider has the global involution $z \mapsto -z$, that is $x(z) = x(-z)$.

Let $\omega_{0,1}(z) := y(z)\mathrm{d}x(z)$, $\omega_{0,2}(z_1, z_2) := B(z_1, z_2)$ and $\omega_{g,n}$ for $2g - 2 + n > 0$ be defined as the topological recursion amplitudes for this initial data [21].

The loop equations for this particular case read:

Theorem 3.1 [6, Proposition 2.8]. *Let $g, n \in \mathbb{N}$. The linear loop equations read:*

$$\omega_{g,n+1}(z, z_1, \ldots, z_n) + \omega_{g,n+1}(-z, z_1, \ldots, z_n) = \delta_{g,0}\delta_{n,1} \frac{\mathrm{d}x(z)\mathrm{d}x(z_1)}{(x(z) - x(z_1))^2}. \tag{12}$$

The quadratic loop equations claim that the following expression

$$\frac{1}{\mathrm{d}x(z)^2}\left(\omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g,\\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\right) \tag{13}$$

is a rational function of $x(z)$ with no poles at the branch points.

We will make use of an immediate consequence of the loop equations:

Corollary 3.2. *For all $g, n \geq 0$,*

$$\begin{aligned} P_{g,n}(x(z); z_1, \ldots, z_n) := \frac{-1}{\mathrm{d}x(z)^2}\Bigg(&\omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) \\ &+ \sum_{\substack{g_1+g_2=g,\\ I_1 \sqcup I_2 = \{z_1,\ldots,z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2)\Bigg) \\ &+ \sum_{i=1}^{n} \mathrm{d}_i\left(\frac{1}{x(z) - x(z_i)} \frac{\omega_{g,n}(z_1, \ldots, -z_i, \ldots, z_n)}{\mathrm{d}x(z_i)}\right) \end{aligned} \tag{14}$$

is a rational function of $x(z)$ that has no poles at the branch points and no poles when $x(z) = x(z_i)$.

Proof. First, the expression (14) is invariant under $z \to -z$ and meromorphic on Σ, hence, a meromorphic function of $x(z) \in \mathbb{C}\mathrm{P}^1$, that is a rational function of $x(z)$. From the loop equations, it has no pole at branch points. Let us study the behavior at $z = z_i$. The only term in (13) that contains a pole at $z = z_i$ is

$$\frac{1}{\mathrm{d}x(z)^2} B(z, z_i)\, \omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n).$$

Remark that $B(-z, z_i)$ has no pole at $z = z_i$ and

$$B(z, z_i) + B(-z, z_i) = \frac{\mathrm{d}x(z)\mathrm{d}x(z_i)}{(x(z) - x(z_i))^2} = \mathrm{d}_i\left(\frac{\mathrm{d}x(z)}{x(z) - x(z_i)}\right).$$

Therefore, we add a term without any poles to the previous one and consider the term with a pole at $z = z_i$ to be

$$\frac{1}{\mathrm{d}x(z)^2} \frac{\mathrm{d}x(z)\mathrm{d}x(z_i)}{(x(z) - x(z_i))^2}\, \omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n).$$

We can write it as

$$\mathrm{d}_i\left(\left(\frac{1}{x(z)-x(z_i)}\right)\left(\frac{\omega_{g,n}(-z,z_1,\ldots,\hat{z}_i,\ldots,z_n)}{\mathrm{d}x(z)}\right.\right.$$
$$\left.\left.-\frac{\omega_{g,n}(-z_i,z_1,\ldots,\hat{z}_i,\ldots,z_n)}{\mathrm{d}x(z_i)}+\frac{\omega_{g,n}(-z_i,z_1,\ldots,\hat{z}_i,\ldots,z_n)}{\mathrm{d}x(z_i)}\right)\right). \tag{15}$$

The sum of the first two terms does not have a pole at $z = z_i$. Therefore, subtracting the last term for all $i = 1, \ldots, n$, we obtain an expression with no poles at $z = z_i$. Since this expression is an even function of z, there is no pole at $z = -z_i$ either. □

Recall that in general $\omega_{0,2} = B$ can have poles only at coinciding points and the $\omega_{g,n}$'s with $2g - 2 + n > 0$ can have poles only at ramification points. Therefore, from the corollary, we see that $P_{g,n}(x(z); z_1, \ldots, z_n)$ as a function of z can only have poles at the poles of $\omega_{0,1} = y\mathrm{d}x$.

3.1. Variational formulas of the topological recursion

We recall one of the most important properties of the topological recursion: We know how the output data behaves when we vary the family of spectral curves to which we apply the topological recursion with respect to some parameters. This result already appeared in the original paper, where topological recursion was introduced [21, Theorem 5.1] and was revisited in terms of generalized cycles in [17, Theorem 3.2]. A detailed proof of it with applications in the context of combinatorics can be found in [3, Section 3.2].

Theorem 3.3. *Let $(\Sigma, x_t, y_t\mathrm{d}x_t, B_\Sigma)$ be a holomorphic family of spectral curves, where B_Σ is the normalized bidifferential of the second kind on Σ, depending on a parameter $t \in \mathbb{C}$. Let $\omega^t_{g,n}$ be the topological recursion differentials associated to that family of spectral curves. For the base topologies, we have $\omega^t_{0,1} = y_t\mathrm{d}x_t$ and $\omega^t_{0,2} = B_\Sigma$. Assume there exists a generalized cycle γ on Σ whose support does not contain the zeroes of* d*x and such that*

$$\partial_t\, \omega^t_{0,1}(z) = \int_\gamma B(z,\cdot),$$

where the derivatives are taken with x fixed. Then,

$$\partial_t\, \omega^t_{g,n}(z_1,\ldots,z_n) = \int_\gamma \omega^t_{g,n+1}(z_1,\ldots,z_n,\cdot), \tag{16}$$

where the derivatives are taken at $x_i = x(z_i)$ fixed.

We remark that equation (16) for $(0, 2)$ follows from the formula for $\omega_{0,3}$ in terms of B_Σ [21, Theorem 4.1] and the Rauch variational formula (see [3, Lemma 3.13] and references therein).

3.2. Relation to time deformation for elliptic curves

Now we restrict ourselves to curves described by polynomials of the form

$$y^2 = x^3 + tx + V.$$

In this case, $\omega_{0,1} = y\mathrm{d}x$ can only have poles at $x(z) = \infty$.

The topological recursion amplitudes for $2g - 2 + n \geq 0$ are analytic away from branch points; in particular, they are analytic at ∞, with the following behavior near $z = \infty$:

$$\omega_{g,n}(z, z_1, \ldots, z_n) = O(z^{-2}).$$

In our case, this implies the following behavior at $x(z) = \infty$:

$$\frac{\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)} = O(x(z)^{\frac{-3}{2}}), \text{ for } 2g - 2 + n \geq 0. \tag{17}$$

Since the only pole can come from terms that contain $\omega_{0,1} = y\mathrm{d}x$, $P_{g,n}$ has the following behavior at $x(z) \to \infty$:

$$P_{g,n}(x(z); z_1, \dots, z_n) = 2\frac{y(z)\mathrm{d}x(z)\,\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)^2} + O(x(z)^{-1}),$$

that is

$$P_{g,n}(x(z); z_1, \dots, z_n) = 2y(z)O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-1}) = O(x(z)^0), \text{ for } 2g - 2 + n \geq 0, \tag{18}$$

where the last behavior comes from the fact that in the elliptic curve case, we have $y \sim x^{\frac{3}{2}}$.

We have seen that $P_{g,n}$ is a polynomial of degree 0, that is independent of z, and can be written:

$$P_{g,n}(x(z); z_1, \dots, z_n) = 2 \lim_{z \to \infty} x(z)^{\frac{3}{2}} \frac{\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)}.$$

Corollary 3.4. *For* $(g, n) \neq (0, 0), (0, 1)$*:*

$$P_{g,n}(x(z); z_1, \dots, z_n) = -\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \dots, z_n) = \frac{\partial}{\partial t}\omega_{g,n}(z_1, \dots, z_n), \tag{19}$$

with $\mathcal{B}_{\infty,1}$ *the second kind of cycle given by* $\frac{1}{2\pi i}\mathcal{C}_\infty\sqrt{x(z)}$*, where* $\mathcal{C}_\infty$ *denotes a small contour around* ∞*.*

Moreover

$$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V = x^3 + tx + \frac{\partial}{\partial t}\omega_{0,0}, \tag{20}$$

$$P_{0,1}(x(z); z_1) = 2\frac{y(z)}{\mathrm{d}x(z)}B(z, z_1) - \mathrm{d}_1\left(\frac{y(z) + y(z_1)}{x(z) - x(z_1)}\right) = \frac{\partial}{\partial t}\omega_{0,1}(z_1). \tag{21}$$

Proof. For $(g, n) = (0, 0)$, we obtain $P_{0,0}(x(z)) = y(z)^2$ from (14), which together with $V = \frac{\partial}{\partial t}\omega_{0,0}$ from (11) gives the expression for this case.

The expression for $(g, n) = (0, 1)$ is a direct computation using (14):

$$P_{0,1}(x(z); z_1) = \frac{y(z)}{\mathrm{d}x(z)}(B(z, z_1) - B(-z, z_1)) - \mathrm{d}_1\left(\frac{y(z_1)}{x(z) - x(z_1)}\right).$$

In order to get the second equality in (21), observe from the first equality that also $P_{0,1}$ has the following behavior at $x(z) \to \infty$:

$$P_{0,1}(x(z); z_1) = 2\frac{y(z)\,B(z, z_1)}{\mathrm{d}x(z)} + O(x(z)^{-1}) = 2y(z)O(x(z)^{\frac{-3}{2}}) + O(x(z)^{-1}) = O(x(z)^0).$$

For $(g,n) \neq (0,0)$, since $P_{g,n}$ is constant with respect to z, we can write

$$\begin{aligned} P(x(z); z_1, \dots, z_n) &= 2 \lim_{x(z)\to\infty} x(z)^{\frac{3}{2}} \frac{\omega_{g,n+1}(z, z_1, \dots, z_n)}{\mathrm{d}x(z)} \\ &= -\mathop{\mathrm{Res}}_{z\to 0} \sqrt{x(z)}\, \omega_{g,n+1}(z, z_1, \dots, z_n) = \\ &= -\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \dots, z_n). \end{aligned} \tag{22}$$

Moreover, since $t = -t_{\infty,1}$, we have $\frac{\partial}{\partial t}\omega_{0,1} = -\frac{\partial}{\partial t_{\infty,1}}\omega_{0,1} = -\int_{\mathcal{B}_{\infty,1}} \omega_{0,2}$, and since $B = \omega_{0,2}$ is the Bergman kernel normalized on the $\mathcal{A}$-cycles, Theorem 3.3 implies for any topology that

$$\oint_{\mathcal{B}_{\infty,1}} \omega_{g,n+1}(z, z_1, \dots, z_n) = \frac{\partial}{\partial t_{\infty,1}} \omega_{g,n}(z_1, \dots, z_n) = -\frac{\partial}{\partial t} \omega_{g,n}(z_1, \dots, z_n). \qquad \square$$

3.3. *Generalization to any plane curve with a global involution*

Our goal is to find the relation between $P_{g,n}$ from loop equations and an operator depending on time deformations acting on the topological recursion amplitudes. In this section, we generalize the relation that we have just found in the Painlevé I case to all plane curves with a global involution.

Consider algebraic curves of the form

$$y^2 = R(x), \tag{23}$$

with $R(x) \in \mathbb{C}(x)$ an arbitrary rational function.

This is parametrized by a pair of meromorphic functions x, y on a Riemann surface Σ. The ramified covering given by $x : \Sigma \to \mathbb{C}\mathrm{P}^1$ is a double cover. Depending on the parity of the behavior of y at $x \to \infty$, x has either one double pole (order $d = -2$) or two simple poles (order $d = -1$). Let us denote σ the global involution which sends $(x, y) \mapsto (x, -y)$.

Let $\{\lambda_l\}_{l=1}^N$ be the set of zeroes of the denominator of $R(x)$. The 1-form $\omega_{0,1} = y\mathrm{d}x$ can have a pole over $x = \infty$ and poles over $\{\lambda_l\}_{l=1}^N$. We call ζ_i the poles of $\omega_{0,1}$ of respective degrees given by m_i.

Let us define $d_i := \mathrm{ord}_{\zeta_i}(x)$. If ζ_i is not a pole of x, we assume that it is not a ramification point, hence, $d_i = 1$. If ζ_i is a pole of x, then d_i can be either -2 or -1, as we commented, depending on ζ_i being a ramification point or not. Notice that if ζ_i is a pole, so is $\sigma(\zeta_i)$, and thus poles come in pairs. Moreover, we can only have $\sigma(\zeta_i) = \zeta_i$, if ζ_i is a double pole of x, which corresponds to the case $d_i = -2$. Near ζ_i, we use the local variable

$$\xi_i = (x - x(\zeta_i))^{\frac{1}{d_i}}, \tag{24}$$

where we define $x(\zeta_i) = 0$, if ζ_i is a pole of x. We write the Laurent expansion:

$$y\mathrm{d}x \sim \sum_{j=0}^{m_i} t_{\zeta_i,j}\, \xi_i^{-1-j} \mathrm{d}\xi_i + \text{analytic at } \zeta_i. \tag{25}$$

This defines the local KP times [17]

$$t_{\zeta_i,j} = \mathop{\mathrm{Res}}_{\zeta_i} \xi_i^j\, y\mathrm{d}x = \frac{1}{2\pi i} \oint_{\mathcal{A}_{\zeta_i,j}} y\mathrm{d}x. \tag{26}$$

Notice that for poles for which $\sigma(\zeta_i) \neq \zeta_i$, we have

$$t_{\sigma(\zeta_i),j} = -t_{\zeta_i,j}. \tag{27}$$

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Again, loop equations imply that the $P_{g,n}(x;z_1,\dots,z_n)$ defined in (14) has no pole at coinciding points or at branch points. It must be a rational function of x, whose poles can be only at the poles of $y\mathrm{d}x$, that is at the poles of $R(x)$ and possibly at $x=\infty$.

