Chapter 1

Basic Definitions and Concepts from Metric Spaces

In this chapter, we gather some basic definitions, concepts, and results from metric spaces which are required throughout the book. For detail study of metric spaces, we refer to [8, 46, 61, 95, 110, 150, 154].

1.1 Definitions and Examples

Definition 1.1 Let X be a nonempty set. A real-valued function $d : X \times X \to \mathbb{R}$ is said to be a *metric* on X if it satisfies the following conditions:

(M1) $d(x, y) \geq 0$ for all $x, y \in X$;
(M2) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(M3) $d(x, y) = d(y, x)$ for all $x, y \in X$; (symmetry)
(M4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$. (triangle inequality)

The set X together with a metric d on X is called a *metric space* and it is denoted by (X, d). If there is no confusion likely to occur we, sometime, denote the metric space (X, d) by X.

Example 1.1 Let X be a nonempty set. For any $x, y \in X$, define

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then d is a metric, and it is called a *discrete metric*. The space (X, d) is called a *discrete metric space*.

The above example shows that on each nonempty set, at least one metric that is a discrete metric can be defined.

Example 1.2 Let $X = \mathbb{R}^n$, the set of ordered n-tuples of real numbers. For any $x = (x_1, x_2, \dots, x_n) \in X$ and $y = (y_1, y_2, \dots, y_n) \in X$, we define

(a) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, (called taxicab metric)

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(b) $d_2(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}},$ (called usual metric)

(c) $d_p(x,y) = \left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1$

(d) $d_\infty(x,y) = \max_{1\leq i\leq n}|x_i - y_i|.$ (called max metric)

Then, d_1, d_2, d_p $(p \geq 1)$, d_∞ are metrics on $\mathbb{R}^n$.

Example 1.3 Let ℓ^∞ be the space of all bounded sequences of real or complex numbers, that is,

$$\ell^\infty = \left\{\{x_n\} \subset \mathbb{R} \text{ or } \mathbb{C} : \sup_{1\leq n<\infty}|x_n| < \infty\right\}.$$

Then,

$$d_\infty(x,y) = \sup_{1\leq n<\infty}|x_n - y_n|, \quad \text{for all } x = \{x_n\},\ y = \{y_n\} \in \ell^\infty,$$

is a metric on ℓ^∞ and (ℓ^∞, d_∞) is a metric space.

Example 1.4 Let s be the space of all sequences of real or complex numbers, that is,

$$s = \{\{x_n\} \subset \mathbb{R} \text{ or } \mathbb{C}\}.$$

Then,

$$d(x,y) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \text{for all } x = \{x_n\},\ y = \{y_n\} \in s,$$

is a metric on s.

Example 1.5 Let ℓ^p, $1 \leq p < \infty$, denote the space of all sequences $\{x_n\}$ of real or complex numbers such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$, that is,

$$\ell^p = \left\{\{x_n\} \subset \mathbb{R} \text{ or } \mathbb{C} : \sum_{n=1}^{\infty}|x_n|^p < \infty\right\}, \quad \text{for } 1 \leq p < \infty.$$

Then,

$$d(x,y) = \left(\sum_{n=1}^{\infty}|x_n - y_n|^p\right)^{\frac{1}{p}}, \quad \text{for all } x = \{x_n\},\ y = \{y_n\} \in \ell^p,$$

is a metric on ℓ^p and (ℓ^p, d) is a metric space.

Example 1.6 Let $B[a,b]$ be the space of all bounded real-valued functions defined on $[a,b]$, that is,

$$B[a,b] = \{f : [a,b] \to \mathbb{R} : |f(t)| \leq k \text{ for all } t \in [a,b] \text{ and for some constant } k \in \mathbb{R}\}.$$

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Then,

$$d(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|, \quad \text{for all } f, g \in B[a,b],$$

is a metric on $B[a,b]$.

Example 1.7 Let $C[a,b]$ be the space of all continuous real-valued functions defined on $[a,b]$. For any $f, g \in C[a,b]$, we define the real-valued functions d_∞ and d_1 on $C[a,b] \times C[a,b]$ as follows:

$$d_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

and

$$d_1(f,g) = \int_a^b |f(t) - g(t)|\, dt,$$

where the integral is the Riemann integral which is possible because the functions f and g are continuous on $[a,b]$. Then, d_∞ and d_1 are metrics on $C[a,b]$.

Definition 1.2 Let X be a nonempty set. A real-valued function $d : X \times X \to \mathbb{R}$ is said to be a *pseudometric* on X if it satisfies the following conditions:

(PM1) $d(x,y) \geq 0$ for all $x, y \in X$;
(PM2) $d(x,y) = 0$ if $x = y$ for all $x, y \in X$;
(PM3) $d(x,y) = d(y,x)$ for all $x, y \in X$; (symmetry)
(PM4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. (triangle inequality)

The set X together with a pseudometric d on X is called a *pseudometric space*.

Example 1.8 Let $X = \mathbb{R}^2$ and $d(x,y) = |x_1 - y_1|$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$. Then, d is not a metric on X; however, it is a pseudometric on X. Indeed, for $x = (0,0)$, $y = (0,1) \in X$, we have $d(x,y) = 0$ but $x \neq y$. Therefore, it is not a metric on X. It can be easily checked that d satisfies the conditions (PM1) – (PM4).

Definition 1.3 Let X be a nonempty set. A real-valued function $d : X \times X \to \mathbb{R}$ is said to be a *quasimetric* on X if it satisfies the following conditions:

(QM1) $d(x,y) \geq 0$ for all $x, y \in X$;
(QM2) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(QM3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. (triangle inequality)

The set X together with a quasimetric d on X is called a *quasimetric space*.

Example 1.9 The real-valued functions $d_1, d_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d_1(x,y) = \begin{cases} y - x, & \text{if } y \geq x, \\ \alpha(x - y), & \text{if } y < x, \end{cases}$$

for $\alpha > 0$, and

$$d_2(x,y) = \begin{cases} e^y - e^x, & \text{if } y \geq x, \\ e^{-y} - x^{-x}, & \text{if } y < x, \end{cases}$$

are quasimetrics on $\mathbb{R}$.

