PRIMITIVE IDEMPOTENT MEASURES ON COMPACT SEMIGROUPS

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1. Introduction

Let S be a compact semigroup (with jointly continuous multiplication) and let $P(S)$ denote the probability measures on S, i.e. the positive regular Borel measures on S with total mass one. Then $P(S)$ is a compact semigroup with convolution multiplication and the weak* topology. Let $\Pi(P(S))$ denote the set of primitive (or minimal) idempotents in $P(S)$. Collins (**2**) and Pym (**5**) respectively have given complete descriptions of $\Pi(P(S))$ when S is a group and when $K(S)$, the kernel of S, is not a group. Choy (**1**) has given some characterizations of $\Pi(P(S))$ for the general case. In this paper we present some detailed and intrinsic characterizations of $\Pi(P(S))$ for various classes of compact semigroups that are not covered by the results of Collins and Pym.

In Section **2** we give a detailed survey of the known results on $\Pi(P(S))$ together with some preliminary results. We include here some facts about maximal simple subsemigroups of compact semigroups. In Section **3** we consider the commutative case. We show that the elements of $\Pi(P(S))$ are then the Haar measures on certain of the maximal subgroups of S together with the Haar measures on the maximal closed subgroups of $K(S)$. (Throughout this paper, Haar measure means normalized Haar measure.) If m is the Haar measure on $K(S)$, then $\Pi(P(S))\cup\{m\}$ is a compact idempotent semigroup in which all distinct products are m and the topology is the one point compactification of $\Pi(P(S))$ with the discrete topology. As an amusing application we obtain an identification of $\Pi(P(S))$ with $\Pi(P(P(S)))$. Similar results obtain for compact semigroups S such that each idempotent of S is central and $K(S)$ is commutative. In Section **4** we describe the central primitive idempotents on an arbitrary compact semigroup S. Under some weak commutativity assumptions we describe the elements of $\Pi(P(S))$ in terms of the maximal simple subsemigroups of S. Some of the results in Section **3** could be deduced as special cases of the results in Section **4**, but it seems simpler for the exposition to consider the commutative case first.

Given $\mu \in P(S)$, we write supp μ for the support of μ, i.e. the unique minimal closed subset of S with μ-mass one. Then for μ, $\nu \in P(S)$, we have

$$\operatorname{supp} \mu\nu = \operatorname{supp} \mu \operatorname{supp} \nu$$

and when μ is idempotent, supp μ is a simple subsemigroup of S (see Pym (**5**)). When S is a group we have $\mu^2 = \mu$ if and only if supp μ is a group and μ is the

Haar measure on supp μ. We shall always identify μ with its restriction to any closed subset of S that contains supp μ. Moreover, given $\mu \in P(E)$, where E is a closed subset of S we also write μ for the natural extension of μ to a probability measure on S. Given $x \in S$ we write δ_x for the point mass at x.

2. Preliminary results

Let T be any semigroup and let $I(T)$ denote the set of idempotents of T. The natural partial order on $I(T)$ is defined by

$$e \leqq f \text{ if } ef = fe = e.$$

If T has a zero element θ, then $\theta \leqq e$ for each idempotent e. An idempotent e of T is *primitive* if it is a minimal non-zero element of $(I(T), \leqq)$. The set of all primitive idempotents of T is denoted by $\Pi(T)$. The elementary description of $\Pi(T)$ varies according as T has a zero element or not. The following lemma is elementary.

Lemma 1. *Let T be a semigroup and let $e \in I(T)$.*

(i) *If T has no zero element, then $e \in \Pi(T)$ if and only if $I(eTe) = \{e\}$.*

(ii) *If T has zero element θ, then $e \in \Pi(T)$ if and only if $I(eTe) = \{e, \theta\}$.*