For every second type cycle $\mathcal{B}_{p,k}$, $k\geq 1$, we define the operator $\partial_{\mathcal{B}_{p,k}}$ as

$$\partial_{\mathcal{B}_{p,k}}\omega_{g,n}(z_1,\dots,z_n) := \int_{\mathcal{B}_{p,k}}\omega_{g,n+1}(\cdot,z_1,\dots,z_n) = \operatorname*{Res}_{x\to x(p)}\frac{\xi_p^{-k}}{k}\omega_{g,n+1}(\cdot,z_1,\dots,z_n).$$

Proposition 3.5. *Let ζ_i be the poles of $\omega_{0,1}$ of respective degrees given by m_i. The operator*

$$L(x) := \sum_{i,x(\zeta_i)=\infty}\sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j}\sum_{0\leq k\leq \frac{1-j}{d_i}-2} x^k\Big(-\frac{j}{d_i}-k-2\Big)\partial_{\mathcal{B}_{\zeta_i,j+d_i(k+2)}} \tag{28}$$

$$+\sum_{i,x(\zeta_i)\neq\infty}\sum_{j=0}^{m_i} t_{\zeta_i,j}\sum_{k=0}^{j}(x-x(\zeta_i))^{-(k+1)}(j+1-k)\partial_{\mathcal{B}_{\zeta_i,j+1-k}}$$

gives

$$P_{g,n}(x;z_1,\dots,z_n) = L(x).\omega_{g,n}(z_1,\dots,z_n). \tag{29}$$

Proof. Let us write the Cauchy formula for $x=x(z)$

$$\begin{aligned}
P_{g,n}(x;z_1,\dots,z_n) &= \operatorname*{Res}_{x'\to x}\frac{\mathrm{d}x'}{x'-x}\,P_{g,n}(x';z_1,\dots,z_n)\\
&= \frac{1}{2}\operatorname*{Res}_{z'\to z}\frac{\mathrm{d}x(z')}{x(z')-x(z)}\,P_{g,n}(x(z');z_1,\dots,z_n)\\
&\quad+\frac{1}{2}\operatorname*{Res}_{z'\to\sigma(z)}\frac{\mathrm{d}x(z')}{x(z')-x(z)}\,P_{g,n}(x(z');z_1,\dots,z_n)\\
&= \frac{1}{2}\sum_i\operatorname*{Res}_{z'\to\zeta_i}\frac{\mathrm{d}x(z')}{x(z)-x(z')}\,P_{g,n}(x(z');z_1,\dots,z_n)\\
&= -\frac{1}{2}\sum_{i,x(\zeta_i)=\infty}\sum_{k\geq 0}x(z)^k\operatorname*{Res}_{z'\to\zeta_i}x(z')^{-(k+1)}\mathrm{d}x(z')\,P_{g,n}(x';z_1,\dots,z_n)\\
&\quad+\frac{1}{2}\sum_{i,x(\zeta_i)\neq\infty}\sum_{k\geq 0}\xi_i(z)^{-(k+1)}\operatorname*{Res}_{z'\to\zeta_i}\xi_i(z')^k\mathrm{d}x(z')\,P_{g,n}(x';z_1,\dots,z_n).
\end{aligned}\tag{30}$$

Since the behavior at the poles is given by the terms containing $\omega_{0,1}=y\mathrm{d}x$ in (14), near any of the poles ζ_i, we have

$$P_{g,n}(x(z);z_1,\dots,z_n)\sim\frac{2y(z)}{\mathrm{d}x(z)}\omega_{g,n+1}(z,z_1,\dots,z_n)+O(\xi_i^{-2(d_i+1)}). \tag{31}$$

First, consider the poles over $x=\infty$, with $d_i=-1$ or $d_i=-2$, which contribute to $P_{g,n}$ as

$$\begin{aligned}
&-\sum_{i,x(\zeta_i)=\infty}\sum_{k\geq 0}x(z)^k\operatorname*{Res}_{z'\to\zeta_i}x(z')^{-(k+1)}y(z')\omega_{g,n+1}(z',z_1,\dots,z_n)\\
&=-\sum_{i,x(\zeta_i)=\infty}\sum_{j=0}^{m_i}t_{\zeta_i,j}\sum_{k\geq 0}x(z)^k\operatorname*{Res}_{z'\to\zeta_i}\xi_i(z')^{-d_i(k+1)}\xi_i(z')^{-j-1}\frac{1}{d_i\,\xi_i(z')^{d_i-1}}\omega_{g,n+1}(z',z_1,\dots,z_n)
\end{aligned}$$

$$= -\sum_{i,x(\zeta_i)=\infty} \frac{1}{d_i} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geq 0} x(z)^k \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^{-d_i(k+2)-j} \omega_{g,n+1}(z', z_1, \dots, z_n)$$
$$= -\sum_{i,x(\zeta_i)=\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geq 0} x(z)^k \left(k+2+\frac{j}{d_i}\right) \int_{\mathcal{B}_{\zeta_i,j+d_i(k+2)}} \omega_{g,n+1}(z', z_1, \dots, z_n)$$
$$= -\sum_{i,x(\zeta_i)=\infty} \sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j} \sum_{0\leq k\leq \frac{1-j}{d_i}-2} x(z)^k \left(k+2+\frac{j}{d_i}\right) \partial_{\mathcal{B}_{\zeta_i,j+d_i(k+2)}} \omega_{g,n}(z_1, \dots, z_n).$$

The finite poles contribute as

$$\frac{1}{2} \sum_{i,x(\zeta_i)\neq\infty} \sum_{k\geq 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^k dx(z')\, P_{g,n}(x'; z_1, \dots, z_n)$$
$$= \sum_{i,x(\zeta_i)\neq\infty} \sum_{k\geq 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^k y(z') \omega_{g,n+1}(z', z_1, \dots, z_n)$$
$$= \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k\geq 0} \xi_i(z)^{-(k+1)} \operatorname*{Res}_{z'\to\zeta_i} \xi_i(z')^{k-j-1} \omega_{g,n+1}(z', z_1, \dots, z_n)$$
$$= \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} \xi_i(z)^{-(k+1)} (j+1-k) \int_{\mathcal{B}_{\zeta_i,j+1-k}} \omega_{g,n+1}(z', z_1, \dots, z_n)$$
$$= \sum_{i,x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} \xi_i(z)^{-(k+1)} (j+1-k) \partial_{\mathcal{B}_{\zeta_i,j+1-k}} \omega_{g,n}(z_1, \dots, z_n). \qquad \square$$

Let us now rewrite the operator $L(x)$ as a differential operator with respect to the moduli of the spectral curve, that is the times appearing in the pole structure of $\omega_{0,1}$ and the poles $\Lambda = \{\lambda_l\}_{l=1}^N$ of $R(x)$.

Lemma 3.6. *Let $\zeta_l^{(1)}$ and $\zeta_l^{(2)}$ be the two preimages of λ_l by x, which are poles of order m_l of $\omega_{0,1}$. We assume that the filling fractions ϵ_i are independent of the poles λ_l, that is*

$$\frac{\partial}{\partial \lambda_l} \epsilon_i = \oint_{\mathcal{A}_i} \frac{\partial}{\partial \lambda_l} \omega_{0,1} = 0.$$

Then we can decompose the operator derivative with respect to λ_l as

$$\frac{\partial}{\partial \lambda_l} \omega_{g,n}(z_1, \dots, z_n) = \sum_{i=1,2} \sum_{j=0}^{m_l} (j+1) t_{\zeta_l^{(i)},j} \partial_{\mathcal{B}_{\zeta_l^{(i)},j+1}} \omega_{g,n}(z_1, \dots, z_n), \tag{32}$$

for all $g, n \geq 0$.

Proof. Let ζ_l be one of the two preimages of λ_l by x and $\xi_l = x - \lambda_l$ be the local variable around it. Then, for $z \to \zeta_l$, we have

$$\frac{\partial}{\partial \lambda_l} \omega_{0,1}(z) \sim \sum_{j=0}^{m_l} (j+1) t_{\zeta_l,j} (\xi_l(z))^{-j-2} \mathrm{d}\xi_l + \text{ analytic}.$$

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On the other hand, we have

$$\partial_{\mathcal{B}_{\zeta_l,j}}\omega_{0,1}(z) = \int_{\mathcal{B}_{\zeta_l,j}} \omega_{0,2}(\cdot,z) = \operatorname*{Res}_{z'=\lambda_l} \frac{\omega_{0,2}(z',z)}{j(\xi_p(z'))^j}$$
$$= -\operatorname*{Res}_{z'=z} \frac{\omega_{0,2}(z',z)}{j(\xi_p(z'))^j} + \text{ analytic} = \frac{\mathrm{d}\xi_l(z)}{(\xi_l(z))^{j+1}} + \text{ analytic}.$$

The last equality follows from the fact that the fundamental differential of the second kind $\omega_{0,2}$ only has a double pole along the diagonal and the next-to-last equality follows from the decomposition

$$\oint_{\gamma_{\lambda_l}} \frac{\omega_{0,2}(\cdot,z)}{j(\xi_p(\cdot))^j} = \oint_{\gamma_{z,\lambda_l}} \frac{\omega_{0,2}(\cdot,z)}{j(\xi_p(\cdot))^j} - \oint_{\gamma_z} \frac{\omega_{0,2}(\cdot,z)}{j(\xi_p(\cdot))^j} = \text{ analytic} - \oint_{\gamma_z} \frac{\omega_{0,2}(\cdot,z)}{j(\xi_p(\cdot))^j},$$

where γ_z and γ_{λ_l} are small counterclockwise contours around z and λ_l, and γ_{z,λ_l} is a counterclockwise contour that surrounds the segment from z to λ_l, which gives rise to an analytic term.

Therefore,

$$\frac{\partial}{\partial\lambda_l}\omega_{0,1}(z) - \sum_{i=1,2}\sum_{j=0}^{m_l}(j+1)t_{\zeta_l^{(i)},j}\partial_{\mathcal{B}_{\zeta_l^{(i)},j+1}}\omega_{0,1}(z)$$

is a holomorphic 1-form with vanishing $\mathcal{A}$-periods, and thus it vanishes.

This yields the equality for $\omega_{0,1}$. Since we chose $\omega_{0,2}$ to be the Bergman kernel with vanishing $\mathcal{A}$-periods, we obtain the equality for the rest of the $\omega_{g,n}$ from Theorem 3.3. □

Corollary 3.7. *Let $\zeta_\infty \in x^{-1}(\infty)$ and $\zeta_l \in x^{-1}(\lambda_l)$ be poles of $\omega_{0,1}$ of orders m_∞ and m_l, $l = 1,\dots,N$, respectively. Let $d_\infty := \operatorname{ord}_{\zeta_\infty}(x)$. The operator (28) is equal to the differential operator $L(x) = L_\infty(x) + L_\Lambda(x)$, with*

$$L_\infty(x) = \sum_{j=1-2d_\infty}^{m_\infty} t_{\zeta_\infty,j}\sum_{k=0}^{\frac{1-j}{d_\infty}-2} x^k\Big(-\frac{j}{d_\infty}-k-2\Big)\frac{\partial}{\partial t_{\zeta_\infty,j+d_\infty(k+2)}} \tag{33}$$

and

$$L_\Lambda(x) = \sum_{l=1}^{N}\left(\frac{1}{x-\lambda_l}\frac{\partial}{\partial\lambda_l} + \sum_{j=1}^{m_l-1}t_{\zeta_l,j}\sum_{k=1}^{j}(x-\lambda_l)^{-(k+1)}(j+1-k)\frac{\partial}{\partial t_{\zeta_l,j+1-k}}\right). \tag{34}$$

Proof. From the variational formulas of topological recursion (see Theorem 3.3), if ζ is a ramified pole of $\omega_{0,1}$ of order m, that is $x(\zeta) = \infty$ and $\operatorname{ord}_\zeta(x) = -2$, then we have

$$\frac{\partial}{\partial t_{\zeta,j}}\omega_{g,n}(z_1,\dots,z_n) = \partial_{\mathcal{B}_{\zeta,j}}\omega_{g,n}(z_1,\dots,z_n), \text{ for } j = 1,\dots,m.$$

On the other hand, if ζ is an unramified pole of $\omega_{0,1}$ of order m, either $x^{-1}(\infty) = \{\zeta,\sigma(\zeta)\}$ and $\operatorname{ord}_\zeta(x) = -1$ or $x^{-1}(\lambda_l) = \{\zeta,\sigma(\zeta)\}$ and $\operatorname{ord}_\zeta(x) = 1$, then the coefficients of the expansion of $\omega_{0,1}$ around $\zeta = \zeta_+$ and the other preimage $\sigma(\zeta) = \zeta_-$ are not independent. Therefore, the variational formulas include residues at the two preimages:

$$\frac{\partial}{\partial t_{\zeta,j}}\omega_{g,n}(z_1,\dots,z_n) = \partial_{\mathcal{B}_{\zeta,j}}\omega_{g,n}(z_1,\dots,z_n) - \partial_{\mathcal{B}_{\sigma(\zeta),j}}\omega_{g,n}(z_1,\dots,z_n), \text{ for } j = 1,\dots,m.$$

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The variational formulas allow to rewrite all the operators $\partial_{\mathcal{B}_{\zeta,j}}$ in (28), for $j = 1, \ldots, m$, in terms of derivatives with respect to the times $t_{\zeta,j}$ appearing as coefficients of the polar part of $\omega_{0,1}$. This is enough to rewrite the first line of (28) as $L_\infty(x)$ in (33).

For $L_\Lambda(x)$, we make use of (32) to express the remaining operators $\partial_{\mathcal{B}_{\zeta,m_l+1}}$ in terms of allowed times $t_{\zeta,j}$, $j = 1, \ldots, m$, and derivatives with respect to the poles λ_l of $R(x)$:

$$(m_l+1)t_{\zeta_l,m_l}\xi_l^{-1}(\partial_{\mathcal{B}_{\zeta_l,m_l+1}} - \partial_{\mathcal{B}_{\sigma(\zeta_l),m_l+1}}) = \xi_l^{-1}\left(\frac{\partial}{\partial\lambda_l} - \sum_{j=0}^{m_l-1}(j+1)t_{\zeta_l,j}\frac{\partial}{\partial t_{\zeta_l,j+1}}\right). \tag{35}$$

The second type of terms of the right-hand side (RHS) of (35) cancels the terms with $k = 0$ from the second line of (28), yielding (34). □

We have just found a differential operator $L(x)$ in the times and the poles of $R(x)$, whose coefficients are rational functions of x, with poles at $x = \infty$ or $x = x(\zeta_i)$, that is the same poles as $R(x)$, with at most the same degrees.