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Definition 1.4 Let (X, d) be a metric space and let A and B be nonempty subsets of X. The *distance between the sets A and B* is given by

$$d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}.$$

Since $d(x, y) = d(y, x)$, we have $d(A, B) = d(B, A)$.

If A is a singleton set $\{x\}$, then

$$d(\{x\}, B) = \inf\{d(x, y) : y \in B\}.$$

It is called the *distance of a point* $x \in X$ *from the set* B, and we write $d(x, B)$ in place of $d(\{x\}, B)$.

Remark 1.1 **(a)** The equation $d(x, B) = 0$ does not imply that x belongs to B.

(b) If $d(A, B) = 0$, then it is not necessary that A and B have common points.

Example 1.10 Let $A = \{x \in \mathbb{R} : x > 0\}$ and $B = \{x \in \mathbb{R} : x < 0\}$ be subsets of $\mathbb{R}$ with the usual metric. Then $d(A, B) = 0$, but A and B have no common point. If $x = 0$, then $d(x, B) = 0$; but $x \notin B$.

Definition 1.5 Let (X, d) be a metric space and A be a nonempty subset of X. The *diameter of* A, denoted by $\operatorname{diam}(A)$, is given by

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

The set A is called *bounded* if there exists a constant k such that $\operatorname{diam}(A) \leq k < \infty$. In other words, A is bounded if its diameter is finite, otherwise it is called *unbounded*.

In particular, the metric space (X, d) is bounded if the set X is bounded.

1.2 Open Sets and Closed Sets

Definition 1.6 Let (X, d) be a metric space. Given a point $x_0 \in X$ and a real number $r > 0$, the sets

$$S_r(x_0) = \{y \in X : d(x_0, y) < r\}$$

and

$$S_r[x_0] = \{y \in X : d(x_0, y) \leq r\}$$

are called *open sphere* (or *open ball*) and *closed sphere* (or *closed ball*), respectively, with center at x_0 and radius r.

Remark 1.2 **(a)** The open and closed spheres are always nonempty, since $x_0 \in S_r(x_0) \subseteq S_r[x_0]$.

(b) Every open (respectively, closed) sphere in $\mathbb{R}$ with the usual metric is an open (respectively, closed) interval. But the converse is not true; for example, $(-\infty, \infty)$ is an open interval in $\mathbb{R}$ but not an open sphere.

Definition 1.7 Let A be a nonempty subset of a metric space X.

(a) A point $x \in A$ is said to be an *interior point* of A if x is the center of some open sphere contained in A. In other words, $x \in A$ is an interior point of A if there exists $r > 0$ such that $S_r(x) \subseteq A$.

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(b) The set of all interior points of A is called *interior* of A and it is denoted by A°, that is,

$$A^\circ = \{x \in A : S_r(x) \subseteq A \text{ for some } r > 0\}.$$

(c) The set A is said to be *open* if each of its points is the center of some open sphere contained entirely in A; that is, A is an open set if for each $x \in A$, there exists $r > 0$ such that $S_r(x) \subseteq A$.

(d) Let $x \in X$. The set A is said to be a *neighborhood* of x if there exists an open sphere centered at x and contained in A, that is, if $S_r(x) \subseteq A$ for some $r > 0$. In case A is an open set, it is called an *open neighborhood* of x.

Remark 1.3 In a metric space, we have the following:

(a) An open sphere $S_r(x)$ with center at x and radius r is a neighborhood of x.
(b) The interior of A is the neighborhood of each of its points.
(c) Every open set is the neighborhood of each of its points.
(d) The set A is open if and only if each of its points is an interior point, that is, $A = A^\circ$.
(e) Arbitrary union of open sets is open.
(f) Finite intersection of open sets is open.
(g) Arbitrary intersection of open sets need not be open.

Theorem 1.1 *Let A and B be two subsets of a metric space X. Then,*

(a) $A \subseteq B$ *implies* $A^\circ \subseteq B^\circ$;
(b) $(A \cap B)^\circ = A^\circ \cap B^\circ$;
(c) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.

Definition 1.8 Let A be a subset of a metric space X. A point $x \in X$ is said to be a *limit point* (*accumulation point* or *cluster point*) of A if each open sphere centered at x contains at least one point of A other than x.

In other words, $x \in X$ is a limit point of A if

$$(S_r(x) - \{x\}) \cap A \neq \emptyset, \quad \text{for all } r > 0.$$

The set of all limit points of A is called *derived set* and it is denoted by A'.

Definition 1.9 A point $x \in X$ is said to be an *isolated point* of A if there exists an open sphere centered at x which contains no point of A other than x itself, that is, if $S_r(x) \cap A = \{x\}$ for some $r > 0$.

Remark 1.4 If a point $x \in X$ is not a limit point of A, then it is an isolated point. Hence every point of a metric space X is either a limit point or an isolated point of X.

Example 1.11 Consider the metric space $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ with the usual metric given by the absolute value. Then, 0 is the only limit point of X while all other points are the isolated points of X.

Definition 1.10 Let A be a subset of a metric space X. The *closure* of A, denoted by $\overline{A}$ or clA, is the union of A and the set of all limit points of A, that is, $\overline{A} = A \cup A'$.

In other words, $x \in \overline{A}$ if every open sphere $S_r(x)$ centered at x and radius $r > 0$ contains a point of A, that is, $x \in \overline{A}$ if and only if $S_r(x) \cap A \neq \emptyset$ for every $r > 0$.

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Remark 1.5 Let A and B be subsets of a metric space X. Then,

(a) $\overline{\emptyset} = \emptyset$;
(b) $\overline{X} = X$;
(c) $\overline{(\overline{A})} = \overline{A}$;
(d) $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$;
(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(f) $\overline{A} = \left(\overline{A}\right)'$;
(g) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, but $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$.