In studying the primitive idempotent measures on a compact semigroup Pym (**5**) modified the definition somewhat. Given a semigroup T, let T_θ be the semigroup obtained by adjoining a zero element θ to T (whether or not T itself has a zero element). Given $e \in I(T)$, we say that $e \in \Pi^*(T)$ if $e \in \Pi(T_\theta)$. (This is Pym's definition of primitive idempotent when $T = P(S)$.) Since $eT_\theta e = eTe \cup \{\theta\}$, it follows from Lemma 1 that $e \in \Pi^*(T)$ if and only if

$$I(eTe) = \{e\}.$$

Thus if T has no zero element then $\Pi^*(T) = \Pi(T)$. On the other hand if T has a zero element, say m, then $\Pi^*(T) = \{m\}$. Now let $T = P(S)$, where S is a compact semigroup. Then T has a zero element if and only if $K(S)$ is a group, in which case the zero element is the Haar measure on $K(S)$. When $K(S)$ is not a group, Pym (**5**) gives a complete description of

$$\Pi(P(S)) = \Pi^*(P(S)).$$

Theorem 2. (Pym) *Let S be a compact semigroup whose kernel $K(S)$ is not a group. Then*

(i) $\Pi(P(S)) = K(P(S))$,

(ii) *each $\mu \in \Pi(P(S))$ is supported in $K(S)$; and if $E \times G \times F$ is a canonical decomposition of $K(S)$, then*

$$\mu = \mu_1 \times m \times \mu_2$$

where $\mu_1 \in P(E)$, $\mu_2 \in P(F)$ and m is the Haar measure on G.

Suppose now that S is a compact semigroup such that $K(S)$ is a group.

Then $\Pi^*(P(S)) = \{m\}$, where m is the Haar measure on $K(S)$, and Pym's result gives no information about $\Pi(P(S))$. When S is a compact group (and so $K(S) = S$), Collins (**2**) gives a complete description of $\Pi(P(S))$.

Theorem 3. (Collins). *Let S be a compact group. Then $\Pi(P(S))$ consists of the Haar measures on maximal closed proper subgroups of S.*

For the general case, Choy (**1**) has shown that the members of $\Pi(P(S))$ can be characterized in terms of the behaviour of the simple subsemigroups of S. In particular the measures in $\Pi(P(S))$ can be characterized in terms of their supports. The following theorem together with Theorem 3 gives a complete description of those μ in $\Pi(P(S))$ that are supported in $K(S)$.

Theorem 4. *Let S be a compact semigroup whose kernel K is a group. Then*

(i) $\Pi(P(K)) \subset \Pi(P(S))$,

(ii) *if* $\mu \in \Pi(P(S))$ *with* $\text{supp}\,\mu \cap K \neq \emptyset$, *then* $\text{supp}\,\mu \subset K$ *and* $\mu \in \Pi(P(K))$.

Proof. (i) Let $\mu \in \Pi(P(K))$, $\nu \in P(S)$. Then

$$\text{supp}\,\mu\nu\mu = \text{supp}\,\mu\,\text{supp}\,\nu\,\text{supp}\,\mu \subset K$$

and so $\mu P(S)\mu \subset P(K)$. It follows that $\mu P(S)\mu = \mu P(K)\mu$ and therefore

$$\mu \in \Pi(P(S))$$

by Lemma 1.

(ii) Let $\mu \in \Pi(P(S))$, $T = \text{supp}\,\mu$, and suppose $H = T \cap K \neq \emptyset$. Then $TH \subset T \cap K = H$, and similarly $HT \subset H$, so that H is an ideal of T. Since T is simple, we have $H = T$, $T \subset K$. Since $\mu P(K)\mu \subset \mu P(S)\mu$, it follows from Lemma 1 that $\mu \in \Pi(P(K))$.

Let e be the identity of the group K and let δ_e be the point mass at e. Choy (**1**) shows that δ_e is central in $P(S)$, i.e. $\delta_e\mu = \mu\delta_e$ for each $\mu \in P(S)$, and the mapping Φ: $P(S) \to P(K)$ defined by

$$\Phi(\mu) = \delta_e\mu$$

is a continuous homomorphism that maps $\Pi(P(S))$ onto $\Pi(P(K)) \cup \{m\}$. We have of course $\Phi(\mu) = \mu$ for each $\mu \in P(K)$. The result below shows that $\Phi(\mu) = m$ for each $\mu \in \Pi(P(S))$ that is supported outside K.