Example 3.1. In the elliptic case of curves of the form $y^2 = x^3 + tx + V$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 5$, with $d_i = -2$. The only nonvanishing times are $t_{\infty,5} = -2$ and $t_{\infty,1} = -t$, and thus only the terms with $j = 5$ and $k = 0$ contribute:

$$\begin{aligned} L(x) &= \sum_{j=1,5} t_{\infty,j} \sum_{0\le k\le -2+(j-1)/2} x^k (j/2-k-2)\frac{\partial}{\partial t_{\infty,j-2(k+2)}} \\ &= t_{\infty,5}(5/2-0-2)\frac{\partial}{\partial t_{\infty,5-2(0+2)}} \\ &= -2\left(\frac{5}{2}-2\right)\frac{\partial}{\partial t_{\infty,1}} \\ &= -\frac{\partial}{\partial t_{\infty,1}} \\ &= \frac{\partial}{\partial t}. \end{aligned}$$

Example 3.2. In the Airy case, $y^2 = x$, we have only one pole, at $\zeta_i = \infty$, of degree $m_i = 3$, with $d_i = -2$. The sum is empty and

$$L(x) = 0.$$

Remark 3.3. More generally, the admissible curves considered in [9] are those for which

$$L(x) = 0.$$

4. PDEs quantizing any hyperelliptic curve

For $r \ge 1$, let $D = \sum_{i=1}^r \alpha_i[p_i]$ be a divisor on Σ, with $p_i \in \Sigma$. We call $\sum_i \alpha_i$ the *degree* of the divisor and denote $\mathrm{Div}_0(\Sigma)$ the set of divisors of degree 0. For $D \in \mathrm{Div}_0(\Sigma)$, we define the integration of a 1-form $\rho(z)$ on Σ as

$$\int_D \rho(z) := \sum_i \alpha_i \int_o^{p_i} \rho(z), \tag{36}$$

where $o \in \Sigma$ is an arbitrary base point. This integral is well-defined locally, meaning that it is independent of the base point o because the degree of the divisor is zero, however, it depends on a choice of homotopy class from o to p_i.

For $(g,n) \neq (0,2)$, consider the functions of D, defined locally:

$$F_{g,0}(D) \coloneqq F_g = \omega_{g,0},$$

$$F_{g,n}(D) \coloneqq \overbrace{\int_D \cdots \int_D}^{n} \omega_{g,n}(z_1,\dots,z_n),$$

$$F'_{g,n}(z;D) \coloneqq \frac{1}{\mathrm{d}x(z)} \overbrace{\int_D \cdots \int_D}^{n-1} \omega_{g,n}(z,z_2\dots,z_n),$$

$$F''_{g,n}(z,\tilde{z};D) \coloneqq \frac{1}{\mathrm{d}x(z)\mathrm{d}x(\tilde{z})} \overbrace{\int_D \cdots \int_D}^{n-2} \omega_{g,n}(z,\tilde{z},z_3,\dots,z_n).$$

Recall that

$$B(z_1,z_2) \coloneqq \mathrm{d}_1\mathrm{d}_2 \log\Big(E(z_1,z_2)\sqrt{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}\Big),$$

with $E(z_1,z_2)$ being the prime form, which is defined in [21], and satisfies that it vanishes only if $z_1 = z_2$ with a simple zero and has no pole.

For $(g,n) = (0,2)$, define:

$$F_{0,2}(D) \coloneqq 2\sum_{i<j} \alpha_i\alpha_j \log\Big(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\Big), \tag{37}$$

$$F'_{0,2}(z;D) \coloneqq \frac{1}{\mathrm{d}x(z)} \mathrm{d}_z\left(\sum_{i=1}^{r} \alpha_i \log\Big(E(z,p_i)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_i)}\Big)\right), \tag{38}$$

$$F''_{0,2}(z,\tilde{z};D) \coloneqq \frac{B(z,\tilde{z})}{\mathrm{d}x(z)\mathrm{d}x(\tilde{z})}. \tag{39}$$

Since the $\omega_{g,n}$ are symmetric, we have the following relations for $(g,n) \neq (0,2)$:

$$\frac{\mathrm{d}}{\mathrm{d}x_i} F_{g,n}(D) = n\alpha_i F'_{g,n}(p_i;D), \tag{40}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\right)^2 F_{g,n}(D) = n(n-1)\alpha_i^2 F''_{g,n}(p_i,p_i;D) + n\alpha_i \left(\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} F'_{g,n}(\tilde{p};D)\right)_{\tilde{p}=p_i}, \tag{41}$$

where $x_i \coloneqq x(p_i)$, $\tilde{x} \coloneqq x(\tilde{p})$, and $\frac{\mathrm{d}}{\mathrm{d}x}$ acts on meromorphic functions by taking the exterior derivative and dividing by $\mathrm{d}x$, which amounts to derivate an analytic expansion of the meromorphic function with respect to a local variable x.

For $(g,n) = (0,2)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_i} F_{0,2}(D) &= 2\alpha_i \lim_{z\to p_i}\left(F'_{0,2}(z;D) - \alpha_i \frac{\mathrm{d}}{\mathrm{d}x(z)} \log\Big(E(z,p_i)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_i)}\Big)\right) \\ &= 2\alpha_i \sum_{j\neq i} \alpha_j \frac{\mathrm{d}}{\mathrm{d}x(p_i)} \log\Big(E(p_i,p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)}\Big). \end{aligned} \tag{42}$$

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Remark 4.1. Integrating the first part of Theorem 3.1 over a divisor D of degree 0, we obtain:

$$F'_{0,2}(z;D) + F'_{0,2}(-z;D) = \sum_{i=1}^{r} \frac{\alpha_i}{x(z) - x(p_i)}. \tag{43}$$

Lemma 4.1. *Let ζ be a pole of $\omega_{0,1}$ and $\mathcal{B}_{\zeta,k}$ the kth second kind cycle around ζ, with $k \geq 1$. Consider $t_{\zeta,k}$ the corresponding KP time. Then,*

$$\int_D \int_D \int_{\mathcal{B}_{\zeta,k}} \omega_{0,3}(z, z_1, z_2) = \int_D \int_D \frac{\partial}{\partial t_{\zeta,k}} \omega_{0,2}(z_1, z_2) = \frac{\partial}{\partial t_{\zeta,k}} F_{0,2}(D). \tag{44}$$

Proof.

$$\int_D \int_D \left(B(z_1, z_2) - \frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1) - x(z_2))^2} \right) + 2 \sum_{i<j} \alpha_i \alpha_j \log(x_i - x_j) = \tag{45}$$

$$2 \sum_{i<j} \alpha_i \alpha_j \log\left(\frac{E(p_i, p_j)\sqrt{\mathrm{d}x_i \mathrm{d}x_j}}{x_i - x_j} \right) + 2 \sum_{i<j} \alpha_i \alpha_j \log(x_i - x_j) + \sum_i \alpha_i^2 \log \frac{\mathrm{d}x_i}{\mathrm{d}x_i} = F_{0,2}(D).$$

Taking the derivative with respect to $t_{\zeta,k}$ of the first line gives the left-hand side of (44) because we are taking this derivative at fixed x. □

Consider $\mathrm{d}E(z, p_1) := \mathrm{d}_z \log\Big(E(z, p_1)\sqrt{\mathrm{d}x(z)\mathrm{d}x(p_1)}\Big)$, and observe that

$$\lim_{z \to p_1} \frac{\mathrm{d}E(z, p_1)}{\mathrm{d}x(z)} - \frac{1}{x(z) - x(p_1)} = 0. \tag{46}$$

With this notation, one can rewrite (38) as

$$F'_{0,2}(z;D) = \sum_{i=1}^{r} \alpha_i \frac{\mathrm{d}E(z, p_i)}{\mathrm{d}x(z)}$$

and $\frac{\mathrm{d}}{\mathrm{d}x_i} F_{0,2}(D) = 2\alpha_i \sum_{j \neq i} \alpha_j \frac{\mathrm{d}E(p_i, p_j)}{\mathrm{d}x(p_i)}$.

We define

$$S_0(D) := \int_D y\mathrm{d}x = F_{0,1}(D), \quad S_1(D) := \log \prod_{i<j} \left(E(p_i, p_j)\sqrt{\mathrm{d}x(p_i)\mathrm{d}x(p_j)} \right)^{\alpha_i \alpha_j} = \frac{F_{0,2}(D)}{2},$$

$$S_m(D) := \sum_{\substack{2g-2+n=m-1 \\ g \geq 0, n \geq 1}} \frac{F_{g,n}(D)}{n!},$$

and

$$\psi = \psi(D, \hbar) := \exp(S(D, \hbar)), \text{ with } S(D, \hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(D). \tag{47}$$

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Theorem 4.2. *Let $F := \sum_{g>0} \hbar^{2g-2} F_g$. For every $k = 1, \ldots, r$, assuming $\alpha_k^2 = 1$, we obtain*

$$\hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x_k^2} - \sum_{i \neq k} \frac{\frac{\mathrm{d}}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_k} \frac{\mathrm{d}}{\mathrm{d}x_k}}{x_k - x_i} - L(x_k) - L(x_k).F + \sum_{\substack{i \neq j \\ i \neq k, j \neq k}} \frac{\alpha_i \alpha_j}{(x_k - x_i)(x_i - x_j)} \right) \psi = R(x_k)\psi. \tag{48}$$

Proof. We will give the proof of the claim for the case $k = 1$, but it works exactly the same for every k.

Let us first consider the generic situation with $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$. Using (14) and (29) for $n = 0$, we can write $P_{g,0}(x)$, with $x = x(z)$, as

$$F''_{g-1,2}(z, z; D) + \sum_{g_1+g_2=g} F'_{g_1,1}(z; D) F'_{g_2,1}(z; D) = L(x).F_{g,0}(D) = L(x).F_g. \tag{49}$$

Setting $z = p_1, x_1 = x(p_1)$ and using (40) and (41), we obtain

$$\frac{1}{2\alpha_1^2} \left(\frac{\mathrm{d}}{\mathrm{d}x_1} \right)^2 F_{g-1,2}(D) - \frac{1}{\alpha_1} \left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})} F'_{g-1,2}(\tilde{p}; D) \right)_{\tilde{p}=p_1} + \sum_{g_1+g_2=g} \frac{1}{\alpha_1^2} \frac{\mathrm{d}F_{g_1,1}(D)}{\mathrm{d}x_1} \frac{\mathrm{d}F_{g_2,1}(D)}{\mathrm{d}x_1} \tag{50}$$
$$= L(x_1).F_g(D).$$

For $n > 0$, we integrate (14) n times over D and use (29) to get

$$F''_{g-1,n+2}(z, z; D) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no } (0,2)} \binom{n}{n_1} F'_{g_1,n_1+1}(z; D) F'_{g_2,n_2+1}(z; D) +$$
$$- 2nF'_{0,2}(-z; D) F'_{g,n}(z; D) - n \sum_{i=1}^{r} \alpha_i \frac{F'_{g,n}(p_i; D) - F'_{g,n}(z; D)}{x(z) - x(p_i)} = L(x).F_{g,n}(D),$$

where the sum of the second term is taken over $g_i, n_i \geq 0$ and "no $(0, 2)$" means we exclude the cases with $(g_i, n_i) = (0, 1)$ for $i = 1, 2$, for which we have used that

$$(F'_{0,2}(z; D) - F'_{0,2}(-z; D)) F'_{g,n}(z; D) = -2F'_{0,2}(-z; D) F'_{g,n}(z; D) + \sum_{i=1}^{r} \alpha_i \frac{F'_{g,n}(z; D)}{x(z) - x(p_i)},$$

which follows from (43).

Letting $z = p_1, x_i = x(p_i)$, dividing by $n!$ and using (40) and (41), we obtain

$$\frac{1}{(n+2)!\alpha_1^2} \left(\frac{\mathrm{d}}{\mathrm{d}x_1} \right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1} \left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})} F'_{g-1,n+2}(\tilde{p}; D) \right)_{\tilde{p}=p_1} +$$
$$\sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}}^{\text{no } (0,2)} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1} \frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1}$$
$$- \frac{1}{n!} \sum_{i=2}^{r} \frac{1}{x(p_1) - x(p_i)} \left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i} - \frac{\alpha_i}{\alpha_1} \frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1} \right) + \frac{\alpha_1}{(n-1)!} \left(\frac{\mathrm{d}F'_{g,n}(\tilde{p}; D)}{\mathrm{d}x(\tilde{p})} \right)_{\tilde{p}=p_1}$$
$$- \frac{2}{n!\alpha_1} \frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1} F'_{0,2}(-p_1; D) = L(x_1).\frac{F_{g,n}(D)}{n!}.$$

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Using (43) again, we obtain the following expression for the left-hand side:

$$\begin{aligned} &\frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})}F'_{g-1,n+2}(\tilde{p};D)\right)_{\tilde{p}=p_1} + \\ &\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n}}^{\text{no }(0,2)} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2}\frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1} \\ &+\frac{\alpha_1}{(n-1)!}\left(\frac{\mathrm{d}F'_{g,n}(\tilde{p};D)}{\mathrm{d}x(\tilde{p})}\right)_{\tilde{p}=p_1} - \frac{1}{n!}\sum_{i=2}^{r}\frac{1}{x(p_1)-x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right) \\ &- \frac{2}{n!\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\left(F'_{0,2}(\tilde{p};D) - \frac{\alpha_1}{x(\tilde{p})-x(p_1)}\right)_{\tilde{p}=p_1} \\ &= \frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})}F'_{g-1,n+2}(\tilde{p};D)\right)_{\tilde{p}=p_1} \\ &+\frac{\alpha_1}{(n-1)!}\left(\frac{\mathrm{d}F'_{g,n}(\tilde{p};D)}{\mathrm{d}x(\tilde{p})}\right)_{\tilde{p}=p_1} + \sum_{\substack{g_1+g_2=g\\ n_1+n_2=n}}\frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2}\frac{\mathrm{d}F_{g_1,n_1+1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{g_2,n_2+1}(D)}{\mathrm{d}x_1} \\ &- \frac{1}{n!}\sum_{i=2}^{r}\frac{1}{x(p_1)-x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right), \end{aligned} \tag{51}$$

where for the last equality, we have used (42) and (46).