Theorem 1.2 *Let (X, d) be a metric space and A be a subset of X. Then, $x \in \overline{A}$ if and only if $d(x, A) = 0$.*

Definition 1.11 Let A be a subset of a metric space X. The set A is said to be *closed* if it contains all its limit points, that is, $A' \subseteq A$.

Remark 1.6 **(a)** Let A be a subset of a metric space X. Then clearly A is closed if and only if $\overline{A} = A$.
(b) Let A be a subset of a metric space X. Then A is closed if and only if the complement of A is an open set.
(c) In a metric space, every finite set, empty set, and whole space are closed sets.
(d) Arbitrary intersection of closed sets is closed.
(e) Finite union of closed sets is closed. However, arbitrary union of closed sets need not be closed.

Definition 1.12 Let A be a subset of a metric space X. A point $x \in X$ is called a *boundary point* of A if it is neither an interior point of A nor of $X \setminus A$, that is, $x \notin A^\circ$ and $x \notin (X \setminus A)^\circ$.

In other words, $x \in X$ is a *boundary point* of A if every open sphere centered at x intersects both A and $X \setminus A$.

The set of all boundary points of A is called the *boundary of* A and it is denoted by $\text{bd}(A)$.

Remark 1.7 It is clear that $\text{bd}(A) = \overline{A} \cap \overline{(X \setminus A)} = \overline{A} \cap \overline{A^c}$.

1.3 Complete Metric Spaces

Definition 1.13 Let (X, d) be a metric space. A sequence $\{x_n\}$ of points of X is said to be *convergent* if there is a point $x \in X$ such that for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_n, x) < \varepsilon, \quad \text{for all } n > N.$$

The point $x \in X$ is called a *limit point* of the sequence $\{x_n\}$.

More preciously, a sequence $\{x_n\}$ in a metric space X converges to a point $x \in X$ if the sequence $\{d(x_n, x)\}$ of real numbers converges to 0.

Since $d(x_n, x) < \varepsilon$ is equivalent to $x_n \in S_\varepsilon(x)$, the definition of convergent sequence can be restated as follows:

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A sequence $\{x_n\}$ in a metric space X *converges to a point* $x \in X$ if and only if for each $\varepsilon > 0$, there exists a positive integer N such that

$$x_n \in S_\varepsilon(x), \quad \text{for all } n > N.$$

For a convergent sequence $\{x_n\}$ to x, we use the following symbols:

$$x_n \to x \quad \text{or} \quad \lim_{n\to\infty} x_n = x$$

and we express it by saying that x_n approaches x or that x_n converges to x.

Definition 1.14 A sequence $\{x_n\}$ in a metric space X is said to be *bounded* if the range set of the sequence is bounded.

Remark 1.8 In a metric space, every convergent sequence is bounded.

Definition 1.15 Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be a *Cauchy sequence* if for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_n, x_m) < \varepsilon, \quad \text{for all } n, m > N.$$

Theorem 1.3 *Every convergent sequence in a metric space is a Cauchy sequence.*

Exercise 1.1 Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$ for all n. Prove that $\{x_n\}$ is a Cauchy sequence.

Proof Let $\varepsilon > 0$ and choose a positive integer N such that $\frac{1}{2^{N-1}} < \varepsilon$. Then for all $n > m > N$, we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} \\ &< \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}} < \frac{1}{2^{N-1}} < \varepsilon. \end{aligned}$$

■

Definition 1.16 A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Example 1.12 The space $\mathbb{R}^n$ with respect to all the metrics given in Example 1.2 is complete. The space $C[0, 1]$ with respect to the metric d_1 given in Example 1.7 is not complete.

Remark 1.9 A metric space (X, d) is complete if and only if every Cauchy sequence in X has a convergent subsequence.

Exercise 1.2 Let (X, d_X) and (Y, d_Y) be metric spaces. Define

$$d_{X\times Y}((x, y), (u, v)) = d_X(x, u) + d_Y(y, v), \quad \text{for all } (x, y), (u, v) \in X \times Y.$$

Prove that $d_{X\times Y}$ is a metric on $X \times Y$. Further, if (X, d_X) and (Y, d_Y) are complete, then prove that $(X \times Y, d_{X\times Y})$ is also complete.

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Theorem 1.4 (Cantor's Intersection Theorem) *Let (X, d) be a complete metric space and $\{A_n\}$ be a decreasing sequence (that is, $A_{n+1} \subseteq A_n$) of nonempty closed subsets of X such that* $\operatorname{diam}(A_n) \to 0$ *as $n \to \infty$. Then, the intersection $\bigcap_{n=1}^{\infty} A_n$ contains exactly one point.*

The converse of the above theorem is the following:

Theorem 1.5 *Let (X, d) be a metric space. If any decreasing sequence $\{A_n\}$ of nonempty closed sets in X with* $\operatorname{diam}(A_n) \to 0$ *as $n \to \infty$ has exactly one point in its intersection, then (X, d) is complete.*

Definition 1.17 A nonempty subset A of a metric space X is said to be *dense* (or *everywhere dense*) in X if $\overline{A} = X$, that is, if every point of X is either a point or a limit point of A.

In other words, a set A is dense in X if for any given point $x \in X$, there exists a sequence of points of A that converges to x.

It can be easily seen that a subset A of X is dense if and only if A^c has empty interior.

Before giving the examples of dense sets, we provide the criteria for being dense.

Theorem 1.6 *Let A be a nonempty subset of a metric space X. The following statements are equivalent:*

(a) *For every $x \in X$, $d(x, A) = 0$.*
(b) $\overline{A} = X$.
(c) *A has nonempty intersection with every nonempty open subset of X.*

Example 1.13 **(a)** The set of all rational numbers $\mathbb{Q}$ is dense in the usual metric space $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$.

(b) Since $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, the set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in the usual metric space $\mathbb{R}$.

(c) The set $A = \{a + ib \in \mathbb{C} : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$ since $\overline{A} = \mathbb{C}$.