Proposition 5. *Let $\mu \in \Pi(P(S))$ with* $\text{supp}\,\mu \cap K = \emptyset$. *Then $\delta_e\mu = m$ and* $e\,\text{supp}\,\mu = K$.

Proof. Let $T = \text{supp}\,\mu$, so that T is simple and hence is a union of subgroups of S. Since e is central in S, it follows that eT is a closed subgroup of K. Let ν be the Haar measure on eT. Since $\delta_e\mu$ is an idempotent measure supported on eT we have $\delta_e\mu = \nu$. Hence $\nu\mu = \delta_e\mu^2 = \delta_e\mu = \nu$ and similarly $\mu\nu = \nu$. We have $\nu \neq \mu$ since $T \cap K = \emptyset$, and since $\mu \in \Pi(P(S))$ it follows that $\nu = m$, $eT = K$.

The primitive idempotent measures on a compact group are described in terms of the maximal closed proper subgroups. Given any compact semigroup

S, we recall that each subgroup of S is contained in a maximal subgroup of S, and the maximal subgroups of S are closed and pairwise disjoint (see e.g. (4), Theorem 1.1.3). Let $\mathcal{G}$ denote the family of maximal subgroups of S, and for each $e \in I(S)$ let $G(e)$ denote the unique maximal subgroup of S that contains e. In the final section we shall require some results about the maximal simple subsemigroups of S. We recall that each simple subsemigroup of S is contained in a maximal simple subsemigroup of S, and the maximal simple subsemigroups are closed, but need not be pairwise disjoint (see (4), p. 42). Let $\mathcal{M}$ denote the family of maximal simple subsemigroups of S. The next result shows that each $M \in \mathcal{M}$ is the (pairwise disjoint) union of members of $\mathcal{G}$.

Proposition 6. *Let S be a compact semigroup, let $M \in \mathcal{M}$ and let $e \in I(S) \cap M$. Then $G(e) \subset M$.*

Proof. Write $G = G(e)$. Since eMe is a group with identity e, we have $eMe \subset G$. Let T be the subsemigroup of S generated by M and G. Since $GMG = GeMeG = G$, it follows that

$$T = M \cup G \cup MG \cup GM \cup MGM.$$

It is easily checked that, for each $x \in T$, $TxT \supset G$ and so

$$TxT \supset T(TxT)T \supset MeM = M$$

since M is simple. Therefore $TxT = T$ for each $x \in T$ and so T is simple. By maximality $T = M$, and therefore $G(e) \subset M$.

Proposition 7. *Let S be a compact semigroup and suppose the idempotents from distinct members of $\mathcal{M}$ commute with each other. Then the members of $\mathcal{M}$ are pairwise disjoint.*

Proof. Let $M, N \in \mathcal{M}$ with $M \cap N \neq \emptyset$, and suppose that $M \neq N$. Let $e \in I(M \cap N)$ and let $f \in I(eM)$. Since $e \in N, f \in M$ we have

$$f = ef = fe = efe.$$

Then efe is an idempotent in eMe and so $e = efe = f$. Thus $I(eM) = \{e\}$ and similarly $I(Me) = \{e\}$. Since M is simple, it follows that $M = G(e)$, and similarly $N = G(e)$. This contradiction completes the proof.

Another sufficient condition for the members of $\mathcal{M}$ to be pairwise disjoint is that each $M \in \mathcal{M}$ should be left simple, i.e. $Me = M$ for each $e \in I(M)$ (see (4), Theorem 1.3.13); or that each $M \in \mathcal{M}$ should be right simple.

3. The commutative case

Throughout this section S denotes a compact commutative semigroup. Thus each simple subsemigroup of S is a group, and in particular the kernel K of S is a group. Let e denote the identity of K and m the Haar measure on K. Then $I(S)$ is a compact semigroup with zero e, and $P(S)$ is a compact semigroup with zero m. The idempotent measures on S are precisely the Haar measures on compact subgroups of S.

Theorem 8. *Let $\mu \in I(P(S))$, let $T = \operatorname{supp}\mu$ with $T \cap K = \varnothing$, and let f be the identity of T. Then the following statements are equivalent.*

(i) $\mu \in \Pi(P(S))$;

(ii) $T \in \mathscr{G}$, $Te = K$, $f \in \Pi(I(S))$.