For all $\ell \geq 3$, we sum this expression for all $g \geq 0, n \geq 1$ such that $2g-2+n = \ell-2$ and the corresponding $P_{g,0}(x(z_1))$ for $n=0$ from (50) for all $g \geq 0$, such that $2g-2 = \ell-2$ to obtain:

$$\begin{aligned} &\sum_{\substack{2g+n=\ell\\ g\geq 0, n\geq 0}}\Bigg(\frac{1}{(n+2)!\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{g-1,n+2}(D) \\ &\qquad - \frac{1}{(n+1)!\alpha_1}\left(\frac{\mathrm{d}}{\mathrm{d}x(\tilde{p})}F'_{g-1,n+2}(\tilde{p};D)\right)_{\tilde{p}=p_1} + \frac{\alpha_1}{(n-1)!}\left(\frac{\mathrm{d}F'_{g,n}(\tilde{p};D)}{\mathrm{d}x(\tilde{p})}\right)_{\tilde{p}=p_1} \\ &\qquad - \frac{1}{n!}\sum_{i=2}^{r}\frac{1}{x(p_1)-x(p_i)}\left(\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{g,n}(D)}{\mathrm{d}x_1}\right)\Bigg) \\ &+\frac{1}{\alpha_1^2}\sum_{\ell_1+\ell_2=\ell}\left(\sum_{\substack{2g_1-2+n_1=\ell_1-1\\ g_1\geq 0, n_1\geq 1}}\frac{1}{n_1!}\frac{\mathrm{d}F_{g_1,n_1}(D)}{\mathrm{d}x_1}\sum_{\substack{2g_2-2+n_2=\ell_2-1\\ g_1\geq 0, n_2\geq 1}}\frac{1}{n_2!}\frac{\mathrm{d}F_{g_2,n_2}(D)}{\mathrm{d}x_1}\right) = \sum_{\substack{2g+n=\ell\\ g\geq 0, n\geq 0}} L(x_1).\frac{F_{g,n}}{n!}. \end{aligned}$$

In this sum over topologies, assuming that $\frac{1}{\alpha_1} = \alpha_1$, all the first terms of the second line cancel against all the second terms except the ones with $n=1$, which complete the terms of the first line, using (41) for the special case $n=1$ in which only the second term of the right-hand side survives.

Therefore, for $\ell \geq 3$, we have proved

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_{\ell-1} + \frac{1}{\alpha_1^2}\sum_{\ell_1+\ell_2=\ell}\frac{\mathrm{d}}{\mathrm{d}x_1}S_{\ell_1}\frac{\mathrm{d}}{\mathrm{d}x_1}S_{\ell_2} - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S_{\ell-1}}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_{\ell-1}}{\mathrm{d}x_1}}{x(p_1)-x(p_i)}$$

$$= L(x_1).S_{\ell-1} + \begin{cases} L(x_1).F_{\ell/2}, & \ell \text{ even},\\ 0, & \ell \text{ odd},\end{cases}$$

that is

$$[\hbar^\ell]\left[\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^r \frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)}\right)\right] = \tag{52}$$

$$[\hbar^\ell]\left[\hbar^2(L(x_1).S + L(x_1).F) + R(x_1)\right].$$

Let us finally consider the special cases before the assumption $\alpha_1^2 = 1$:

- For $(g,n) = (0,0)$, we get

$$\frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\right)^2 = R(x_1), \quad \text{i.e.} \quad \frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\right)^2 = R(x_1), \quad \text{or} \tag{53}$$

$$[\hbar^0]\left(\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^r \frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} - L(x_1).S - L(x_1).F\right) - R(x_1)\right) = 0.$$

- For $(g,n) = (0,1)$, we get

$$2F'_{0,1}(z;D)F'_{0,2}(z;D) - \sum_{i=1}^r \alpha_i \frac{F'_{0,1}(p_i;D) + F'_{0,1}(z;D)}{x(z)-x(p_i)} = L(x).F_{0,1}(D).$$

Thus

$$2F'_{0,1}(z;D)F'_{0,2}(z;D) - 2\alpha_1\frac{F'_{0,1}(z;D)}{x(z)-x(p_1)} - \alpha_1\frac{F'_{0,1}(p_1,D) - F'_{0,1}(z;D)}{x(z)-x(p_1)} - \sum_{i=2}^r \alpha_i \frac{F'_{0,1}(p_i;D) + F'_{0,1}(z;D)}{x(z)-x(p_i)} = L(x).F_{0,1}(D).$$

At $z = p_1$, this gives

$$2F'_{0,1}(p_1;D)\left(F'_{0,2}(z;D) - \alpha_1\frac{1}{x(z)-x(p_1)}\right)_{z=p_1} + \alpha_1\left(\frac{\mathrm{d}F'_{0,1}(z;D)}{\mathrm{d}x}\right)_{z=p_1} - \sum_{i=2}^r \frac{1}{x(p_1)-x(p_i)}\left(\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\right) = L(x_1).F_{0,1}(D),$$

and equivalently:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{0,1}(D) + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,1}(D)\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,2}(D) - \sum_{i=2}^r \frac{\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} = L(x_1).F_{0,1}(D),$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_0 + \frac{2}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\frac{\mathrm{d}}{\mathrm{d}x_1}S_1 - \sum_{i=2}^r \frac{\frac{\mathrm{d}S_0}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_0}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} = L(x_1).S_0,$$

$$[\hbar]\left(\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^r \frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1)-x(p_i)} - L(x_1).S - L(x_1).F\right) - R(x_1)\right) = 0.$$

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• For $(g,n)=(0,2)$, we first rewrite (14):

$$P_{0,2}(x(z),z_1,z_2)-2y(z)\frac{\omega_{0,3}(z,z_1,z_2)}{\mathrm{d}x}= \tag{54}$$

$$-\frac{B(z,z_1)B(-z,z_2)}{(\mathrm{d}x)^2}-\frac{B(-z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}+\mathrm{d}_1\frac{B(z_2,-z_1)}{(x-x_1)\mathrm{d}x_1}+\mathrm{d}_2\frac{B(z_1,-z_2)}{(x-x_2)\mathrm{d}x_2}=$$
$$2\frac{B(z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}+\mathrm{d}_1\frac{1}{x-x_1}\left(\frac{B(z_2,-z_1)}{\mathrm{d}x_1}-\frac{B(z,z_2)}{\mathrm{d}x}\right)+\mathrm{d}_2\frac{1}{x-x_2}\left(\frac{B(z_1,-z_2)}{\mathrm{d}x_2}-\frac{B(z,z_1)}{\mathrm{d}x}\right)=$$
$$2\frac{B(z,z_1)B(z,z_2)}{(\mathrm{d}x)^2}-\mathrm{d}_1\frac{1}{x-x_1}\left(\frac{B(z_2,z_1)}{\mathrm{d}x_1}+\frac{B(z,z_2)}{\mathrm{d}x}\right)-\mathrm{d}_2\frac{1}{x-x_2}\left(\frac{B(z_1,z_2)}{\mathrm{d}x_2}+\frac{B(z,z_1)}{\mathrm{d}x}\right)$$
$$+\mathrm{d}_1\mathrm{d}_2\frac{1}{x-x_1}\frac{1}{x-x_2}.$$

Now we integrate twice over D:

$$\int_D\int_D P_{0,2}(x(z),z_1,z_2)-2F'_{0,1}(z;D)F'_{0,3}(z;D)= \tag{55}$$
$$2(F'_{0,2}(z;D))^2-2\sum_{i=1}^r\alpha_i\frac{1}{x-x_i}F'_{0,2}(z;D)-\sum_{i=1}^r\frac{2\alpha_i}{x-x_i}\sum_{j\neq i}\alpha_j\left(\frac{\mathrm{d}E(p_i,p_j)}{\mathrm{d}x_i}-\frac{1}{x_i-x_j}\right).$$

We introduce $\hat F'_{0,2}(z;D)=F'_{0,2}(z;D)-\alpha_1\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}$ and obtain:

$$2(\hat F'_{0,2}(z;D))^2+4\alpha_1\hat F'_{0,2}(z;D)\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}+2\alpha_1^2\frac{(\mathrm{d}E(z,p_1))^2}{(\mathrm{d}x)^2}-\frac{2\alpha_1}{x-x_1}\hat F'_{0,2}(z;D)- \tag{56}$$
$$\frac{2\alpha_1^2}{x-x_1}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}-2\sum_{i\neq 1}\frac{\alpha_i}{x-x_i}\hat F'_{0,2}(z;D)-2\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{x-x_i}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}-$$
$$\frac{1}{x-x_1}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,2}(D)-\sum_{i\neq 1}\frac{1}{x-x_i}\frac{\mathrm{d}}{\mathrm{d}x_i}F_{0,2}(D)+2\sum_{\substack{i\neq j\\ i\neq 1}}\frac{\alpha_i\alpha_j}{(x-x_i)(x_i-x_j)}+2\sum_{j\neq 1}\frac{\alpha_1\alpha_j}{(x-x_1)(x_1-x_j)}.$$

Observe that

$$\lim_{z\to p_1}\left(-\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{x-x_i}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x}+\sum_{j\neq 1}\frac{\alpha_1\alpha_j}{(x-x_1)(x_1-x_j)}\right)=\sum_{i\neq 1}\frac{\alpha_1\alpha_i}{(x_1-x_i)^2}. \tag{57}$$

Using this, at $z=p_1$, we obtain

$$\frac{1}{2\alpha_1^2}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2F_{0,2}(D)-\sum_{i\neq 1}\frac{1}{x_1-x_i}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_i}+\frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)+ \tag{58}$$
$$2\alpha_1^2\mathcal{S}(p_1)+2\sum_{\substack{i\neq j\\ i\neq 1,j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1-x_i)(x_i-x_j)},$$

where we have called $\mathcal{S}(p_1)$ the limit

$$\lim_{z\to p_1}\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x(z)}\left(\frac{\mathrm{d}E(z,p_1)}{\mathrm{d}x(z)}-\frac{1}{x(z)-x(p_1)}\right).$$

Using (29) in this special case, together with Lemma 4.1, we obtain

$$\int_D \int_D P_{0,2}(x; z_1, z_2) = \int_D \int_D L(x).\omega_{0,2}(z_1, z_2) = L(x).F_{0,2}(D). \tag{59}$$

For the first term of (55), we thus obtain:

$$L(x_1).\frac{F_{0,2}(D)}{2} = \frac{1}{3\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,1}(D)\frac{\mathrm{d}}{\mathrm{d}x_1}F_{0,3}(D) + \frac{1}{4\alpha_1^2}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 F_{0,2}(D) + \alpha_1^2\mathcal{S}(p_1) - \frac{1}{2}\sum_{i\neq 1}\frac{1}{x_1 - x_i}\left(\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}F_{0,2}(D)}{\mathrm{d}x_1}\right) + \sum_{\substack{i\neq j\\ i\neq 1, j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1 - x_i)(x_i - x_j)}. \tag{60}$$

- For $(g, n) = (1, 0)$, we get

$$\frac{-B(z,-z)}{\mathrm{d}x(z)^2} + 2F'_{0,1}(z; D)F'_{1,1}(z; D) = L(x_1).F_{1,0}(D).$$

At $z = p_1$, this gives

$$\frac{-B(p_1,-p_1)}{\mathrm{d}x_1^2} + 2\frac{1}{\alpha_1^2}\frac{\mathrm{d}F_{0,1}(D)}{\mathrm{d}x_1}\frac{\mathrm{d}F_{1,1}(D)}{\mathrm{d}x_1} = L(x_1).F_{1,0}(D).$$

Using that $\frac{B(p_1,-p_1)}{\mathrm{d}x_1^2} = \mathcal{S}(p_1)$ and summing the expressions for $(0, 2)$ and $(1, 0)$, we obtain:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S_1 + \frac{1}{\alpha_1^2}\left(\frac{\mathrm{d}}{\mathrm{d}x_1}S_1\right)^2 + \frac{2}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S_0\frac{\mathrm{d}}{\mathrm{d}x_1}S_2 - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S_1}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S_1}{\mathrm{d}x_1}}{x(p_1) - x(p_i)} + (\alpha_1^2 - 1)\mathcal{S}(p_1) + \sum_{\substack{i<j\\ i\neq 1}}\frac{\alpha_i\alpha_j}{(x_1 - x_i)(x_1 - x_j)} = L(x_1).S_1 + L(x_1).F_1, \tag{61}$$

that is

$$[\hbar^2]\left[\hbar^2\left(\left(\frac{\mathrm{d}}{\mathrm{d}x_1}\right)^2 S + \frac{1}{\alpha_1^2}\frac{\mathrm{d}}{\mathrm{d}x_1}S\frac{\mathrm{d}}{\mathrm{d}x_1}S - \sum_{i=2}^{r}\frac{\frac{\mathrm{d}S}{\mathrm{d}x_i} + \frac{\alpha_i}{\alpha_1}\frac{\mathrm{d}S}{\mathrm{d}x_1}}{x(p_1) - x(p_i)} + (\alpha_1^2 - 1)\mathcal{S}(p_1) + (\star)\right)\right] = [\hbar^2]\left[\hbar^2(L(x_1).S + L(x_1).F) + R(x_1)\right], \tag{62}$$

with

$$(\star) = \sum_{\substack{i\neq j\\ i\neq 1, j\neq 1}}\frac{\alpha_i\alpha_j}{(x_1 - x_i)(x_i - x_j)}. \tag{63}$$

From the assumption $\alpha_1 = \frac{1}{\alpha_1}$, and summing over all topologies, we get the claim. □

Remark 4.2. Very often in the literature a different convention is used to regularize the $(0, 2)$ term of the wave function:

$$\tilde{\psi}(D, t, \hbar) := \exp\left(\tilde{S}_1(D, t) + \sum_{m\geq 0, m\neq 1}\hbar^{m-1}S_m(D, t)\right),$$

where $\tilde{S}_1(D,t) := \frac{1}{2}\int_D\int_D \left(B(z_1,z_2) - \frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)$. Using (45), we obtain that the relation to our wave function is the following

$$\psi(D,t,\hbar) = \tilde{\psi}(D,t,\hbar)\cdot\prod_{i<j}(x_i - x_j)^{\alpha_i\alpha_j}.$$

4.1. PDE for the Airy curve

In this particular case, we had $P_{g,n} = 0$.