(d) The set $\mathbb{Q}^n = \underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{n\text{-times}}$ is dense in $\mathbb{R}^n$ with the usual metric.

(e) The set

$$A = \{x = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) : a_i \in \mathbb{Q} \text{ for all } 1 \leq i \leq n \text{ and } n \in \mathbb{N}\}$$

is dense in the space ℓ^p, $1 \leq p < \infty$, with the following metric:

$$d_p(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p},$$

where $x = \{x_1, x_2, \ldots\}$ and $y = \{y_1, y_2, \ldots\}$ in ℓ^p.

(f) The set $P[a, b]$ of all polynomials defined on $[a, b]$ with rational coefficients is dense in $C[a, b]$.

(g) Let (X, d) be a discrete metric space. Since every subset of X is closed, the only dense subset of X is itself.

Definition 1.18 A metric space X is said to be *separable* if there exists a countable dense set in X.

A metric space which is not separable is called *inseparable*.

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Example 1.14 **(a)** The usual metric space $\mathbb{R}$ is separable since the set of all rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

(b) The usual metric space $\mathbb{C}$ is separable since the set $A = \{a + ib \in \mathbb{C} : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

(c) The Euclidean space $\mathbb{R}^n$ is separable since the set $\mathbb{Q}^n = \underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{n\text{-times}}$ is countable and dense in $\mathbb{R}^n$.

(d) The space ℓ^p, $1 \le p < \infty$, is separable as the set

$$A = \left\{x = (a_1, a_2, \dots, a_n, 0, 0, \dots) : a_i \in \mathbb{Q}, 1 \le i \le n \text{ and for all } n \in \mathbb{N}\right\}$$

is countable and dense in the space ℓ^p.

(e) The space $C[a, b]$ is separable since the set $P[a, b]$ of all polynomials defined on $[a, b]$ with rational coefficients is countable and dense in $C[a, b]$.

(f) A discrete metric space X is separable if and only if the set X is countable.

Example 1.15 The space ℓ^∞ of all bounded sequences of real or complex numbers with the metric

$$d_\infty(x, y) = \sup_{1 \le n < \infty} |x_n - y_n|,$$

where $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ^∞, is not separable.

Definition 1.19 Two metrics d_1 and d_2 on the same underlying set X are said to be *equivalent* if for every sequence $\{x_n\}$ in X and $x \in X$,

$$\lim_{n \to \infty} d_1(x_n, x) = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} d_2(x_n, x) = 0,$$

that is, a sequence converges to x with respect to the metric d_1 if and only if it converges to x with respect to the metric d_2.

The metric spaces (X, d_1) and (X, d_2) are said to be *equivalent* if the metrics d_1 and d_2 are equivalent.

Remark 1.10 If two metrics are equivalent, then the families of open sets are same in (X, d_1) and (X, d_2).

The following result provides a sufficient condition for two metrics on a set to be equivalent.

Theorem 1.7 *Two metrics d_1 and d_2 on a nonempty set X are equivalent if there exist constants k_1, $k_2 > 0$ such that*

$$k_1 d_2(x, y) \le d_1(x, y) \le k_2 d_2(x, y), \quad \textit{for all } x, y \in X. \tag{1.1}$$

1.4 Compact Spaces

Definition 1.20 Let X be a metric space and Λ be any index set.

(a) A collection $\mathscr{F} = \{G_\alpha\}_{\alpha \in \Lambda}$ of subsets of X is called a *cover* of X if $\bigcup_{\alpha \in \Lambda} G_\alpha = X$, that is, every element of X belongs to at least one member of $\mathscr{F}$. If each member of $\mathscr{F}$ is an open set in X, then it is called an *open cover* of X.

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(b) A subcollection $\mathscr{C}$ of a cover $\mathscr{F}$ of X is called a *subcover* if $\mathscr{C}$ is itself a cover of X. $\mathscr{C}$ is called a *finite subcover* if it consists only a finite number of members.
In other words, if there exist $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n} \in \mathscr{F}$ such that $\bigcup_{k=1}^{n} G_{\alpha_k} = X$, then the subcollection $\mathscr{C} = \left\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\right\}$ is called a finite subcover of X.
In this case, $\mathscr{F}$ is said to be *reducible to a finite cover* or contains a *finite subcover*.

Definition 1.21 Let X be a metric space and Y be a subset of X. A collection $\mathscr{F} = \{G_\alpha\}_{\alpha \in \Lambda}$ of subsets of X is said to be a *cover* of Y if $Y \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha$.

Definition 1.22 A metric space X is said to be *compact* if every open cover of X has a finite subcover.

A nonempty subset Y of a metric space (X, d) is compact if it is a compact metric space with the metric induced on it by d.

Theorem 1.8 *Every closed subset of a compact metric space is compact.*

Definition 1.23 A collection $\mathcal{C} = \left\{C_1, C_2, \dots\right\}$ of subsets of a metric space X is said to have the *finite intersection property* if every finite subcollection of $\mathcal{C}$ has nonempty intersection, that is, for every finite collection $\left\{C_1, C_2, \dots, C_n\right\}$ of $\mathcal{C}$, we have $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

Theorem 1.9 *A metric space X is compact if and only if every collection of closed sets in X having finite intersection property has nonempty intersection.*

Definition 1.24 A metric space X is said to have the *Bolzano–Weierstrass property* if every infinite subset of X has a limit point.

Definition 1.25 A metric space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

A subset A of a metric space X is said to be *sequentially compact* if every sequence in A contains a subsequence which converges to a point in A.

It is well known that

$$\text{compactness} \Leftrightarrow \text{Bolzano–Weierstrass property} \Leftrightarrow \text{sequentially compactness}$$

Definition 1.26 Let (X, d) be a metric space and $\varepsilon > 0$ be given. A subset A of X is called an *ε-net* if A is finite and $X = \bigcup_{x \in A} S_\varepsilon(x)$, that is, if A is finite and its points are scattered through X in such a way that each point of X is distant by less than ε from at least one point of A.

