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ON AN ALGORITHM FOR ORDERING OF GRAPHS

BY
MILAN SEKANINA(1)

Let (G, ρ) be a finite connected (undirected) graph without loops and multiple edges. So x, y being two elements of G (vertices of the graph (G, ρ)), $\langle x, y\rangle \in \rho$ means that x and y are connected by an edge. Two vertices $x, y \in G$ have the distance $\mu(x, y)$ equal to n, if n is the smallest number with the following property: there exists a sequence $x_0, x_1, \ldots, x_n$ of vertices such that $x_0 = x$, $x_n = y$ and $\langle x_{i-1}, x_i\rangle \in \rho$ for $i = 1, \ldots, n$. If $x \in G$, we put $\mu(x, x) = 0$.

If $G_1 \subset G$, (G_1, ρ) will denote the full subgraph in (G, ρ) on G_1, i.e. $x, y \in G_1$ are connected in (G_1, ρ) by an edge whenever they are connected by an edge in (G, ρ).

If (G_1, ρ) is a full subgraph of (G, ρ), then $\mathscr{R}(G_1)$ denotes the decomposition of (G_1, ρ) into connected components.

In [1] (see also [2] and [3]) the following theorem was proved:

If a, b are vertices of (G, ρ), $a \neq b$, then it is possible to order G in a sequence $a = x_1, x_2, \ldots, x_g = b$ (where $g = \text{card } G$) such that

$$\mu(x_i, x_{i+1}) \leq 3.$$

In this note there is given an algorithm for finding a sequence of this sort, but not having given b in advance. We only require the end of such a sequence to have the distance 1 from the starting point (if $g > 1$, of course).

Let us consider our graph as a collection of points with edges considered as ways connecting the points. We shall give an algorithm according to which one can successively proceed from one point to another using each edge at most once in each direction and labelling some of the passed points successively with numbers $1, 2, \ldots, g$ in such a manner that between two labellings at most two points can be passed without labelling. The last label will be given to some neighboring point to the starting point and every point will have some label.

First, let us suppose (G, ρ) is a tree.

ALGORITHM I

(1) Choose some point a and label it with 1.

(2) Let us be in a point x which has just been labelled with n.

(2.1) If there is a way $\langle x, y_1\rangle$ not passed in any direction as yet, we proceed to y_1.

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(2.2) If there exists no such y_1 described in (2.1), choose some $\langle x, y_1\rangle$ not passed in the direction from x to y_1 and we proceed to y_1.

(3.1) If (2.1) occurred then:

(3.1.1) in the case there exists $\langle y_1, y_2\rangle$ not passed in any direction we proceed to y_2 and label it with $n+1$.

(3.1.2) If y_2 from (3.1.1) does not exist, we label y_1 with $n+1$.

(3.2) If (2.2) occurred then:

(3.2.1) if (3.1.1) is valid we pass to y_2.

(3.2.2) If (3.1.1) does not hold then:

(3.2.2.1) if y_1 is not labelled we label it with $n+1$.

(3.2.2.2) If y_1 is labelled we proceed to some y_2 where $\langle y_1, y_2\rangle$ has not been passed in the direction from y_1 to y_2.

(4) If (3.2.1) occured then, if there exists $\langle y_2, y_3\rangle$ not passed in any direction, we proceed to y_3 and label it with $n+1$. In other case y_2 is labelled with $n+1$.

If (3.2.2.2) occurred, and if there exists $\langle y_2, y_3\rangle$ not passed in any direction, we pass to y_3 and label it with $n+1$. Otherwise y_2 is labelled with $n+1$.

We must stop if we are to label some point already having some label or when we cannot proceed further. We shall prove that following our algorithm we shall stop at a and the last labelling concerns some z with $\mu(z, a)=1$. The last used way is from z to a. After that all ways were passed in each direction just once and every point has a label.

The proof will be done by induction on g.

The assertion is trivial for $g=1$. Let $g>1$.

(A) Suppose that a is not an end vertex of (G, ρ). Let $\{(G_1, \rho), \ldots, (G_k, \rho)\} = \mathscr{R}(G-\{a\})$. Let card $G_i=g_i$. First we label a with 1. Then we proceed by (2.1) to some y_1. Let us choose our notation so that $y_1 \in G_1$. Then we can apply our assumption to the tree $(G_1 \cup \{a\}, \rho)$. So the algorithm brings us to a after passing all ways in $(G_1 \cup \{a\}, \rho)$ in both directions just once, after having given a label to each point from $G_1 \cup \{a\}$ and the last label g_1+1 has been given to some y at the distance 1 from a. So now we are in the situation described by (3.2) and we can proceed by (3.2.1). Let us use the way $\langle a, z\rangle$, $z \in (G_2, \rho)$. After that we proceed by (4). So our situation is similar to what it would be if we were describing $(G_2 \cup \{a\}, \rho)$ starting from a. Thus we can use our assumption for $(G_2 \cup \{a\}, \rho)$ and we end the labelling as we did for $(G_1 \cup \{a\}, \rho)$ but as the first label we use g_1+2, the last g_1+g_2+1. Then we are labelling (G_3, ρ) (under suitable notation), etc., and we finish the labelling of the whole graph (G, ρ) by labelling of (G_k, ρ) and stopping in a after having labelled with the last label g some vertex $v \in G_k$ with $\mu(a, v)=1$ and having used then $\langle v, a\rangle$ in the direction from v to a.