Therefore, in this case, we obtain the following system of PDEs:

$$\hbar^2\left(\frac{d^2}{dx_k^2} - \sum_{i\neq k}\frac{\frac{d}{dx_i} + \frac{\alpha_i}{\alpha_k}\frac{d}{dx_k}}{x_k - x_i} + \sum_{\substack{i\neq j\\ i\neq k, j\neq k}}\frac{\alpha_i\alpha_j}{(x_k - x_i)(x_i - x_j)}\right)\psi = x_k\psi, \tag{64}$$

for every $k = 1,\ldots,r$.

Example 4.3. Considering the divisor $D = [z_1] - [z_2]$, sending $z_2 \to \infty$ and regularizing the $(0,1)$ factor of the wave function, we recover the Airy quantum curve from our PDE for $k = 1$, with $x = x_1$:

$$\left(\hbar^2\frac{d^2}{dx^2} - x\right)\psi = 0.$$

4.2. PDE for the Painlevé case

In this case, we have $P_{g,n} = \frac{\partial}{\partial t}\omega_{g,n}(z_1,\ldots,z_n)$.

Therefore, we obtain the following PDE:

$$\hbar^2\left(\frac{d^2}{dx_k^2} - \sum_{i\neq k}\frac{\frac{d}{dx_i} + \frac{\alpha_i}{\alpha_k}\frac{d}{dx_k}}{x_k - x_i} - \frac{\partial}{\partial t} - \frac{\partial}{\partial t}F + (\star)\right)\psi(D) = (x_k^3 + tx_k + V)\psi = \left(x_k^3 + tx_k + \frac{\partial}{\partial t}\omega_{0,0}\right)\psi,$$

for every $k = 1,\ldots,r$, where $(\star)$ is given by (63).

4.3. Reduced equation

Consider a divisor $D = [z] - [z']$ with two points, and call $x = x(z), x' = x(z')$. The equation we have obtained is a PDE: it involves both d/dx and d/dx', as well as partial derivatives with respect to times when $L(x) \neq 0$. Let us show here that it is possible to eliminate d/dx' and arrive to an equation involving only d/dx, as well as possibly times derivatives.

Define

$$\tilde{\psi}(z,z') := (x - x')\psi([z]-[z'],t,\hbar)e^F. \tag{65}$$

Define the differential operators

$$\mathcal{D} := \hbar^2\frac{d^2}{dx^2} - \hbar^2 L(x) - R(x), \tag{66}$$

$$\mathcal{D}' := \hbar^2\frac{d^2}{dx'^2} - \hbar^2 L(x') - R(x'). \tag{67}$$

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Equation (48) is equivalent to

$$\mathcal{D}\tilde{\psi} = \frac{\hbar^2}{x-x'}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\mathrm{d}}{\mathrm{d}x'}\right)\tilde{\psi} = -\mathcal{D}'\tilde{\psi}. \tag{68}$$

In particular, this implies

$$\hbar^2\frac{\mathrm{d}}{\mathrm{d}x'}\tilde{\psi} = -\hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\psi} + (x-x')\mathcal{D}\tilde{\psi}, \tag{69}$$

and applying $\mathrm{d}/\mathrm{d}x'$ again, we find

$$\begin{aligned}\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x'^2}\tilde{\psi} &= -\mathcal{D}\tilde{\psi} - \hbar^2\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x'}\tilde{\psi} + (x-x')\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x'}\tilde{\psi}\\ &= \hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{\psi} - 2\mathcal{D}\tilde{\psi} - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}+\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x} - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\tilde{\psi}.\end{aligned}$$

Therefore

$$\begin{aligned}0 &= (\mathcal{D}+\mathcal{D}')\tilde{\psi}\\ &= \mathcal{D}\tilde{\psi} + \left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x'^2} - R(x') - \hbar^2 L(x')\right)\tilde{\psi}\\ &= \left(\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mathcal{D} - R(x') - \hbar^2 L(x') - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}+\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x} - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\right)\tilde{\psi}\\ &= \left(R(x)-R(x')+\hbar^2(L(x)-L(x')) - (x-x')\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}+\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x} - \hbar^{-2}\mathcal{D}(x-x')\mathcal{D}\right)\right)\tilde{\psi}\\ &= \left(R(x)-R(x')+\hbar^2(L(x)-L(x')) - (x-x')\left(-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{D}+\mathcal{D}\frac{\mathrm{d}}{\mathrm{d}x} - \hbar^{-2}(x-x')\mathcal{D}^2\right)\right)\tilde{\psi}\\ &= \left(R(x)-R(x')+\hbar^2(L(x)-L(x')) - (x-x')\left(\frac{\mathrm{d}R(x)}{\mathrm{d}x}+\hbar^2\frac{\mathrm{d}L(x)}{\mathrm{d}x} - \hbar^{-2}(x-x')\mathcal{D}^2\right)\right)\tilde{\psi},\end{aligned}$$

and thus

$$\frac{R(x)-R(x')}{x-x'}\tilde{\psi} + \hbar^2\frac{L(x)-L(x')}{x-x'}\tilde{\psi} = \left(\frac{\mathrm{d}R(x)}{\mathrm{d}x}+\hbar^2\frac{\mathrm{d}L(x)}{\mathrm{d}x} - \hbar^{-2}(x-x')\mathcal{D}^2\right)\tilde{\psi}. \tag{70}$$

Finally,

$$\mathcal{D}^2\tilde{\psi} = \frac{\hbar^2}{x-x'}\left(\frac{R(x)-R(x')}{x-x'} + \hbar^2\frac{L(x)-L(x')}{x-x'} - \frac{\mathrm{d}R(x)}{\mathrm{d}x} - \hbar^2\frac{\mathrm{d}L(x)}{\mathrm{d}x}\right)\tilde{\psi}. \tag{71}$$

This equation is a PDE, with rational coefficients $\in \mathbb{C}(x)$, involving $\mathrm{d}/\mathrm{d}x$ and $\partial/\partial t_k$s but no $\mathrm{d}/\mathrm{d}x'$ anymore.

Notice that the right-hand side is of order $O(\hbar^2)$ in the limit $\hbar \to 0$, and $\mathcal{D} \to \hat{y}^2 - R(x)$, where $\hat{y} = \hbar\mathrm{d}/\mathrm{d}x$.

5. Quantum curves

The goal now is to prove that $\psi(D)$ obeys an isomonodromic system of differential equations, and in particular, this implies the existence of a quantum curve $\hat{P}(\hat{x},\hat{y},\hbar)$ that annihilates $\psi(D)$. To this purpose, we first prove that $\psi([z]-[z'])$ coincides with the integrable kernel of an isomonodromic system. The way to prove it generalizes the method of [2], that is first proving that the ratio of ψ and the integrable kernel has to be a formal series of the form $1 + O(z'^{-1})$ and then showing that the only solution of equation (48) which has that behavior implies that the ratio must be 1.

5.1. Painlevé I (genus zero case)

We shall prove that $\psi([z]-[z'])$ coincides with the integrable kernel associated to the Painlevé I kernel.

Consider a solution of the Painlevé system (8), written as

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}, \quad \det\Psi(x) = 1,$$

that is $\Psi(x)$ satisfying (8):

$$\left(\hbar\frac{\partial}{\partial x} - \mathcal{L}(x,t;\hbar)\right)\Psi(x) = 0\,, \quad \left(\hbar\frac{\partial}{\partial t} - \mathcal{R}(x,t;\hbar)\right)\Psi(x) = 0.$$

Define $A(x), \tilde{A}(x), B(x), \tilde{B}(x)$ as WKB $\hbar$-formal series solutions, with leading orders

$$\begin{aligned} A(x) &\sim \frac{i}{\sqrt{2z}}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)), \\ B(x) &\sim \frac{i}{\sqrt{2z}}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)), \\ \tilde{A}(x) &\sim i\sqrt{\frac{z}{2}}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)), \\ \tilde{B}(x) &\sim -i\sqrt{\frac{z}{2}}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1+O(\hbar)), \end{aligned}$$

and with each coefficient of higher powers of $\hbar$ in $(1+O(\hbar))$ being a polynomial of $1/z$ that tends to 0 as $z \to \infty$. The choice of normalization constants for the WKB solutions $C_A = C_B = \frac{i}{\sqrt{2}}$ is made, such that $\det\Psi = -2C_AC_B = 1$ and $C_A = C_B$. The integrable kernel is defined as (a WKB formal series of $\hbar$):

$$K(x,x') := \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x-x'}. \tag{72}$$

From the isomonodromic system (8), one can verify that this kernel obeys the same equation (48) as $\psi([z]-[z'])$. In fact, they are equal:

Theorem 5.1. *We have, as formal WKB power series of* $\hbar$

$$\psi([z]-[z'],t,\hbar) = \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x-x'}, \tag{73}$$

where $x = x(z)$ *and* $x' = x(z')$.

Proof. Define the ratio

$$H(z,z') := \frac{(x-x')\psi([z]-[z'],t,\hbar)}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}. \tag{74}$$

It is a formal series of $\hbar$ whose coefficients are rational functions of z and z'. The leading orders show that

$$H(z,z') = 1 + O(\hbar).$$

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More explicitly, we have

$$H(z,z') = \frac{(z^2 - z'^2)\frac{1}{(z-z')\sqrt{2z2z'}}\sum \hbar^k c_k\left(\frac{1}{z},\frac{1}{z'}\right)}{\frac{1}{2}\sqrt{\frac{z'}{z}}\left(\sum \hbar^k \alpha_k\left(\frac{1}{z}\right)\right)\left(\sum \hbar^\ell \tilde{\beta}_\ell\left(\frac{1}{z'}\right)\right) + \frac{1}{2}\sqrt{\frac{z}{z'}}\left(\sum \hbar^k \tilde{\alpha}_k\left(\frac{1}{z}\right)\right)\left(\sum \hbar^\ell \beta_\ell\left(\frac{1}{z'}\right)\right)}$$
$$= \frac{\sum \hbar^k c_k\left(\frac{1}{z},\frac{1}{z'}\right)}{\left(\frac{1}{z}\left(\sum \hbar^k \alpha_k\left(\frac{1}{z}\right)\right)\left(\sum \hbar^\ell \tilde{\beta}_\ell\left(\frac{1}{z'}\right)\right) + \frac{1}{z'}\left(\sum \hbar^k \tilde{\alpha}_k\left(\frac{1}{z}\right)\right)\left(\sum \hbar^\ell \beta_\ell\left(\frac{1}{z'}\right)\right)\right)\Big/\left(\frac{1}{z}+\frac{1}{z'}\right)}. \tag{75}$$

Since we are in the hyperelliptic case, we have that $\beta_k\left(\frac{1}{z}\right) = \alpha_k\left(\frac{-1}{z}\right)$ and $\tilde{\beta}_k\left(\frac{1}{z}\right) = \tilde{\alpha}_k\left(\frac{-1}{z}\right)$, which implies that the first factor of the denominator is divisible by $\frac{1}{z} + \frac{1}{z'}$. Hence, the series at the denominator is invertible.

Therefore, at each order of $\hbar$, the coefficient is a polynomial of $1/z$, $1/z'$, which tends to 0 at $z, z' \to \infty$:

$$H(z,z') - 1 \in \frac{1}{zz'}\mathbb{C}[z^{-1}, z'^{-1}][[\hbar]].$$

We shall prove that $H = 1$ by following the method of [2]. Let us assume that $H \neq 1$, and write

$$H(z,z') = 1 + H_M(z)z'^{-M} + O(z'^{-M-1}),$$

where $M \geq 1$ is the smallest possible power of z' whose coefficient H_M would be $\neq 0$ as a formal series of $\hbar$. Let $\tilde{K}(x,x') \coloneqq A(x)\tilde{B}(x') - \tilde{A}(x)B(x')$. Using equation (68) for $\tilde{\psi}(z,z') = H(z,z')\tilde{K}(x,x')e^F$, together with the fact that $\overline{K}(x,x') \coloneqq \tilde{K}(x,x')e^F$ satisfies the same equation (68) (which is proved in Appendix A in general), we obtain the following equation for $H = H(z,z')$:

$$\frac{\mathrm{d}^2 H}{\mathrm{d}x'^2} + 2\frac{\mathrm{d}\ln B(x')}{\mathrm{d}x'}\frac{\mathrm{d}H}{\mathrm{d}x'} + 2\frac{\mathrm{d}\ln(\tilde{A}(x)/A(x) - \tilde{B}(x')/B(x'))}{\mathrm{d}x'}\frac{\mathrm{d}H}{\mathrm{d}x'}$$
$$- \tilde{K}(x,x')(HL(x').\tilde{K}(x,x') - H\tilde{K}(x,x')L(x').F) = \frac{1}{x'-x}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x'}\right)H, \tag{76}$$

whose leading power of z' comes only from the second term and is

$$2y(z')H_M(z)z'^{-M-2} + O(z'^{-M-2}) = 2H_M(z)z'^{-M-1} + O(z'^{-M-2}) = 0, \tag{77}$$

implying that $H_M = 0$, and thus contradicting the hypothesis that $H \neq 1$. This proves the theorem. □

As a corollary, this implies that

$$\lim_{z'\to\infty} \frac{(x(z) - x(z'))\psi([z]-[z'],t,\hbar)}{\tilde{B}(x(z'))} = A(x(z)). \tag{78}$$

In other words

$$\frac{i}{2\sqrt{z}} e^{\hbar^{-1}\int_0^z \omega_{0,1}} e^{\sum_{(g,n)\neq(0,1),(0,2)} \frac{\hbar^{2g-2+n}}{n!}\int_\infty^z \cdots \int_\infty^z \omega_{g,n}} = A(x(z)). \tag{79}$$