In other words, a finite subset $A = \left\{x_1, x_2, \dots, x_n\right\}$ of X is an ε-net for X if for every point $y \in X$, there exists an $x_{i_0} \in A$ such that $d\left(y, x_{i_0}\right) < \varepsilon$.

Example 1.16 Let $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < 4\}$, that is, X is the open sphere centered at the origin and radius 2. If $\varepsilon = \frac{3}{2}$, then the set

$$A = \left\{(1, -1), (1, 0), (1, 1), (0, -1), (0, 0), (0, 1), (-1, -1), (-1, 0), (-1, 1)\right\}$$

is an ε-net for X.

On the other hand, if $\varepsilon = 1/2$, then A is not an ε-net for X. For example, the point $y = \left(\frac{1}{2}, \frac{1}{2}\right)$ belongs to X but the distance between y and any point in A is greater than $\frac{1}{2}$.

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Definition 1.27 A metric space (X, d) is said to be *totally bounded* if it has an ε-net for each $\varepsilon > 0$.

Remark 1.11 Every totally bounded metric space X is bounded but the converse is not true in general.

Since X is totally bounded, it has an ε-net $A = \{x_1, x_2, \dots, x_n\}$ for each $\varepsilon > 0$. Then, $X = \bigcup_{i=1}^{n} S_\varepsilon(x_i)$. Since finite union of bounded sets is bounded, it follows that X is bounded.

Example 1.17 Under the usual metric $d(x, y) = |x - y|$, the real line $\mathbb{R}$ is neither bounded nor totally bounded. Under the metric $d^*(x, y) = \min\{|x - y|, 1\}$, the real line $\mathbb{R}$ is bounded but not totally bounded.

Theorem 1.10 *Every totally bounded and complete metric space is compact.*

Theorem 1.11 *Every totally bounded metric space is separable.*

Remark 1.12 A discrete metric space is compact if and only if it is finite.

1.5 Continuous Functions

Definition 1.28 Let X be a nonempty set. A function $f : X \to \mathbb{R}$ is said to be

(a) *bounded above* if there exists a real number k such that $f(x) \leq k$ for all $x \in X$;
(b) *bounded below* if there exists a real number k such that $k \leq f(x)$ for all $x \in X$;
(c) *bounded* if it is both bounded above as well as bounded below.

Definition 1.29 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is said to be *continuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$,

$$d_X(x, x_0) < \delta \quad \text{implies} \quad d_Y(f(x), f(x_0)) < \varepsilon,$$

that is,

$$x \in S_\delta(x_0) \quad \text{implies} \quad f(x) \in S_\varepsilon(f(x_0)),$$

(see Figure 1.1). In other words, f is continuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f(S_\delta(x_0)) \subseteq S_\varepsilon(f(x_0)).$$

The function f is said to be *continuous on* X if it is continuous at every point of X.

Theorem 1.12 *Let X and Y be metric spaces and $f : X \to Y$ be a function. The following statements are equivalent:*

(a) *f is continuous on X.*
(b) *For every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ implies $f(x_n) \to f(x)$.*
(c) *$f^{-1}(B)$ is open in X wherever B is open in Y.*
(d) *$f^{-1}(D)$ is closed in X wherever D is closed in Y.*
(e) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.*

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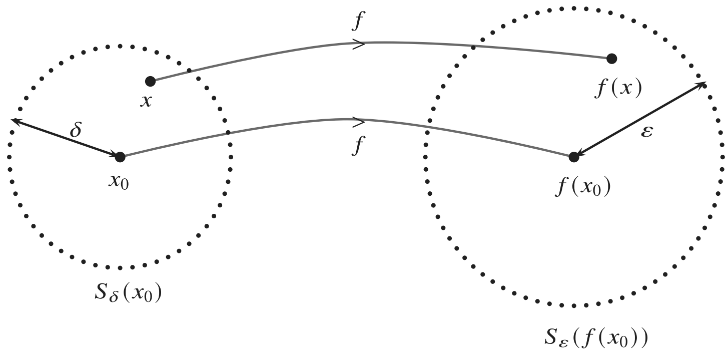

Figure 1.1 A continuous function

Theorem 1.13 *Let X and Y be metric spaces and $f : X \to Y$ be a continuous function. If A is a compact subset of X, then $f(A)$ is compact in Y.*

Exercise 1.3 Prove that a continuous real-valued function defined on a compact set is bounded and it assumes maximum and minimum values.

Proof Let $f : X \to \mathbb{R}$ be continuous and A be a compact subset of a metric space X. By Theorem 1.13, we see that $f(A)$ is a compact subset of $\mathbb{R}$. By Heine-Borel Theorem "A subset of $\mathbb{R}$ is closed and bounded if and only if it is compact", $f(A)$ is closed and bounded. Thus, $\sup f(A)$ and $\inf f(A)$ exist and belong to $f(A)$. Therefore, there exist $\hat{x}, \tilde{x} \in A$ such that for all $y \in A$, $\inf f(A) = f(\hat{x}) \le f(y) \le f(\tilde{x}) = \sup f(A)$. ■

Exercise 1.4 Let (X, d) be a metric space and A be a nonempty compact subset of X. Prove that for every $x_0 \in X$, there exists a $y_0 \in A$ such that

$$d(x_0, y_0) = d(x_0, A) = \inf_{y \in A} d(x_0, y).$$

Proof Consider the real-valued function $f : A \to \mathbb{R}_+$ defined by $f(x) = d(x, x_0)$ for all $x \in A$. Now $|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \le d(x, y)$, so f is continuous on A. But A is compact, so f has a minimum on A by Exercise 1.3. That is, there exists a $y_0 \in A$ such that $f(y_0) = d(x_0, y_0) = \inf_{y \in A} d(x_0, y) = d(x_0, A)$. ■

Definition 1.30 Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if for each $\varepsilon > 0$, there exists a $\delta > 0$ (depends only on ε) such that for every $x, y \in X$,

$$d(x, y) < \delta \quad \text{implies} \quad \rho(f(x), f(y)) < \varepsilon.$$

Remark 1.13 Every uniform continuous function is continuous but the converse need not be true in general.

