ON A GEOMETRICAL THEOREM IN EXTERIOR ALGEBRA

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To my friend Donald Coxeter
on the occasion of his sixtieth birthday

In this paper we shall give necessary and sufficient conditions for three lines, passing respectively through the vertices of a proper triangle PQR in the real Euclidean plane, to be concurrent. Of course, the theorem of Ceva deals with this problem, but it is useful to have a criterion which involves only vectors localized at a point O of the plane, and the exterior products of these vectors. Applications are made to theorems which are not easily proved by other methods. We begin with a brief discussion of the exterior product of two vectors.

1. Let V be a two-dimensional vector space over the field of the real numbers. We construct a vector space Λ^2 over the field of the reals. We call this space the space of 2-vectors on V.

It consists of all sums $\sum x_i(u_i \wedge v_i)$, where the x_i are real numbers, and the u_i, v_i are vectors in V. The following rules are to hold for the "wedge" product, or "exterior product" $\wedge$:

$$(x_1 u_1 + x_2 u_2) \wedge v = x_1(u_1 \wedge v) + x_2(u_2 \wedge v),$$

$$u \wedge (y_1 v_1 + y_2 v_2) = y_1(u \wedge v_1) + y_2(u \wedge v_2),$$

$$u \wedge u = 0,$$

and

$$u \wedge v + v \wedge u = 0.$$

The fourth rule is derivable from the third. Since the exterior product $u \wedge v$ is a member of a vector space over the field of the reals, the equation $ku \wedge v = 0$, where the vector on the right is the zero vector of the vector space Λ^2, leads to the two possibilities $k = 0$ or $u \wedge v = 0$.

If u and v are dependent, so that $v = ku$, then

$$u \wedge v = u \wedge (ku) = k(u \wedge u) = 0.$$

We shall prove below that if this is not the case, then $u \wedge v \neq 0$.

A geometrical interpretation of the exterior product is easy to find. Let e_1, e_2 be orthogonal unit vectors, and let

$$u = x_1 e_1 + x_2 e_2 \quad \text{and} \quad v = y_1 e_1 + y_2 e_2.$$

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Then

$$u \wedge v = (x_1 e_1 + x_2 e_2) \wedge (y_1 e_1 + y_2 e_2)$$
$$= (x_1 y_2 - x_2 y_1) e_1 \wedge e_2.$$

The number multiplying the exterior product of e_1 and e_2 is twice the measure of the area of the triangle formed by the origin and the points (x_1, x_2), (y_1, y_2). The equation for $u \wedge v$ also shows us that the exterior product $e_1 \wedge e_2$ of the basis vectors e_1 and e_2 is a basis for the space Λ^2 of 2-vectors on V. Since we do not wish the vector space Λ^2 to consist only of the zero vector, we assume that $e_1 \wedge e_2 \neq 0$. It then follows that $u \wedge v = 0$ *if and only if u and v are linearly dependent*, which means that $(x_1, x_2) = k(y_1, y_2)$.

It is now easy to give a criterion for three points to be collinear. Let $a = (x_1, x_2)$, $b = (y_1, y_2)$, and $c = (z_1, z_2)$ be three points in the plane, and at the same time let a, b, and c also represent the vectors issuing from the origin to these points. Then we have:

THEOREM I. *The points a, b, and c are collinear if and only if*

$$a \wedge b + b \wedge c + c \wedge a = 0.$$

Proof. As vectors, we write $a = x_1 e_1 + x_2 e_2$, $b = y_1 e_1 + y_2 e_2$, and $c = z_1 e_1 + z_2 e_2$. Then, evaluating, our criterion gives us

$$[(x_1 y_2 - x_2 y_1) + (y_1 z_2 - y_2 z_1) + (z_1 x_2 - z_2 x_1)] e_1 \wedge e_2 = 0,$$

and therefore an equivalent criterion is the determinantal expression:

$$\begin{vmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 1 \end{vmatrix} = 0,$$

which we know is a necessary and sufficient condition for the collinearity of the points (x_1, x_2), (y_1, y_2), and (z_1, z_2).