In the Painlevé system, the function $A(x)$ is annihilated by the quantum curve

$$\hat{y}^2 - \left((x-U)^2(x+2U) + \frac{\hbar^2}{2}\ddot{U}(x-U) + \frac{\hbar^2}{4}\dot{U}^2\right) + \frac{\hbar^2}{2(x-U)}\dot{U} - \frac{\hbar}{x-U}\hat{y}. \tag{80}$$

5.2. General genus zero case

The same argument applies to any isomonodromic system, thanks to the fact that both the 2-point wave function $\tilde{\psi} = (x - x')\psi e^F$ constructed from topological recursion and the integrable kernel $\overline{K}(x, x') = \tilde{K}(x, x')e^F$ satisfy the same PDE (68). The first claim is the main result of the article, and the second claim was proved in Appendix A in general, since we could not find this result in the literature for a general isomonodromic system. Consider that we have a Lax system of the type

$$\begin{cases} \hbar\frac{\partial}{\partial x}\Psi(x) = \mathcal{L}(x;\hbar)\Psi(x), \\ \hbar\frac{\partial}{\partial t_k}\Psi(x) = \mathcal{R}_k(x;\hbar)\Psi(x), \end{cases} \tag{81}$$

whose spectral curve in the limit $\hbar \to 0$ is a genus zero curve of the form $\det(y - \mathcal{L}_0(x)) = y^2 - R(x) = 0$, and which has a WKB formal power series solution of the form

$$\begin{aligned} A(x) &\sim \frac{i}{\sqrt{2\mathrm{d}x/\mathrm{d}z}}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\ B(x) &\sim \frac{i}{\sqrt{2\mathrm{d}x/\mathrm{d}z}}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\ \tilde{A}(x) &\sim i\sqrt{\frac{1}{2}\mathrm{d}x/\mathrm{d}z}\, e^{\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \\ \tilde{B}(x) &\sim -i\sqrt{\frac{1}{2}\mathrm{d}x/\mathrm{d}z}\, e^{-\hbar^{-1}\int_0^x y\mathrm{d}x}(1 + O(\hbar)), \end{aligned}$$

where each coefficient of higher powers of $\hbar$ in $(1 + O(\hbar))$ is a rational function of z that tends to 0 at poles $z \to \zeta$, where we have chosen a pole $x(\zeta) = \infty$. This pole has degree $-d = 1$ or $-d = 2$, and $\xi = x^{1/d}$ is a local coordinate near the pole. We emphasize that for all genus zero spectral curves, where $y^2 = P_{\mathrm{odd}}(x)$ with $P_{\mathrm{odd}}(x)$ an odd polynomial of x, such systems are explicitly known as Gelfand–Dikii systems [22] described in Section 5.4 below, and for more general spectral curves (genus zero, possibly with nonempty Newton polygon), it was proved in [28] that a 2×2 autonomous system $\mathcal{L}_0$ always admits an $\hbar$-deformation $\mathcal{L}$ with this property.

We define the following formal series of $\hbar$:

$$H(z, z') := \frac{(x(z) - x(z'))\psi([z] - [z'])}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')} = 1 + O(\hbar).$$

We have proved that the integrable kernel $\overline{K}(x, x')$ associated to the Lax system satisfies equation (68) (see Theorem A.1), and thus, H satisfies the PDE (76).

Moreover, the coefficients of H are analytic functions of z', which tend to 0 at $z' \to \zeta$. Let us write $H(z, z') = 1 + O(x'^{1/d})$. The subleading coefficient of $H = 1 + H_M(z)x'^{M/d} + O(x'^{M/d-1})$ must satisfy $y(z')H_M(z)x'^{M/d-1} = O(x'^{M/d-2})$, and therefore, $H = 1$.

This implies that $\psi([z] - [z'])$ coincides with the integrable kernel

$$\psi([z] - [z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z) - x(z')}.$$

Then, taking the limit $z' \to \zeta$, this implies that

$$\lim_{z' \to \zeta} \frac{(x(z) - x(z'))\psi([z] - [z'])}{\tilde{B}(x(z'))} = A(x(z)).$$

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Knowing that $A(x), \tilde{A}(x)$ satisfy an isomonodromic system with first equation

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} = \mathcal{L}(x,t,\hbar) \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix}, \tag{82}$$

with

$$\mathcal{L}(x,t,\hbar) = \begin{pmatrix} \alpha(x,t,\hbar) & \beta(x,t,\hbar) \\ \gamma(x,t,\hbar) & \delta(x,t,\hbar) \end{pmatrix}, \tag{83}$$

where $\alpha, \beta, \gamma, \delta$ are rational functions of x, with coefficients being formal power series of $\hbar$, we get the quantum curve annihilating $A(x)$:

$$\hat{y}^2 - (\alpha+\delta)\hat{y} + (\alpha\delta - \beta\gamma) - \hbar\left(\mathrm{d}\alpha/\mathrm{d}x - \alpha \frac{\mathrm{d}\beta/\mathrm{d}x}{\beta} + \frac{\mathrm{d}\beta/\mathrm{d}x}{\beta} \hat{y} \right). \tag{84}$$

Its classical part $\hbar \to 0$ is indeed the spectral curve

$$y^2 - (\alpha(x,t,0) + \delta(x,t,0))y(\alpha(x,t,0)\delta(x,t,0) - \beta(x,t,0)\gamma(x,t,0)) = \det(y\mathrm{Id} - \mathcal{L}_0(x)). \tag{85}$$

5.3. Higher genus case

If the curve $y^2 = R(x)$ has genus $\hat{g} > 0$, it was verified in [8] (for $\hat{g} = 1$), and argued in [4], that the perturbative wave function cannot satisfy the quantum curve: in fact, just because it is not a function (order by order in $\hbar$) on the spectral curve. Indeed, multiple integrals of type $\int_o^z \cdots \int_o^z \omega_{g,n}$ are not invariant after z goes around a cycle, and do not transform as Abelian differentials. It was argued in [4, 17, 21] that only the nonperturbative wave function of [16, 20] can be a wave function and can obey a quantum curve, and this was proved up to the 3rd nontrivial powers of $\hbar$ for arbitrary curves in [4], and verified to many orders for elliptic curves in [8].

It is also useful to introduce the partition function, which is independent of any divisor: $Z(\hbar) = \psi(D = \emptyset, \hbar)$, namely

$$Z(\hbar) \coloneqq \exp\Big(\sum_{g \geq 0} \hbar^{2g-2} \omega_{g,0} \Big),$$

where the $\omega_{g,0}$ were also denoted by F_g. From now on, we omit the dependence on $\hbar$ of both partition and wave functions, and we will introduce the dependence on the filling fractions.

Consider a curve $y^2 = R(x)$ with genus $\hat{g} > 0$, and let $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$ be a symplectic basis of cycles of $H_1(\Sigma, \mathbb{Z})$ (i.e., integer cycles). We choose the Bergman kernel normalized on $\mathcal{A}$-cycles.

Recall the first kind times

$$\epsilon_i = \frac{1}{2\pi i} \oint_{\mathcal{A}_i} y\mathrm{d}x, \quad i = 1, \ldots, \hat{g}, \tag{86}$$

and that the $\mathcal{B}_i$-period of $\omega_{g,n+1}$ is the variation of $\omega_{g,n}$ with respect to ϵ_i:

$$\frac{\partial}{\partial \epsilon_i} \omega_{g,n}(z_1, \ldots, z_n) = \oint_{\mathcal{B}_i} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n). \tag{87}$$

Since (48) is a linear PDE, any linear combination of solutions is a solution. Moreover, since the coefficients of the PDE do not involve the times ϵ_i, we remark that shifting $\epsilon_i \to \epsilon_i + n_i$ is another solution, which we denote as follows

$$\psi((\epsilon_i \to \epsilon_i + n_i)_i; [z] - [z']). \tag{88}$$

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The transseries linear combination introduced in [4, 16, 17, 20]

$$\hat{\psi}([z]-[z']) := \frac{1}{\hat{\mathcal{T}}} \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} \psi((\epsilon_i \to \epsilon_i + n_i)_i; [z]-[z']) Z((\epsilon_i \to \epsilon_i + n_i)_i), \tag{89}$$

where

$$\hat{\mathcal{T}} := \sum_{n_1,\dots,n_{\hat{g}} \in \mathbb{Z}^{\hat{g}}} Z((\epsilon_i \to \epsilon_i + n_i)_i), \tag{90}$$

is thus also a solution of the same PDE. The construction of the transseries (89) is actually a sum over integer first kind $\mathcal{B}$-cycles, since

$$\psi((\epsilon_i); [z+\mathcal{B}_i]-[z']) = e^{\frac{\partial}{\partial \epsilon_i}} \psi((\epsilon_i)_i; [z]-[z']) = \psi((\epsilon_1, \dots, \epsilon_i \to \epsilon_i + 1, \dots, \epsilon_{\hat{g}}); [z]-[z']).$$

The combination $\hat{\psi}$, hence, remains invariant [17] if we modify the integration homotopy class from z' to z by adding first kind $\mathcal{B}$-cycles, and it is, order by order as a transseries of $\hbar$, a function of z and z' on the spectral curve. Shifts by other kinds of cycles of the spectral curve were already trivial for ψ, in the sense that they result in the multiplication by simple factors, which one expects from a solution of an isomonodromic system.

From there, the same argument as for the genus zero case applies. Assume that there is an isomonodromic system

$$\begin{cases} \hbar \frac{\partial}{\partial x} \Psi(x) &= \mathcal{L}(x;\hbar)\Psi(x), \\ \hbar \frac{\partial}{\partial t_k} \Psi(x) &= \mathcal{R}_k(x;\hbar)\Psi(x), \end{cases} \tag{91}$$

whose associated spectral curve is our spectral curve, with

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}$$

a formal transseries solution. Then the formal transseries

$$H(z,z') = \frac{(x(z)-x(z'))\hat{\psi}([z]-[z'])}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')} = 1 + O(\hbar) \tag{92}$$

satisfies the PDE (76), and is such that $H(z,z') = 1 + O(x'^{1/d})$. The subleading order of $H = 1 + H_M(z)x'^{M/d} + O(x'^{M/d-1})$ must satisfy $y(z')H_M(z)x'^{M/d-1} = O(x'^{M/d-2})$, and therefore $H = 1$. This implies that

$$\hat{\psi}([z]-[z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z)-x(z')}. \tag{93}$$

This also implies that

$$\lim_{z' \to \zeta} \frac{(x(z)-x(z'))\hat{\psi}([z]-[z'])}{\tilde{B}(x(z'))} = A(x(z)). \tag{94}$$

Since $A(x), \tilde{A}(x)$ satisfy the isomonodromic system

$$\hbar \frac{\partial}{\partial x} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} = \mathcal{L}(x) \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix},$$

where

$$\mathcal{L}(x,t,\hbar) = \begin{pmatrix} \alpha(x,t,\hbar) & \beta(x,t,\hbar) \\ \gamma(x,t,\hbar) & \delta(x,t,\hbar) \end{pmatrix},$$

we find the quantum curve annihilating $A(x)$:

$$\hat{y}^2 - (\alpha(x)+\delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) - \hbar\left(\alpha'(x) - \alpha(x)\frac{\beta'(x)}{\beta(x)} + \frac{\beta'(x)}{\beta(x)}\hat{y}\right). \tag{95}$$

Its classical part $\hbar \to 0$ is indeed the equation

$$\det(y\mathrm{Id} - \mathcal{L}(x,t,0)) = 0. \tag{96}$$

5.4. Examples: Gelfand–Dikii systems

These systems generalize the Painlevé I equation; they appear in the enumeration of maps in the large size limit [22]. For these Gelfand–Dikii systems, the proof that $\psi([z]-[z'])$ coincides with the integrable kernel (which then implies the quantum curve) can be found in [22, Chapter 5], by another method. Here, let us provide another proof with our current method.

The Gelfand–Dikii polynomials are defined as differential polynomials of a function $U(t)$, by the recursion

$$R_0(U) = 2\,, \qquad \frac{\partial}{\partial t}R_{k+1}(U) = -2U\frac{\partial R_k(U)}{\partial t} - R_k(U)\frac{\partial U}{\partial t} + \frac{\hbar^2}{4}\frac{\partial^2 R_k(U)}{\partial t^2}. \tag{97}$$

At each step, the integration constant is chosen so that $R_k(U)$ is homogeneous in powers of U and $\partial^2/\partial t^2$. The first few are given by

$$\begin{aligned} R_0 &= 2, \\ R_1 &= -2U, \\ R_2 &= 3U^2 - \frac{\hbar^2}{2}\ddot{U}, \\ R_3 &= -5U^3 + \frac{5\hbar^2}{2}U\ddot{U} + \frac{5\hbar^2}{4}\dot{U}^2 - \frac{\hbar^4}{8}\frac{\partial^4 U}{\partial t^4}. \end{aligned} \tag{98}$$

Let $m \geq 1$ be an integer, and let $\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_m$ be a set of “times.” Let $U(t;\tilde{t}_0,\tilde{t}_1,\tilde{t}_2,\dots,\tilde{t}_m)$ be a solution of the following nonlinear ODE:

$$\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(U) = t. \tag{99}$$

Notice that, formally, $t = -2\tilde{t}_{-1}$. For $m = 1$, the equation (99) is the Painlevé I equation. The case $m = 2$ is called the Lee–Yang equation. The case $m = 0$ is simply $U(t) = -\frac{t}{2\tilde{t}_0}$.