Example 1.18 **(a)** Let X be a discrete metric space and Y be any metric space. Then, any function $f : X \to Y$ is uniformly continuous.

(b) Let $X = (0, 1)$ be a metric space with the metric induced by the usual metric on $\mathbb{R}$ and $Y = \mathbb{R}$ with the usual metric. The function $f : X \to Y$ defined by $f(x) = \frac{1}{x}$, for all $x \in X$, is not uniformly continuous.

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(c) No polynomial function of degree greater than 1 is uniformly continuous on the usual metric space $\mathbb{R}$. Note that any polynomial function is continuous.

(d) The logarithmic function is not uniformly continuous on the usual metric space $X = (0, \infty)$.

Exercise 1.5 Let (X, d) be a metric space. Prove that the function $y \mapsto d(x, y)$ is uniformly continuous.

Theorem 1.14 *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. If X is compact, then f is uniformly continuous.*

Theorem 1.15 *Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be an uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.*

The following example shows that a continuous function may not map a Cauchy sequence into a Cauchy sequence.

Example 1.19 Let $X = (0, \infty)$ with the induced usual metric on $\mathbb{R}$ and $Y = \mathbb{R}$ with the usual metric. The function $f : X \to Y$ defined by $f(x) = \frac{1}{x}$, for all $x \in X$, is continuous on X. Clearly, $\left\{x_n : x_n = \frac{1}{n}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. But $\left\{f\left(\frac{1}{n}\right)\right\}_{n\in\mathbb{N}} = \{n\}_{n=1}^{\infty}$ is not a Cauchy sequence in Y. Indeed, the absolute difference of any two distinct points is at least as large as 1.

Exercise 1.6 Show that the function $f(x) = e^x$ defined on the usual metric space $\mathbb{R}$ is not uniformly continuous.

Exercise 1.7 Let (X, d) be a metric space and A be a nonempty subset of X. Prove that the function $f : X \to \mathbb{R}$ defined by

$$f(x) = d(x, A), \quad \text{for all } x \in X,$$

is uniformly continuous.

In view of Theorem 1.12 (b), a function $f : X \to Y$ from a metric space X to a metric space Y is continuous at a point $x \in X$ if and only if for every sequence $\{x_n\}$ that converges to $x \in X$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Definition 1.31 Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be

(a) *lower semicontinuous* at a point $x \in X$ if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x$ as $n \to \infty$, equivalently,

$$f(x) \leq \liminf_{y\to x} f(y);$$

(b) *upper semicontinuous at a point* $x \in X$ if $f(x) \geq \limsup_{n\to\infty} f(x_n)$ whenever $x_n \to x$ as $n \to \infty$, equivalently,

$$f(x) \geq \limsup_{y\to x} f(y);$$

(c) *upper semicontinuous* (respectively, *lower semicontinuous*) *on* X if it is upper semicontinuous (respectively, lower semicontinuous) at each point of X.

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Example 1.20 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Then, f is upper semicontinuous at $x = 0$ but not lower semicontinuous at $x = 0$ (see Figure 1.2).

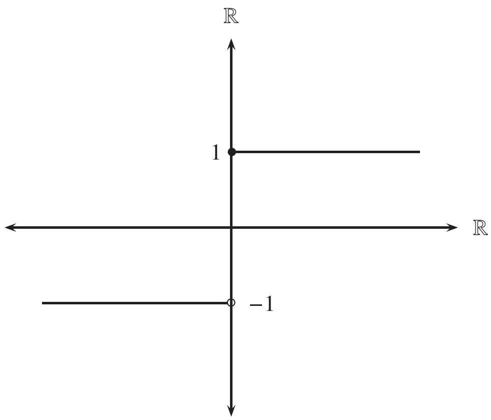

Figure 1.2 An upper semicontinuous function

Example 1.21 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -1, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then, f is lower semicontinuous at $x = 0$ but not upper semicontinuous at $x = 0$ (see Figure 1.3).

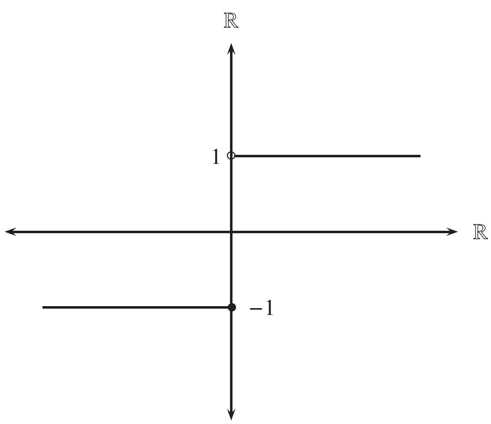

Figure 1.3 A lower semicontinuous function

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Remark 1.14 **(a)** A function $f : X \to \mathbb{R}$ is lower (respectively, upper) semicontinuous on X if and only if the *lower level set* $\{x \in X : f(x) \leq \alpha\}$ (respectively, the *upper level set* $\{x \in X : f(x) \geq \alpha\}$) is closed in X for all $\alpha \in \mathbb{R}$. Equivalently, f is lower (respectively, upper) semicontinuous on X if and only if the set $\{x \in X : f(x) > \alpha\}$ (respectively, $\{x \in X : f(x) < \alpha\}$) is open in X for all $\alpha \in \mathbb{R}$.

(b) A function $f : X \to \mathbb{R}$ is lower (respectively, upper) semicontinuous on X if and only if the *epigraph* $\{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ (respectively, the *hypograph* $\{(x, \alpha) \in X \times \mathbb{R} : f(x) \geq \alpha\}$) of f is closed in X.

In view of the above remark, Penot and Théra [142] gave the following definition of lower semicontinuous functions.