This theorem is well known, and is merely given here for completeness.

As a final result, let $|u$ denote the *supplement* of the vector u, that is, the vector u rotated from the Ox_1-axis towards the Ox_2-axis through a right angle. If $u = x_1 e_1 + x_2 e_2$, then $|u = -x_2 e_1 + x_1 e_2$. It is immediate that if:

$$v = y_1 e_1 + y_2 e_2,$$

then

$$u \wedge |v = v \wedge |u = (x_1 y_1 + x_2 y_2) e_1 \wedge e_2$$
$$= (u.v) e_1 \wedge e_2,$$

where $u.v$ stands for the inner product of the vectors u and v.

2. Let us now suppose that the points P, Q, and R, which are respectively end points of the localized vectors p, q, and r, form a proper triangle; that through the point P, and originating at P, there passes a free vector p^*;

that through the point Q, and originating at Q, there passes a free vector q^*; and that through the point R, and originating at R, there passes the free vector r^*.

These free vectors determine lines through P, Q, and R, and we seek conditions for these lines to be concurrent. We replace the vectors p^*, q^*, and r^* by free vectors p', q', and r' respectively, along the same lines and also originating at the respective points P, Q, and R. We now have

THEOREM II. *A necessary and sufficient condition for the lines of action of the vectors p^*, q^*, and r^* to be concurrent is the existence of vectors p', q', and r' such that*

$$p \wedge p' + q \wedge q' + r \wedge r' = 0, \qquad \textit{where } p' + q' + r' = 0.$$

Proof. Necessity: We suppose that the lines of action of the vectors p^*, q^*, and r^* do intersect in a point W. Then if W is the end point of the localized vector w, we may write

$$w = xp + yq + zr, \qquad \text{where } x + y + z = 1.$$

Since p^*, q^*, and r^* are along the lines PW, QW, and RQ, respectively, we may choose $p' = x(w - p)$, $q' = y(w - q)$, and $r' = z(w - r)$. On adding, we find that

$$\begin{aligned} p' + q' + r' &= (x + y + z)w - xp - yq - zr \\ &= w - xp - yq - zr = 0, \end{aligned}$$

and we also find that

$$p \wedge p' + q \wedge q' + r \wedge r' = (px + qy + rz) \wedge w = w \wedge w = 0,$$

so that the necessity of the theorem is established.

Sufficiency: We are told that vectors p', q', and r' satisfying the given conditions exist. Let the line of action of p^* through P intersect the line of action of q^* through Q in the point W. Then if W is the end point of the localized vector w, we may determine x and y so that

$$p' = x(w - p) \quad \text{and} \quad q' = y(w - q),$$

since, by hypothesis, p' is along p^* and q' is along q^*. Since

$$p \wedge p' + q \wedge q' + r \wedge r' = 0,$$

$$px \wedge (w - p) + yq \wedge (w - q) + r \wedge r' = 0.$$

Also, $$p' + q' = (x + y)w - (px + qy) = -r',$$

so that $$px + qy = (x + y)w + r',$$

and on substituting in

$$\begin{aligned} px \wedge (w - p) + yq \wedge (w - q) + r \wedge r' \\ = (px + yq) \wedge w + r \wedge r' = 0, \end{aligned}$$

we obtain the equation

$$[(x + y)w + r'] \wedge w + r \wedge r' = 0,$$

which simplifies to

$$r' \wedge w + r \wedge r' = 0,$$

which can be written

$$r' \wedge (w - r) = 0.$$

Hence, since the exterior product of two non-zero vectors is only the zero vector when they are linearly dependent, we must have $r' = z(w - r)$, for some definite z, and since r' is along r^*, the line of action of r^* passes through W. The sufficiency of the theorem is therefore established.

Remarks. The theorem continues to hold if W is not a finite point. If p' is parallel to q', the condition $p' + q' + r' = 0$ shows that r' is also parallel to p' and to q', so that p^*, q^*, and r^* are all parallel. If we are given, on the other hand, that p^*, q^*, and r^* are all parallel, it is simple to prove the necessity of the given conditions.