Consider the Lax pair (adopting the normalizations of [22]) given by

$$\mathcal{R}(x,t,\hbar) = \begin{pmatrix} 0 & 1 \\ x + 2U(t) & 0 \end{pmatrix} \tag{100}$$

and

$$\mathcal{L}(x,t,\hbar) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{L}_j(x,t,\hbar), \tag{101}$$

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where

$$\mathcal{L}_j(x,t,\hbar) = \begin{pmatrix} \alpha_j(x,t) & \beta_j(x,t) \\ \gamma_j(x,t) & -\alpha_j(x,t) \end{pmatrix} \text{ with,}$$

$$\beta_j(x,t) = \frac{1}{2}\sum_{k=0}^{j} x^{j-k} R_k(U), \quad \alpha_j(x,t) = -\frac{\hbar}{2}\frac{\partial}{\partial t}\beta_j(x,t), \quad \gamma_j(x,t) = (x+2U)\beta_j(x,t) + \hbar\frac{\partial}{\partial t}\alpha_j(x,t).$$

One can easily verify that the Gelfand–Dikii polynomials are such that the zero curvature equation is satisfied

$$\hbar\frac{\partial}{\partial t}\mathcal{L}(x,t,\hbar) + \hbar\frac{\partial}{\partial x}\mathcal{R}(x,t,\hbar) = [\mathcal{R}(x,t,\hbar), \mathcal{L}(x,t,\hbar)]. \tag{102}$$

The differential equation (99) admits a formal power series solution $U(t,\hbar)$ with only even powers of $\hbar$:

$$U(t,\hbar) = \sum_k \hbar^{2k} u_k(t), \tag{103}$$

whose first term $u(t) = u_0(t)$ satisfies an algebraic equation

$$\sum_{j=0}^{m} \frac{(2j+1)!}{j!(j+1)!}\tilde{t}_j(-u/2)^{j+1} = -\frac{1}{4}t.$$

In the Painlevé I case, $m=1$, $\tilde{t}_k = \delta_{k,1}$, we recover $t = -3u^2$.

The spectral curve, in the limit $\hbar \to 0$:

$$\det(y - \mathcal{L}(x,t,0)) = 0 \tag{104}$$

is always a genus zero curve. It admits the rational parametrization

$$\begin{cases} x(z) = z^2 - 2u(t) \\ y(z) = \sum_{j=0}^{m} \tilde{t}_j \left(z^{2j+1}(1-2u(t)/z^2)^{j+\frac{1}{2}}\right)_+ \end{cases}, \tag{105}$$

where $()_+$ means the positive part in the Laurent series expansion near $z=\infty$, that is

$$y(z) = \sum_{j=0}^{m} \tilde{t}_j \sum_{k=0}^{j} (-u)^k \frac{(2j+1)!!}{(2j-2k+1)!!}\, z^{2j-2k+1}.$$

In the Painlevé I case, $\tilde{t}_j = \delta_{j,1}$, we recover $y(z) = z^3 - 3uz$. In the case $m=0$, with $\tilde{t}_0 = 1$, we recover the Airy system

$$\mathcal{L}(x,t,\hbar) = \begin{pmatrix} 0 & 1 \\ x-t & 0 \end{pmatrix} \tag{106}$$

with spectral curve $y^2 = x - t$.

Let $\Psi(x,t,\hbar)$ as follows

$$\Psi(x,t,\hbar) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}$$

be a WKB $\hbar$ formal series solution of

$$\hbar \frac{\partial}{\partial x} \Psi(x,t,\hbar) = -\mathcal{L}(x,t,\hbar)\Psi(x,t,\hbar), \qquad \hbar \frac{\partial}{\partial t} \Psi(x,t,\hbar) = \mathcal{R}(x,t,\hbar)\Psi(x,t,\hbar). \tag{107}$$

Our previous results show that the formal series $\psi([z]-[\infty],\tilde{t},\hbar)$ coincide with

$$A(x,t,\hbar) = \frac{1}{\sqrt{2z}} e^{\hbar^{-1}\int_0^z y\,dx} e^{\sum_{(g,n)\neq(0,1),(0,2)} \frac{\hbar^{2g-2+n}}{n!} \int_\infty^z \cdots \int_\infty^z \omega_{g,n}}, \tag{108}$$

and is annihilated by the quantum curve

$$\hat{y}^2 - (\alpha(x)+\delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) - \hbar\left(\alpha'(x) - \alpha(x)\frac{\beta'(x)}{\beta(x)} + \frac{\beta'(x)}{\beta(x)}\hat{y}\right). \tag{109}$$

Appendix A. PDE for any isomonodromic system

The goal of this appendix is to show that the integrable kernel $\overline{K}(x,x') := \tilde{K}(x,x')e^F$ of an isomonodromic system satisfies the same PDE (68) that we obtained for the 2-point wave function built from topological recursion. Here, $F = \log \mathcal{T}$, with $\mathcal{T}$ the tau function of the isomonodromic system.

Consider a solution of a 2×2 isomonodromic system written as:

$$\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \tilde{A}(x) & \tilde{B}(x) \end{pmatrix}, \qquad \det \Psi(x) = 1,$$

that is $\Psi(x)$ satisfying the (compatible) system of equations:

$$\begin{cases} \hbar \frac{\partial}{\partial x} \Psi = \mathcal{L}(x;\hbar)\Psi, \\ \hbar \frac{\partial}{\partial t_k} \Psi = \mathcal{R}_k(x;\hbar)\Psi. \end{cases} \tag{110}$$

The equations with respect to the isomonodromic times t_k can be seen as isomonodromic deformations of the first equation.

Consider the deformed spectral curve

$$P(x,y;\hbar) = \det(y\,\mathrm{Id} - \mathcal{L}(x;\hbar)) = y^2 + R(x) + \sum_{m\geq 1} \hbar^m P_m(x,y) = P_0(x,y) + O(\hbar). \tag{111}$$

We will make use of the following technical assumption:

Assumption A.1. *Let $\mathcal{N}$ be the Newton polygon associated to $P_0(x,y) = y^2 - R(x)$, and let $\mathring{\mathcal{N}}$ be its interior. For $m \geq 1$, we only allow $P_m(x,y) = \sum_{i,j} P_{i,j} x^i y^j$ whose only nonzero coefficients $P_{i,j} \neq 0$ are such that $(i+1, j+1) \in \mathring{\mathcal{N}}$, which, in our case, only allows for $j = 1$, so $P_m(x,y) = P_m(x)$.*

Remark A.2. This assumption is to ensure that P has the same Casimirs as P_0. It is always possible to transform a deformed spectral curve into one satisfying this assumption, by redefining what we mean by Casimirs of P_0, but, here, we assume we already have this shape for simplicity.

The associated classical spectral curve $\Sigma = \{(x,y) \mid y^2 = R(x)\}$ is presented as a two-sheeted covering of the Riemann sphere $x : \Sigma \to \mathbb{P}^1$.

We assume that the entries of the matrix $\mathcal{L}$ are rational functions of x:

$$\mathcal{L}(x;\hbar) = \sum_{l=1}^{N} \sum_{j=0}^{m_l} \frac{\mathcal{L}_j^{(l)}}{(x-\lambda_l)^{j+1}} - \sum_{j=1}^{m_\infty} \mathcal{L}_{j-1}^{(\infty)} x^{j-1}.$$

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We often omit the dependence on the variables $\hbar$ and even x for simplicity. The solutions $\Psi(x)$ of the first linear differential equation of (110) have essential singularities at $x = \lambda_l$ and $x = \infty$. Let us define $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We know that around each pole of $\mathcal{L}(x)$, the function $\Psi(x)$ admits an asymptotic expansion of the form

$$\hat{\Psi}(x)e^{\hbar^{-1}\sigma_3 T(x)}, \tag{112}$$

where $\hat{\Psi}(x)$ is regular at the poles $P \in \{\infty, \lambda_1, \ldots, \lambda_l\}$ of $\mathcal{L}(x)$ and

$$T(x) \underset{x\to P}{\sim} \tilde{t}_{P,0} \log \xi_P + \sum_{j=1}^{m_P} \frac{\tilde{t}_{P,j}}{\xi_P^j},$$

where $\xi_P = x^{d_\infty}$, with d_∞ equal to -1 (unramified) or -2 (ramified), for $P = \infty$, and $\xi_P = x - \lambda_l$, for $P = \lambda_l$.

The isomonodromic deformation parameters $t_k = T_{\alpha,j}$ of (110), using the multiindex notation $k = (\alpha, j)$, include:

- $\alpha = \infty$: $T_{\infty,j} := \tilde{t}_{\infty,j} = t_{\zeta_\infty^+,j}$, for $j = 1, \ldots, m_\infty$.
  - If ∞ is an unramified critical value of x, that is $d_\infty = -1$ and $x^{-1}(\infty) = \{\zeta_\infty^+, \zeta_\infty^-\}$, then $t_{\zeta_\infty^+,j} = -t_{\zeta_\infty^-,j}$.
  - If ∞ is a ramified critical value of x, that is $d_\infty = -2$ and $\zeta_\infty^+ = \zeta_\infty^-$.
- $\alpha = l = 1, \ldots, N$:
  - For $j = 1, \ldots, m_l$, $T_{l,j} := \tilde{t}_{\lambda_l,j} = t_{\zeta_l^+,j} = -t_{\zeta_l^-,j}$.
  - For $j = -1$, $T_{l,-1} := \lambda_l$.

We will also use this multiindex notation for the matrices $\mathcal{R}_k = \mathcal{R}_{\alpha,j}$ of the system (110).

Remark A.3. From our Assumption A.1, we have that all t_i are independent of $\hbar$ and coincide with the moduli of the classical spectral curve $y^2 = R(x)$.

Remark A.4. Since $\operatorname{Tr}\mathcal{L} = 0$, we have

$$\mathcal{L}^2 = -\det \mathcal{L} \cdot \mathrm{Id}. \tag{113}$$

Let $\mathbb{K}(x, x') := \Psi^{-1}(x')\Psi(x)$. Observe that $\tilde{K}(x, x') = A(x)\tilde{B}(x') - \tilde{A}(x)B(x') = \operatorname{Tr}\left(\mathbb{K}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$. We called ζ_i the poles of $y\mathrm{d}x$. Recall the operator defined by:

$$\mathcal{D} := \hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 L(x) - R(x),$$

where

$$L(x) := \sum_{i, x(\zeta_i)=\infty} \sum_{j=1-2d_i}^{m_i} t_{\zeta_i,j} \sum_{0\le k\le \frac{1-j}{d_i}-2} x^k \left(-\frac{j}{d_i} - k - 2\right) \partial_{\mathcal{B}_{\zeta_i, j+d_i(k+2)}} \tag{114}$$
$$+ \sum_{i, x(\zeta_i)\neq\infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} (x - x(\zeta_i))^{-(k+1)} (j + 1 - k) \partial_{\mathcal{B}_{\zeta_i, j+1-k}},$$

with

$$\partial_{\mathcal{B}_{p,k}} \omega_{g,n}(z_1, \ldots, z_n) := \int_{\mathcal{B}_{p,k}} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n) = \operatorname*{Res}_{x\to p} \frac{\xi_p^k}{k} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n),$$

for every second type cycle $\mathcal{B}_{p,k}$, $k \geq 1$, which we introduced in (3).

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From Corollary 3.7, the operator $L(x)$ can be rewritten in terms of derivatives with respect to the isomonodromic parameters: $t_{\zeta_i,k}$, $k = 1, \dots, m_i$, and λ_l, for $l = 1, \dots, N$.

Theorem A.1. *The operator*

$$\mathcal{O} := \mathcal{D} - \hbar^2 L(x).F - \frac{\hbar^2}{x - x'}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right) \tag{115}$$

annihilates the integrable kernel:

$$\mathcal{O}\,\mathbb{K}(x, x') = 0. \tag{116}$$

The differential equation (116) is equivalent to

$$\left(\mathcal{D} - \frac{\hbar^2}{x - x'}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)\right)(\mathbb{K}(x, x')e^F) = 0.$$

The proof of this theorem will follow from the three following lemmas.

Lemma A.2. *Let $\mathcal{T}$ be the tau function of the isomonodromic system and $F := \log \mathcal{T}$. Then,*

$$\frac{1}{2}\mathrm{Tr}\,\mathcal{L}(x;\hbar)^2 = -\det \mathcal{L}(x;\hbar) = \hbar^{-2}R(x) + L.F. \tag{117}$$

Proof. It was proved by Jimbo–Miwa–Ueno [27] that there exists a tau function $\mathcal{T}(\{t_k\})$ of the isomonodromic times $t_k = T_{\alpha,j}$, such that

$$\frac{\partial}{\partial t_k}\log \mathcal{T} = -\operatorname*{Res}_{x=P}\mathrm{Tr}\left(\frac{\partial}{\partial t_k}T(x)\sigma_3\Psi^{-1}(x)\frac{\partial}{\partial x}\Psi(x)\right), \tag{118}$$

where $P = \infty$, if $\alpha = \infty$, and $P = \lambda_l$, if $\alpha = l$, for $l = 1, \dots, N$. Let y_s be the singular part of y. We have

$$\frac{\partial}{\partial x}\Psi = \frac{\partial}{\partial x}\hat{\Psi}e^{\hbar^{-1}\sigma_3 T} + \hbar^{-1}\hat{\Psi}\sigma_3 y_s e^{\hbar^{-1}\sigma_3 T},$$

which implies

$$\mathcal{L} = \frac{\partial}{\partial x}\Psi \cdot \Psi^{-1} = \frac{\partial}{\partial x}\hat{\Psi}\cdot\hat{\Psi}^{-1} + \hbar^{-1}\hat{\Psi}\sigma_3 y_s\hat{\Psi}^{-1}.$$

Therefore

$$\mathrm{Tr}\,\mathcal{L}^2 = \mathrm{Tr}\left(\frac{\partial}{\partial x}\hat{\Psi}\cdot\hat{\Psi}^{-1}\right)^2 + 2\hbar^{-2}y_s^2 + 2\mathrm{Tr}\left(y_s\sigma_3\hat{\Psi}^{-1}\frac{\partial}{\partial x}\hat{\Psi}\right). \tag{119}$$

For every $P \in \{\infty, \lambda_1, \dots, \lambda_l\}$, $\zeta_i \in x^{-1}(P)$, we have

$$\frac{\partial}{\partial t_{\zeta_i,k}}T(x) = -\frac{\xi_P^{-k}}{k}, \tag{120}$$

for $k = 1, \dots, m_i$. Around $\zeta_l \in x^{-1}(\lambda_l)$, we have

$$\frac{\partial}{\partial \lambda_l}T(x) = -\sum_{j=0}^{m_l} -t_{\zeta_l,j}\xi_{\lambda_l}^{-j-1}, \tag{121}$$

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which behaves as $-y(z)$ around $z = \zeta_l$. Substituting (119) in (118), we get

$$\frac{\partial}{\partial t_k} F = -\operatorname*{Res}_{x=P} \frac{1}{2y_s(x)} \frac{\partial}{\partial t_k} T(x) \left(\operatorname{Tr} \mathcal{L}(x)^2 - \operatorname{Tr} \left(\frac{\partial}{\partial x} \hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1} \right)^2 - 2\hbar^{-2} y_s(x)^2 \right).$$