Definition 1.32 Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be

(a) *lower semicontinuous* on X if for each $x \in X$ and each $\alpha \in \mathbb{R}$ such that $f(x) > \alpha$, there exists $\delta > 0$ such that $f(y) > \alpha$ for all $y \in S_\delta(x)$;

(b) *upper semicontinuous* on X if for each $x \in X$ and each $\alpha \in \mathbb{R}$ such that $f(x) < \alpha$, there exists $\delta > 0$ such that $f(y) < \alpha$ for all $y \in S_\delta(x)$.

The following theorem shows that the conditions for lower semicontinuity on a metric space X given in Definition 1.31 and Definition 1.32 are equivalent.

Theorem 1.16 *Let (X, d) be a metric space, $x \in X$ and $f : X \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(a) *f is lower semicontinuous at x.*

(b) *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $f(x) - f(y) < \varepsilon$ whenever $d(x, y) < \delta$.*

Proof (a) $\Rightarrow$ (b) Suppose that f is lower semicontinuous at x. Set $\lambda := \liminf_{y \to x} f(y)$, and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$\left|\lambda - \inf f(S_\delta(x))\right| < \varepsilon.$$

By the definition of the lower semicontinuity of f at x, we have

$$f(x) \leq \lambda.$$

Thus, for every $y \in S_\delta(x)$, we have

$$f(x) - f(y) \leq \lambda - f(y) \leq \lambda - \inf_{z \in S_\delta(x)} f(z) < \varepsilon.$$

(b) $\Rightarrow$ (a) Let $\varepsilon > 0$ and choose a positive δ_0 such that $f(x) - f(y) < \frac{\varepsilon}{2}$ for all $y \in S_\delta(x)$. Assume that $\bar{x} \in S_{\delta_0}(x)$ such that

$$f(\bar{x}) < \inf_{z \in S_{\delta_0}(x)} f(z) + \frac{\varepsilon}{2}.$$

Hence,

$$\alpha := f(x) - \inf_{z \in S_\delta(x)} f(z) < f(x) - f(\bar{x}) + \frac{\varepsilon}{2}, \quad \text{for all positive } \delta \leq \delta_0. \tag{1.2}$$

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Note that the number α in (1.2) is positive. It follows that

$$\left| f(x) - \inf_{z \in S_\delta(x)} f(z) \right| < \varepsilon, \quad \text{for all positive } \delta \leq \delta_0.$$

Therefore, $f(x) \leq \liminf_{y \to x} f(y)$, that is, f is lower semicontinuous at x. ■

Similarly, we can have the following result for upper semicontinuous functions.

Theorem 1.17 *Let (X, d) be a metric space, $x \in X$ and $f : X \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(a) *f is upper semicontinuous at x.*
(b) *For every $\varepsilon > 0$, there is a $\delta > 0$ such that $f(y) - f(x) < \varepsilon$ whenever $d(x, y) < \delta$.*

The following theorem establishes that every lower semicontinuous function attains its minimum in any compact set.

Theorem 1.18 *Let K be a nonempty compact subset of a metric space (X, d) and $f : X \to \mathbb{R}$ be a lower semicontinuous function. Then, f attains its minimum on K.*

Proof Set $\lambda := \inf_{z \in K} f(z)$. If $\lambda = -\infty$, then there exists a sequence $\{x_n\}$ in K such that $\lim_{n \to \infty} f(x_n) = -\infty$. Without loss of generality, we may assume that $\{x_n\}$ converges to $x \in K$ by the compactness of K. By the lower semicontinuity of f, we have

$$f(x) \leq \lim_{n \to \infty} f(x_n) = -\infty,$$

which is not possible. Thus, $\lambda \in \mathbb{R}$. Now take a sequence $\{x_n\}$ in K such that $\lim_{n \to \infty} f(x_n) = \lambda$. Assume that $\{x_n\}$ converges to $x_0 \in K$ by the compactness of K. Again, by the lower semicontinuity of f, we have

$$f(x_0) \leq \lim_{n \to \infty} f(x_n) = \lambda = \inf_{z \in K} f(z).$$

This means that x_0 is a minimizer of f on K. ■

Exercise 1.8 Let K be a nonempty compact subset of a metric space (X, d) and $f : X \to \mathbb{R}$ be an upper semicontinuous function. Prove that f attains its maximum on K.

Chen et. al. [58] introduced the following concept of lower semicontinuity from above.

Definition 1.33 Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be

(a) *lower semicontinuous from above* at a point $x \in X$ if for any sequence $\{x_n\}$ in X converging to x and satisfying $f(x_{n+1}) \leq f(x_n)$ for all $n \in \mathbb{N}$, we have $f(x) \leq \lim_{n \to \infty} f(x_n)$;
(b) *upper semicontinuous from below* at a point $x \in X$ if for any sequence $\{x_n\}$ in X converging to x and satisfying $f(x_{n+1}) \geq f(x_n)$ for all $n \in \mathbb{N}$, we have $f(x) \geq \lim_{n \to \infty} f(x_n)$;
(c) *lower semicontinuous from above* (respectively, *upper semicontinuous from below*) on X if it is lower semicontinuous from above (respectively, upper semicontinuous from below) at every point of X.

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Obviously, lower (respectively, upper) semicontinuity implies lower semicontinuity from above (respectively, upper semicontinuity from below), but the converse implications do not hold.

Example 1.22 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x + \frac{1}{2}, & \text{if } x < 0, \\ x^2 + 1, & \text{if } x \geq 0. \end{cases}$$

Then, f is lower semicontinuous from above at $x = 0$, but not lower semicontinuous at this point.

Example 1.23 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x - 1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ x + 1, & \text{if } x > 0. \end{cases}$$

Then, f is lower semicontinuous from above as well as upper semicontinuous from below at $x = 0$, but it is neither lower nor upper semicontinuous at this point.

By Theorem 1.18, we have that every bounded below and lower semicontinuous real-valued function has a minimum on a compact set. However, Chen et al. [58] showed that the Weierstrass's theorem still holds for bounded below and lower semicontinuous from above functions.