Proof. (i)⇒(ii). Let $\mu \in \Pi(P(S))$. Then $Te = K$ by Proposition 5. Let $T \subset G \in \mathscr{G}$ and let ν be the Haar measure on G. Since ν is the zero of $P(G)$, we have $\mu\nu = \nu$, $\nu \neq m$. Since $\mu \in \Pi(P(S))$, it follows that $\mu = \nu$ and $T = G$. Let $j \in I(fI(S)f)$ with $j \neq e$. Then $jf = j$ and jG is a subgroup of S so that $jG \subset G(j)$. Let ρ be the Haar measure on jG. Since $\delta_j\mu$ is idempotent with support jG, we have $\delta_j\mu = \rho$ and so $\rho\mu = \mu$. Since $\mu \in \Pi(P(S))$, it follows that $\mu = \rho$ and so $j = f$. Therefore $f \in \Pi(I(S))$.

(ii)⇒(i). Let condition (ii) hold. Let $\nu \in I(P(S))$ with $\mu\nu = \nu$ and let $H = \operatorname{supp}\nu$. Suppose first that $H \cap K \neq \varnothing$. Then $e \in H \cap K$ and $H = TH \supset Te = K$. Since $K \in \mathscr{G}$, we have $H = K$, $\nu = m$. Suppose now that $H \cap K = \varnothing$, and that j is the identity of H. Since fj is an idempotent and $TH = H$, we have $j = fj$ and so $j \in I(fI(S)f)$. Since $f \in \Pi(I(S))$, it follows that $j = f$, $H \subset T$. But then $\mu\nu = \mu$ and so $\nu = \mu$. Therefore $\mu \in \Pi(P(S))$.

Corollary 9. *If S has a zero, then $K = \{e\}$ and there is a one-one correspondence between $\Pi(I(S))$ and $\Pi(P(S))$ in which $j \in \Pi(I(S))$ corresponds to the Haar measure on $G(j)$.*

We remark that it is easy to show by examples that the three conditions (i) $T \in \mathscr{G}$ (ii) $Te = K$ (iii) $f \in \Pi(I(S))$ are independent of each other.

Theorem 8 together with Theorems 5 and 3 gives a complete intrinsic characterization of the elements of $\Pi(P(S))$. The next result describes the structure of $\Pi(P(S))$. The simple proof below was suggested by the referee.

Theorem 10. *$\Pi(P(S)) \cup \{m\}$ is a compact idempotent semigroup with $\mu\nu = m$ ($\mu \neq \nu$) and with discrete topology on $\Pi(P(S))$.*

Proof. $\Pi(P(S)) \cup \{m\}$ is a closed subset of $P(S)$ by (1), Lemma 2.2. Let $\mu, \nu \in \Pi(P(S))$ with $\mu \neq \nu$. Then $\mu\nu$ is idempotent, and $\mu\nu = \mu(\mu\nu)\mu \in \{\mu, m\}$ as μ is primitive. Similarly, $\mu\nu \in \{\nu, m\}$. Hence $\mu\nu = m$. That $\Pi(P(S))$ now has the discrete topology is shown in (1), Lemma 2.3.

Given any (commutative) compact semigroup S, $I(S)$ is a compact idempotent semigroup and its kernel is a single point, i.e. $I(S)$ always has a zero element, whereas S need not have a zero element.

Proposition 11. *If S has a zero element, then $\Pi(S) = \Pi(I(S))$.*

Proof. Note that S and $I(S)$ have the same zero element. Let $j \in I(S)$. It is clearly sufficient to show that $I(jS) = I(jI(S))$. Let $f \in I(jS)$. Then $f = jf \in I(jI(S))$. The other inclusion is trivial and so the proof is complete.

Corollary 12. *If S has a zero element, there is a one-one correspondence between* $\Pi(S)$ *and* $\Pi(P(S))$, $\Pi(S)\leftrightarrow\Pi(P(S))$. *In particular,*

$$\Pi(P(S))\leftrightarrow\Pi(P(P(S))).$$

Proof. Apply Corollary 9 and Proposition 11. For the final statement, recall that $P(S)$ has zero element m.