Since $\hat{\Psi}(x)$ is analytic at $x = P$, we have $\frac{\partial}{\partial x} \hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1} = O(\xi_P^2)$ for $x \to P$. Thus, the middle term does not contribute to the residue:

$$\frac{\partial}{\partial t_k} F = -\operatorname*{Res}_{x=P} \frac{1}{2y_s(x)} \frac{\partial}{\partial t_k} T(x) \big(\operatorname{Tr} \mathcal{L}(x)^2 - 2\hbar^{-2} R(x) \big).$$

Using the behaviors (120) and (121), we get that $\operatorname{Tr} \mathcal{L}(x)^2 = 2(L(x).F + \hbar^{-2} R(x))$. □

Lemma A.3. *Let $\zeta_\infty \in x^{-1}(\infty)$ and $\zeta_l \in x^{-1}(\lambda_l)$ be poles of $\omega_{0,1}$ of orders m_∞ and m_l, $l = 1, \dots, N$, respectively. Let $d_\infty := \operatorname{ord}_{\zeta_\infty}(x)$. We have*

$$\mathcal{O}\, \mathbb{K}(x, x') = \left(\Psi(x')^{-1} \hbar^2 \left(\frac{\partial \mathcal{L}(x;\hbar)}{\partial x} - \frac{\mathcal{L}(x;\hbar) - \mathcal{L}(x';\hbar)}{x - x'} - \mathcal{R}(x, x') \right) \Psi(x') \right) \mathbb{K}(x, x'), \tag{122}$$

with

$$\begin{aligned} \mathcal{R}(x, x') := & \sum_{j=1-2d_\infty}^{m_\infty} t_{\zeta_\infty, j} \sum_{s=1}^{j+2d_\infty} \xi_\infty^{s-j-2d_\infty} \left(\frac{-s}{d_\infty} \right) (\mathcal{R}_{\infty,s}(x) - \mathcal{R}_{\infty,s}(x')) \\ & + \sum_{l=1}^{N} \left(\xi_{\lambda_l}^{-1} (\mathcal{R}_{l,-1}(x) - \mathcal{R}_{l,-1}(x')) + \sum_{s=1}^{m_l - 1} \sum_{j=s}^{m_l - 1} t_{\zeta_l, j}\, \xi_{\lambda_l}^{-j+s-2} s (\mathcal{R}_{l,s}(x) - \mathcal{R}_{l,s}(x')) \right). \end{aligned}$$

Proof. On the one hand, we have

$$\frac{\partial}{\partial x} \mathbb{K}(x, x') = \Psi^{-1}(x') \frac{\partial}{\partial x} \Psi(x) = \Psi^{-1}(x') \mathcal{L}(x;\hbar) \Psi(x) = \Psi^{-1}(x') \mathcal{L}(x;\hbar) \Psi(x') \mathbb{K}(x, x'); \tag{123}$$

$$\frac{\partial}{\partial x'} \mathbb{K}(x, x') = -\Psi^{-1}(x') \frac{\partial}{\partial x'} \Psi(x') \Psi^{-1}(x') \Psi(x) = -\Psi^{-1}(x') \mathcal{L}(x';\hbar) \Psi(x') \mathbb{K}(x, x'); \tag{124}$$

$$\frac{\partial^2}{\partial x^2} \mathbb{K}(x, x') = \Psi^{-1}(x') \left(\frac{\partial}{\partial x} \mathcal{L}(x;\hbar) + \mathcal{L}(x;\hbar)^2 \right) \Psi(x') \mathbb{K}(x, x'). \tag{125}$$

On the other hand,

$$\frac{\partial}{\partial t_k} \mathbb{K}(x, x') = \Psi^{-1}(x') \frac{\partial}{\partial t_k} \Psi(x) + \frac{\partial}{\partial t_k} \Psi^{-1}(x') \Psi(x) = \Psi^{-1}(x') (\mathcal{R}_k(x;\hbar) - \mathcal{R}_k(x';\hbar)) \Psi(x). \tag{126}$$

Using (123), (124), (125), and (113), we can rewrite

$$\begin{aligned} & (\mathcal{D} + L(x).F) \mathbb{K}(x, x') - \frac{\hbar^2}{x - x'} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \mathbb{K}(x, x') = \\ & \left(\Psi(x')^{-1} \hbar^2 \left(\frac{\partial \mathcal{L}(x;\hbar)}{\partial x} - \det \mathcal{L}(x;\hbar) - \frac{\mathcal{L}(x;\hbar) - \mathcal{L}(x';\hbar)}{x - x'} \right) \Psi(x') \right. \\ & \left. - (\hbar^2 L(x) + R(x) + \hbar^2 L(x).F) \right) \mathbb{K}. \end{aligned}$$

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Making use of (117), we obtain

$$\mathcal{O}\,\mathbb{K} = \left(\Psi(x')^{-1}\hbar^2\left(\frac{\partial\mathcal{L}(x;\hbar)}{\partial x} - \frac{\mathcal{L}(x;\hbar)-\mathcal{L}(x';\hbar)}{x-x'}\right)\Psi(x') - \hbar^2 L(x)\right)\mathbb{K}. \tag{127}$$

Using the shape of the operator $L(x)$ from Corollary 3.7 in terms of derivatives with respect to the isomonodromic parameters and the action on $\mathbb{K}$ by these derivatives given by (126), we deduce

$$L(x)\mathbb{K} = \Psi^{-1}(x')\mathcal{R}(x,x')\Psi(x')\mathbb{K}.$$

□

Let us define $T(x) := \int^x y(x)\mathrm{d}x$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We can write

$$\mathbb{K}(x,x') = e^{-\hbar^{-1}\sigma_3 T(x')}\hat{\mathbb{K}}(x,x')e^{\hbar^{-1}\sigma_3 T(x)}$$

with $\hat{\mathbb{K}}(x,x')$ analytic when $x' \to \zeta_i$, where the ζ_i are the poles of $y\mathrm{d}x$. Let us use the notation $\sigma := \hbar^{-1}\sigma_3$.

Lemma A.4. *The expression*

$$e^{\sigma T(x')}(\mathcal{O}\mathbb{K}(x,x'))\hat{\mathbb{K}}(x,x')^{-1}e^{-\sigma T(x')} \tag{128}$$

is a rational function of x and x' with no poles at $x' \to \infty$ or $x' \to \lambda_l$.

Proof. The idea now is to rewrite the operator acting on $\hat{\mathbb{K}} = \hat{\mathbb{K}}(x,x')$, instead of on $\mathbb{K} = \mathbb{K}(x,x')$, and analyze the poles of the equation, as a function of x'. We have:

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbb{K} &= e^{-\sigma T(x')}\left(\frac{\partial}{\partial x}\hat{\mathbb{K}} + \hat{\mathbb{K}}\sigma\, y(x)\right)e^{\sigma T(x)}.\\
\frac{\partial}{\partial x}\mathbb{K}\cdot\mathbb{K}^{-1} &= e^{-\sigma T(x')}\left(\frac{\partial}{\partial x}\hat{\mathbb{K}}\,\hat{\mathbb{K}}^{-1} + y(x)\hat{\mathbb{K}}\,\sigma\hat{\mathbb{K}}^{-1}\right)e^{\sigma T(x')}.\\
\frac{\partial}{\partial x'}\mathbb{K}\cdot\mathbb{K}^{-1} &= e^{-\sigma T(x')}\left(-y(x')\sigma + \frac{\partial}{\partial x'}\hat{\mathbb{K}}\,\hat{\mathbb{K}}^{-1}\right)e^{\sigma T(x')}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial^2 x}\mathbb{K}\cdot\mathbb{K}^{-1} &= e^{-\sigma T(x')}\left(\frac{\partial^2}{\partial^2 x}\hat{\mathbb{K}} + 2y(x)\frac{\partial}{\partial x}\hat{\mathbb{K}}\,\sigma + \frac{\partial}{\partial x}y(x)\hat{\mathbb{K}}\,\sigma + \hbar^{-2}y(x)^2\hat{\mathbb{K}}\right)\hat{\mathbb{K}}^{-1}e^{\sigma T(x')}.\\
L(x).\mathbb{K}\cdot\mathbb{K}^{-1} &= e^{-\sigma T(x')}\left(-L(x).T(x')\sigma + L(x).T(x)\hat{\mathbb{K}}\sigma\hat{\mathbb{K}}^{-1} + L(x).\hat{\mathbb{K}}\hat{\mathbb{K}}^{-1}\right)e^{\sigma T(x')}.
\end{aligned}$$

Therefore, we can write

$$e^{\sigma T(x')}(\mathcal{O}\mathbb{K}(x,x'))\hat{\mathbb{K}}(x,x')^{-1}e^{-\sigma T(x')} = \frac{y(x')\sigma}{x-x'} + L(x).T(x')\sigma + F(x,x'), \tag{129}$$

with $F(x,x')$ a rational function of x and x' with no poles at $x' \to \infty$ or $x' \to \lambda_l$. Let z' be one of the preimages of x': $x' = x(z')$.

- Behavior at $x' \to \infty$: We consider the behavior of $L(x).T(x')$ at $z' \to \zeta_\infty$, with $\zeta_\infty \in x^{-1}(\infty)$. Here, we call $d = -d_\infty$ and $m = m_\infty$ to simplify the notations, and use $\xi = x^{-\frac{1}{d}}, \xi' = x'^{-\frac{1}{d}}$ as local coordinates. The only terms of $L_\infty(x)$ that may bring poles at $z' \to \zeta_\infty$ when applied to $T(x')$ read

$$L_\infty(x) = \sum_{j=1+2d}^{m} t_{\zeta_\infty,j}\sum_{k=0}^{\frac{j-1}{d}-2}\xi^{-dk}\left(\frac{j}{d}-k-2\right)\frac{\partial}{\partial t_{\zeta_\infty,j-d(k+2)}}.$$

We have

$$\frac{s}{d}\frac{\partial}{\partial t_{\zeta_\infty,s}}T(x') = -\frac{1}{d}\xi'^{-s}.$$

Therefore, around $z' = \zeta_\infty$, we get

$$L_\infty(x).T(x') = -\frac{1}{d}\sum_{j=1+2d}^{m} t_{\zeta_\infty,j}\sum_{k=0}^{\frac{j-1}{d}-2}\xi^{-dk}\xi'^{-j+(k+2)d}, \tag{130}$$

which is indeed singular when $z' \to \zeta_\infty$. On the other hand, we consider the local behavior of the first term of the RHS of (129) around $z' = \zeta_\infty$

$$\frac{y(x')}{x-x'} = -\frac{y(x')}{x'-x} = \frac{1}{d}\sum_{j=0}^{m} t_{\zeta_\infty,j}\sum_{k\geq 0}\xi'^{-j+(k+2)d}\xi^{-dk},$$

whose singular terms around $z' = \zeta_\infty$ cancel exactly (130).

- Behavior at $x' \to \lambda_l$: We consider the behavior of $L(x).T(x')$ at $z' \to \zeta_l$, with $\zeta_l \in x^{-1}(\lambda_l)$. The only terms of L_Λ that may bring poles at $z' \to \zeta_l$ applied to $y(x')\mathrm{d}x'$ read

$$\sum_{s=1}^{m_l+1}\sum_{j=s-1}^{m_l} t_{\zeta_l,j}(x-\lambda_l)^{-j+s-2}s\,\partial_{\mathcal{B}_{\zeta_l,s}}\omega_{0,1}(z') = \sum_{s=1}^{m_l+1}\sum_{j=s-1}^{m_l} t_{\zeta_l,j}(x-\lambda_l)^{-j+s-2}s\oint_{\mathcal{B}_{\zeta_l,s}}\omega_{0,2}(\cdot,z')$$
$$= \sum_{s=1}^{m_l+1}\sum_{j=s-1}^{m_l}\frac{t_{\zeta_l,j}(x-\lambda_l)^{s-1}}{(x-\lambda_l)^{-j+1}}\operatorname*{Res}_{p_1=\zeta_l}\frac{\omega_{0,2}(p_1,z')}{(x(p_1)-\lambda_l)^s} = \operatorname*{Res}_{p_1=\zeta_l}\sum_{j=0}^{m_l}\frac{t_{\zeta_l,j}\,B(p_1,z')}{(x-\lambda_l)^{j+1}}\sum_{s=1}^{j+1}\frac{(x-\lambda_l)^{s-1}}{(x_1-\lambda_l)^s},$$

where we denoted $x_1 = x(p_1)$ and $\omega_{0,2} = B$ is the Bergman kernel. Developing the last sum and dropping the terms which are regular at $p_1 \to \zeta_l$, we obtain

$$\operatorname*{Res}_{p_1=\zeta_l}\frac{y(p_1)}{x_1-x}B(p_1,z'), \tag{131}$$

which is indeed irregular when $z' \to \zeta_l$. Observe that this is the local behavior around $z' = \zeta_l$ of the derivative of the second term of the RHS of (129) with respect to x'. Now taking the derivative of the first term as well, we get

$$\frac{\mathrm{d}}{\mathrm{d}x'}\frac{y(x')}{x-x'} = \operatorname*{Res}_{p_1=z'}\frac{y(p_1)}{x-x_1}B(z',p_1). \tag{132}$$

Finally, adding (131) and (132), we obtain an expression which is regular at $z' \to \zeta_l$:

$$\frac{1}{2\pi i}\oint_\gamma\frac{y(\cdot)}{x-x_1}B(z',\cdot),$$

where γ is a contour around z' and ζ_l.

Thus, (129) is a rational function of x and x' with no poles at $x' \to \infty$ or $x' \to \lambda_l$. □

Proof of Theorem A.1. Using the shape of Lemma A.3, we deduce that $\mathcal{O}\,\mathbb{K}(x,x') = 0$ when $x' \to x$ and hence also the expression given by (128) vanishes when $x' \to x$. In Lemma A.4, we have shown that (128) has no poles as a rational function of x'. Thus, it must be constant, and since it vanishes at $x' \to x$, it must be zero, which implies $\mathcal{O}\,\mathbb{K}(x,x') = 0$. □

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