Theorem 1.19 *Let K be a nonempty compact subset of a metric space X and $f : K \to \mathbb{R}$ be bounded below and lower semicontinuous from above. Then, there exists $\bar{x} \in K$ such that $f(\bar{x}) = \inf_{y \in K} f(y)$.*

Proof Since K is compact and f is bounded below, there exists a sequence $\{x_n\}$ in K such that $x_n \to \bar{x} \in K$, $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ and $f(x_n) \to \inf_{y \in K} f(y)$.

By the lower semicontinuity from above, we have

$$f(\bar{x}) \leq \lim_{n \to \infty} f(x_n) = \inf_{y \in K} f(y).$$

Hence, $f(\bar{x}) = \inf_{y \in K} f(y)$. ■

Chen et al. [58] also showed that Ekeland's variational principle and Caristi's fixed point theorem hold for lower semicontinuity from above functions.

Definition 1.34 A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be *right upper semicontinuous,* also called *upper semicontinuous from the right,* if $\varphi(t) \geq \limsup_{r \to t^+} \varphi(r)$ for all $t \geq 0$.

Example 1.24 Define a function $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \begin{cases} \sqrt{t}, & \text{if } t \in [0, 1), \\ \sqrt{t} + 1, & \text{if } t \in [1, \infty). \end{cases}$$

We see that $\varphi(1) = 2$ and the function ψ is discontinuous at $t = 1$. Note that $\lim_{t \to 1^-} \varphi(t) = 1$ and $\limsup_{t \to 1^+} \varphi(t) = 2$. Thus, the function φ is right upper semicontinuous.

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Lemma 1.1 [89] *Let* $\varphi : [0, \infty) \to [0, \infty)$ *be a right upper semicontinuous function such that* $\varphi(t) < t$ *for all* $t > 0$. *Then,* $\lim_{n\to\infty} \varphi^n(t) = 0$.

Proof Since for each $t > 0$, $\varphi(t) < t$, we have $\varphi^2(t) = \varphi(\varphi(t)) < \varphi(t) < t$. By induction, we obtain a nonincreasing sequence $\{\varphi^n(t)\}$. So we can assume that $\{\varphi^n(t)\}$ decreases to a nonnegative number c. If $c > 0$, then

$$c > \varphi(c) \geq \limsup_{n\to\infty} \varphi(\varphi^n(t)) = \lim_{n\to\infty} \varphi^{n+1}(t) = c,$$

which is a contraction. Hence $c = 0$, and $\lim_{n\to\infty} \varphi^n(t) = 0$. ■

Lemma 1.2 [89] *Let* $\varphi : [0, \infty) \to [0, \infty)$ *be a right upper semicontinuous function such that* $\varphi(t) < 1$ *for all* $t > 0$. *Then the function* $\Phi : [0, \infty) \to [0, \infty)$, *defined by* $\Phi(t) = \varphi(t)t$, *is right upper semicontinuous and* $\Phi(t) < t$ *for all* $t > 0$.

Proof Since for each $t > 0$, $\varphi(t) < 1$, we have $\Phi(t) = \varphi(t)t < t$, and

$$\begin{aligned}\Phi(t) = \varphi(t)t &\geq \left(\limsup_{r\to t^+} \varphi(r)\right) t \\ &= \limsup_{r\to t^+} \left(\varphi(r)r\right) \\ &= \limsup_{r\to t^+} \Phi(r).\end{aligned}$$

Hence, $\Phi(t)$ is right upper semicontinuous. ■

Lemma 1.3 [89] *Let* $\varphi : [0, \infty) \to [0, 1)$ *be such that* $\limsup_{r\to t^+} \varphi(r) < 1$ *for all* $t > 0$, *and let* $\Psi(t) = \max\left\{\varphi(t), \limsup_{r\to t^+} \varphi(r)\right\}$ *for all* $t > 0$. *Then, the function* $\Psi : [0, \infty) \to [0, 1)$ *is right upper semicontinuous and* $\Psi(t) \geq \varphi(t)$ *for all* $t \geq 0$.

Proof Since for each $t > 0$, $\varphi(t) < 1$, and $\limsup_{r\to t^+} \varphi(r) < 1$ for all $t > 0$, we have $\Psi(t) = \max\left\{\varphi(t), \limsup_{r\to t^+} \varphi(r)\right\} < 1$, and Ψ is a function from $[0, \infty)$ to $[0, 1)$.

Now we prove that Ψ is right upper semicontinuous. Let $\alpha = \limsup_{r\to t^+} \Psi(r)$. Then by the definition of upper limit, there exists a nonincreasing sequence $\{t_n\}$ with limit t such that $\lim_{n\to\infty} \Psi(t_n) = \alpha$. Denote $\alpha_n = \Psi(t_n)$, then $\lim_{n\to\infty} \alpha_n = \alpha$. For each $\varepsilon > 0$, $n = 1, 2, ...$, if $\Psi(t_n) = \varphi(t_n)$, take $t'_n = t_n$, and if $\Psi(t_n) = \limsup_{r\to t^+} \varphi(r)$, by the definition of upper limit, we can choose t'_n, $t_{n-1} > t'_n \geq t_n$ such that

$$\varphi(t'_n) \geq \limsup_{r\to t'_n} \varphi(r) - \varepsilon = \alpha_n - \varepsilon.$$

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In both the cases, we have $\{t'_n : t_{n-1} > t'_n \geq t_n\}$ such that $\varphi(t'_n) \geq \alpha_n - \varepsilon$ and $t'_n \to t^+$. Since

$$\limsup_{r \to t^+} \varphi(r) \geq \lim_{n \to \infty} \varphi(t'_n) \geq \lim_{n \to \infty} (\alpha_n - \varepsilon) = \alpha - \varepsilon,$$

we have

$$\Psi(t) = \max\left\{\varphi(t), \limsup_{r \to t^+} \varphi(r)\right\} \geq \alpha - \varepsilon, \quad \text{for all } \varepsilon > 0.$$

Hence, $\Psi(t) \geq \alpha = \limsup_{r \to t^+} \Psi(r)$. ■

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