We can add a little to Corollary 12 by describing the maximal subgroups of $P(S)$ that support a primitive idempotent measure. Glicksberg (3) has shown that an arbitrary closed subgroup Γ of $P(S)$ consists of the G-translates of Haar measure on H, where G is a closed subgroup of S and H a closed subgroup of G. It is clear that Γ is maximal if and only if G is maximal. Given $\mu\in\Pi(P(S)$ with $\operatorname{supp}\mu\cap K=\varnothing$, it is then clear that $G(\mu)=\{\mu\}$. Given $\mu\in\Pi(P(S))$ with $\operatorname{supp}\mu\subset K$, we see that $G(\mu)$ consists of the K-translates of μ and also $G(\mu)$ may be identified with $K/\operatorname{supp}\mu$.

Remarks. (1) Theorem 8 remains true (with identical proof) under the weaker hypothesis that S is a compact semigroup in which each idempotent is central, and so, in particular, Theorem 8 holds for compact inverse semigroups. Theorem 10 also remains true if the idempotents of S are central and $K(S)$ is commutative. Note from (1), Example 2.8 (iii) that if S is the symmetric group of order 3 then $\Pi(P(S))\cup\{m\}$ is not a subsemigroup of $P(S)$. Hence for an extension of Theorem 10 we need some restriction on $K(S)$.

(2) Suppose now that S is a compact semigroup in which all the idempotents of S commute with each other. Then again every simple subsemigroup of S is a group and the idempotents of $P(S)$ are Haar measures on compact subgroups of S. In Theorem 8 we have (ii)⇒(i), but (i)⇒(ii) is false as the following example shows. Let S be the 2×2 matrix semigroup consisting of the four matrix units, the zero matrix, the identity matrix and the matrix

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

It is readily verified that S is a semigroup with zero in which $I(S)$ is commutative and

$$\Pi(I(S))=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

However it is not difficult to check that the Haar measure μ on the subgroup

$$G=\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$$

is a primitive idempotent measure. In fact $\Pi(P(S))$ consists of μ and the point masses on $\Pi(I(S))$. Note that $\Pi(P(S))\cup\{m\}$ fails to be a subsemigroup of $P(S)$.

4. Some non-commutative cases

Let S be an arbitrary compact semigroup and let $C\Pi(P(S))$ denote the set of central primitive idempotent measures on S. If the kernel K of S is not a group, we easily deduce from Theorem 2 that $C\Pi(P(S)) = \emptyset$. For the rest of this section we suppose that S is a compact semigroup whose kernel K is a group. Recall that a subset E of S is *normal* if $xE = Ex$ for each $x \in S$. Choy (**1**), Theorem 3.2 shows that the central idempotent measures on S are precisely the Haar measures on compact normal subgroups of S. The theorem below shows that the description of $C\Pi(P(S))$ is closely related to the commutative case. We need first a simple lemma.

Lemma 13. *Let G be a normal subgroup of S and let $j \in I(S)$. Then jG is a subgroup of S.*

Proof. We note that j is central in G. In fact given $x \in G$, there is $y \in G$ with $jx = yj$ and then $jx = jxj$ and similarly $xj = jxj$. Let f be the identity of G. Then jf is an idempotent. Since $jG = Gj$ we have $jG = jGj$ and so

$$jGjG = jGGj = jGj = jG.$$

Thus jG is a semigroup with identity jf. Given $x \in G$, we have $x^{-1}j \in jG$ and

$$jxx^{-1}j = jfj = jf.$$

This completes the proof.

Theorem 14. *Let $\mu \in I(P(S))$, let $T =$ supp μ with $T \cap K = \emptyset$, and let f be the identity of T. Then the following statements are equivalent.*

(i) $\mu \in C\Pi(P(S))$;

(ii) $T \in \mathscr{G}$, *T is normal*, $Te = K$, $I(fSf) = \{f, e\}$.

Proof. Argue as in Theorem 8. For the proof of (i)⇒(ii) we need to use Lemma 13 above, together with the fact that if j is central in T then $\delta_j\mu = \mu\delta_j$ and so $\delta_j\mu$ is idempotent.