(B) Let a be an end vertex of (G, ρ). The case $g=2$ is clear. Suppose $g \geq 3$. So after labelling of a with 1 we proceed by (2.1) to some y_1 and then by (3.1.1) to some y_2 and this is labelled with 2. Let $G_1 \in \mathscr{R}(G-\{a, y_1\})$ and $y_2 \in G_1$. Then we can apply our algorithm to (G_1, ρ). We end in y_2 after labelling with the last label some $y \in G_1$ with $\mu(y, y_2)=1$ and using for the last time the way $\langle y, y_2\rangle$ in the direction from y to y_2 or the last label belongs to y_2 (in the case $G_1=\{y_2\}$). So we are now in the situation described in (3.2.2.2) or (2.2). We must use (there is no other choice) $\langle y_2, y_1\rangle$ in the direction from y_2 to y_1. If there is no other set in $\mathscr{R}(G-\{a, y_1\})$ besides G_1 we must label (by (4) or (3.2.2.1)) y_1 and then proceed to a. Then we are done. So let $\{G_1\} \neq \mathscr{R}(G-\{a, y_1\})$.

From y_1 (we did not label it) we proceed to $y_3 \in G_2$, $G_2 \in \mathscr{R}(G-\{a, y_1\})$, $G_2 \neq G_1$ (by (4) or (3.2.1)).

(a) Let the first case occur (so (4) was used). Then y_3 must be labelled in this step and the case of G_2 is the same as for G_1. Then we use our algorithm for (G_2, ρ).

(b) If (3.2.1) was used then we must label y_3 if $\{y_3\}=G_2$ (by (4), first part). If $\{y_3\} \neq G_2$ we choose some way $\langle y_3, y_4\rangle$ not passed before. Clearly $y_4 \in G_2$ and we label y_4. We can use our algorithm for $(G_2 \cup \{y_1\}, \rho)$ where y_1 is considered as a starting point but without being labelled.

In the case (a) we end in y_3, the last vertex labelled being y_3 (if $\{y_3\}=G_2$) or some $z \in G_2$ with $\mu(z, y_3)=1$ and passing then along $\langle z, y_3\rangle$ in the direction from z to y_3.

In the case (b) we end in y_1, the last vertex labelled being y_3 and having passed then from y_3 to y_1.

If $\{G_1, G_2\}=\mathscr{R}(G-\{a, y_1\})$ then by (3.2.2.1) or (4) next labelling concerns y_1 and then we proceed to a (we have no other choice).

If $\{G_1, G_2\} \neq \mathscr{R}(G-\{a, y_1\})$ we do not label y_1 and we proceed by (3.2.1) or (4) to some $v \in G' \in \mathscr{R}(G-\{a, y_1\})$, $G_1 \neq G' \neq G_2$. Further, we proceed as in G_2.

After finite number of steps all vertices of sets in $\mathscr{R}(G-\{a, y_1\})$ are labelled, then by (3.2.2.1) or (4) y_1 is labelled and we proceed in the only possible way, $\langle y_1, a\rangle$ from y_1 to a.

Let (G, ρ) now be a quite arbitrary finite connected (undirected) graph without loops and multiple edges with card $G>1$.

Algorithm II

Let Algorithm II differ from Algorithm I by adding the following rule:

Let us be in the point x and let $\langle x, y\rangle$ be such a way which was not passed before in any direction but in some previous step we were already in y (y can be still without label). Then $\langle x, y\rangle$ cannot be used.

So by this rule the cases (2.1), (3.1.1), (3.2.1), and (4, first part) of Algorithm I are modified.

Now we shall show that Algorithm II has the same effect as Algorithm I for trees with one exception: some ways may not be passed.

Let us suppose that Algorithm II stopped. First of all, it is clear that the passed ways (maybe some of them have been passed only in one direction) and passed vertices form a subgraph of (G, ρ) which is a tree. Denote this tree as (H, ρ_1). Using of Algorithm II for (G, ρ) can now be considered as the using of Algorithm I for (H, ρ_1). So we must end in the point a having labelled in the previous step a vertex y with $\mu(a, y)=1$. Suppose $G-H\neq \varnothing$. Let $z\in G-H$ and let z be connected by an edge with some $x\in H$ (such z clearly exists). $\langle x, z\rangle$ could be used and we had to use it at least when we were leaving x for the last time if $x\neq a$ and if $x=a$ we could continue our procedure to z by using $\langle a, z\rangle$. So $G-H=\varnothing$.

Let us add not so formal but more transparent description of Algorithm II.

(1) Choose some point a and label it with 1.

(2) Non-passed way (in any direction) to a passed point (although without label) cannot be used.

(3) Non-passed usable way is to be preferred to a way passed already in the opposite direction and every way can be passed in each direction at most once.

(4) Before continuing along one direction passed way we must label our point if it is still without any label.

(5) Only two non-passed ways between two successive labellings can be used and if possible they must be used.

(6) At most two points can be passed between two successive labellings.

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University of Manitoba,
Winnipeg, Manitoba

University of J. E. Purkyně,
Brno, Czechoslovakia