Applications. We naturally test the theorem on the medians of triangle PQR. The medians are parallel to the vectors

$$p' = p - (q + r)/2, \qquad q' = q - (r + p)/2, \qquad r' = r - (p + q)/2.$$

Evidently $p' + q' + r' = 0$, and we also have

$$p \wedge p' + q \wedge q' + r \wedge r' = 0.$$

If we consider the perpendiculars from P, Q, and R respectively onto the sides QR, RP, and PQ, these are parallel to

$$p' = |(q - r), \qquad q' = |(r - p), \qquad \text{and } r' = |(p - q),$$

and $p' + q' + r' = |(0) = 0$. On the other hand, using the theorem that $u \wedge |v = (u.v)e_1 \wedge e_2$, we find that $p \wedge p' + q \wedge q' + r \wedge r' = ke_1 \wedge e_2$, where

$$k = p \cdot (q - r) + q \cdot (r - p) + r \cdot (p - q) = 0,$$

so that the three perpendiculars are concurrent.

A more severe test of the theorem is its application to the theory of *orthologic triangles*. ABC, PQR are two triangles such that the sides QR, RP, and PQ of PQR are respectively perpendicular to DA, DB, and DC, for some point D. Then we wish to show that lines through P, Q, and R which are respectively perpendicular to BC, CA, and AB are concurrent at a point S. With small letters representing localized vectors whose end points are at the points considered, we have

$$(q - r) \cdot (d - a) = (r - p) \cdot (d - b) = (p - q) \cdot (d - c) = 0.$$

If we add these equations, we obtain the equation

$$(q - r) \cdot a + (r - p) \cdot b + (p - q) \cdot c = 0,$$

which may be written as $p \cdot (c - b) + q \cdot (a - c) + r \cdot (b - a) = 0$. But this last equation may now be written in the form

$$p \wedge |(c - b) + q \wedge |(a - c) + r \wedge |(b - a) = 0.$$

Hence if we choose $p' = |(c - b)$, $q' = |(a - c)$, and $r' = |(b - a)$, we not only have $p' + q' + r' = 0$, but also $p \wedge p' + q \wedge q' + r \wedge r' = 0$, so that vectors in the directions of p', q', and r' through the vertices P, Q, and R respectively of PQR are concurrent. But these vectors are perpendicular to BC, CA, and AB, respectively, which is what we wish to prove.

Similar methods give an immediate proof of a theorem Forder ascribes to Casey: If triangles $A_1 A_2 A_3$ and $B_1 B_2 B_3$ are orthologic, and C_1, C_2, and C_3 divide $A_1 B_1$, $A_2 B_2$, and $A_3 B_3$ in the same ratio, then the triangle $C_1 C_2 C_3$ is orthologic both to $A_1 A_2 A_3$ and to $B_1 B_2 B_3$.

3. This paper owes its inspiration to the remarkable book by H. G. Forder, *The Calculus of Extension* (New York: Chelsea Publishing Company, 1960). Theorem I can be found there, but not Theorem II. In any case, the interpretation given here in terms of linear algebra is different. Forder introduces many concepts which I find difficult to bring down to earth. But the methods developed in his book are powerful ones, and it is evident that much work can usefully be done in simplifying and interpreting some of the concepts he uses.

For those fortunate beings, like Coxeter and myself, who have survived the Cambridge Mathematical Tripos, and are therefore among the small number of mathematicians who have studied the science of Statics, my Theorem II may have a familiar ring! If the sum of three coplanar forces acting on a rigid body is zero, their resultant is zero, and *if they all pass through a point*, their effect on a rigid body is zero: that is, they are in equilibrium. But three forces may add to zero, and yet be equivalent to a couple. The moment of a couple about any point is the same constant. The first condition of Theorem II is merely that the sum of the moments of the free vectors p', q', and r', acting at the points P, Q, and R, taken about the origin, should be zero. This, together with the condition that $p' + q' + r' = 0$ is sufficient to ensure equilibrium, which ensures the three free vectors being concurrent or parallel.

Curiously enough, the theorem was not discovered via Statics, but subconsciously a knowledge of Statics made itself felt!

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