Theorem 14 describes the elements of $C\Pi(P(S))$ that are supported outside the kernel K. The elements of $C\Pi(P(S))$ that are supported in the kernel are precisely the Haar measures on the maximal closed subgroups of K that are normal in S.

Theorem 15. *$C\Pi(P(S)) \cup \{m\}$ is a compact idempotent semigroup with $\mu\nu = m$ ($\mu \neq \nu$) and with discrete topology on $\Pi(P(S))$.*

Proof. Argue as in Theorem 10.

Now let S be a compact semigroup (whose kernel K is a group) such that

$$M, N \in \mathscr{M}, M \neq N, j \in I(M) \Rightarrow jN = Nj. \tag{*}$$

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Note that j is then central in N, and, by Proposition 7, the members of $\mathcal{M}$ are pairwise disjoint. Thus each simple subsemigroup of S is contained in a unique maximal simple subsemigroup. In particular, the support of any idempotent measure on S is contained in a unique maximal simple subsemigroup of S. When $M \in \mathcal{M}$ is a group with $M \cap K = \emptyset$ we make a convenient abuse of notation by writing $\Pi(P(M))$ for the set consisting of the Haar measure on M.

Theorem 16. *Let S satisfy (*), let $\mu \in I(P(S))$ and let $T = \text{supp}\, \mu$ with $T \subset M \in \mathcal{M}$, $T \cap K = \emptyset$. Then the following statements are equivalent.*

(i) $\mu \in \Pi(P(S))$;

(ii) $\mu \in \Pi(P(M))$, $Te = K$, $I(jSj) = \{j, e\}$ *for each* $j \in I(T)$.

Proof. (i)⇒(ii). Let $\mu \in \Pi(P(S))$. Then $\mu \in \Pi(P(M))$, and, by Proposition 5, $Te = K$. Let $j \in I(T)$, $f \in I(jSj)$ and suppose $f \neq e$, $f \notin M$. Then f is central in T and so $\mu\delta_f = \delta_f\mu$. Let $\rho = \mu\delta_f$ and then $\rho^2 = \rho$, $\mu\rho = \rho\mu = \rho$. Therefore $f = jf \in \text{supp}\, \rho$. But $\mu \in \Pi(P(S))$ gives $\rho = m$ or $\rho = \mu$. This contradiction shows that $I(jSj) \subset M \cup \{e\}$. If $f \in M$, then $f = jfj \in jMj = G(j)$ and so $f = j$.

(ii)⇒(i). Let condition (ii) hold, let $v \in I(P(S))$ with $\mu v = v\mu = v$ and let $R = \text{supp}\, v$. Then R is simple and there is $N \in \mathcal{M}$ with $R \subset N$. If $N = K$, then

$$R = TR \supset Te = K$$

and so $v = m$. If $N = M$, then $v = \mu$ since $\mu \in \Pi(P(M))$. Suppose finally that $N \cap M \cap K = \emptyset$. Let $j \in I(T)$. Then j is central in R, and since $RT = TR = R$, it follows that there is an idempotent in $R \cap jSj$. This contradiction completes the proof.

As an illustration of Theorem 16, let R be the disjoint union of a family $\{S_\lambda\colon \lambda \in \Lambda\}$ of compact simple semigroups and let the union of the topologies on the S_λ be a base for a topology on R. Then R is locally compact. Now let $S = R \cup \{\theta\}$ with the topology of the one point compactification of R. Extend the multiplications on the S_λ to a multiplication on S by defining all new products to be θ. It is now easily checked that S is a compact semigroup. A simple application of Theorem 16 gives

$$\Pi(P(S)) = \cup\{\Pi(P(S_\lambda))\colon \lambda \in \Lambda\} = \cup\{K(P(S_\lambda))\colon \lambda \in \Lambda\}.$$

The results of this paper indicate that the size of $\Pi(P(S))$ reflects the degree of non-commutativity of S. As the extreme example, let S be a compact space with multiplication

$$xy = x \quad (x, y \in S).$$

Then S is a compact simple semigroup and $P(S)$ has multiplication

$$\mu v = \mu \quad (\mu, v \in P(S)).$$

Therefore $\Pi(P(S)) = P(S)$.

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