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INEQUALITIES WITH WEIGHTS FOR POWERS OF GENERALISED INVERSES

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Recently the authors extended to positive semi-definite matrices converses of matrix inequalities corresponding to ratios and differences of means and of Hölder's inequality. Here, further generalisations are given to the case of several vectors with appropriate weights.

INTRODUCTION

Generalisations of some well-known converses of means inequalities involving powers of generalised inverses were given in [**3**]. Here we give further generalisations in the case of several vectors.

NOTATION AND PREVIOUS RESULTS.

Given any matrix A with complex entries, there exists a unique matrix A^+ satisfying

$$AA^+A = A,\ A^+AA^+ = A^+,\ AA^+ = \left(AA^+\right)^*,\ A^+A = \left(A^+A\right)^*$$

where * denotes conjugate transpose. A^+ is called the (Moore-Penrose) generalised inverse of A.

Let A be a positive semi-definite Hermitian matrix of order n and rank k. Let those roots of A which are strictly positive be ordered as $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$. For simplicity of notation, if $r < 0$, we shall use $A^{(r)}$ for $\left(A^+\right)^{-r}$. Note that, with $r < 0$, $\left(A^+\right)^{-r} = \left(A^{-r}\right)^+$. Also if $k = n$, $A^+ = A^{-1}$ and $A^{(r)} = A^r$.

In our proofs we need the following result from [**3**].

LEMMA 1. *Let r and s be real numbers, $r < 0$, $s > 0$. Let A be a positive semi-definite Hermitian matrix of order n and rank k. Let those roots of A which are strictly positive be ordered as $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$, $\lambda_k < \lambda_1$. Set*

$$a = \frac{\lambda_1^r - \lambda_k^r}{\lambda_1^s - \lambda_k^s}, \quad b = \frac{\lambda_1^s\lambda_k^r - \lambda_1^r\lambda_k^s}{\lambda_1^s - \lambda_k^s} \tag{1}$$

Let B be a Hermitian matrix that satisfies

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$$A^{-r}BA^{-r} = A^{-r},\ A^{-r}B = BA^{-r}$$

and has $n-k$ non-positive roots. Then for any x such that $x^*x = 1$

$$x^*Bx - ax^*A^sx - b \leqslant 0. \tag{2}$$

Strict equality holds if and only if $x = u_1 + u_k$ where $Au_1 = \lambda_1 u_1$ and $Au_k = \lambda_k u_k$.

RESULTS FOR MEANS.

First we state a theorem that is equivalent to Lemma 1.

THEOREM 1. *Let r, s, A and B be defined as in Lemma 1. Further, let w_j, $j = 1, \ldots, m$ be positive numbers with $\sum_{j=1}^{m} w_j = 1$ and let x_j, $j = 1, \ldots, m$, be such that $x_j^* x_j = 1$, $j = 1, \ldots, m$. Then*

$$\sum_{j=1}^{m} w_j x_j^* B x_j - a \sum_{j=1}^{m} w_j x_j^* A^s x_j - b \leqslant 0. \tag{3}$$

Strict equality holds if and only if $x_j = u_{j1} + u_{jk}$ where $Au_{j1} = \lambda_1 u_{j1}$ and $Au_{jk} = \lambda_k u_{jk}$.

PROOF: By (2) we have

$$x_j^* B x_j - a x_j^* A^s x_j - b \leqslant 0.$$

Multiplying this inequality by w_j, and adding, we get (3). □

THEOREM 2. *Let r, s, A, B, w_j, x_j, $j = 1, \ldots, m$ be as in Theorem 1 and $\gamma = \lambda_1/\lambda_k$. If $\sum_{j=1}^{m} w_j x_j^* B x_j \geqslant 0$, then*

$$\left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)^{1/s} \left(\sum_{j=1}^{m} w_j x_j^* B x_j\right)^{-1/r} \leqslant \Omega, \tag{4}$$

where

$$\Omega = \left\{\frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)}\right\}^{1/s} \left\{\frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)}\right\}^{-1/r}. \tag{5}$$

Strict equality holds if and only if $x_j = u_{j1} + u_{jk}$ where $Au_{j1} = \lambda_1 u_{j1}$ and $Au_{jk} = \lambda_k u_{jk}$ and

$$\sum_{j=1}^{m} w_j u_{j1}^* u_{j1} = [r/(\gamma^r - 1) - s/(\gamma^s - 1)]/(s-r). \tag{6}$$

PROOF: By Theorem 1, with a and b defined by (1),

$$\left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)^{1/s} \left(\sum_{j=1}^{m} w_j x_j^* B x_j\right)^{-1/r}$$
$$\leqslant \left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)^{1/s} \left[a\sum_{j=1}^{m} w_j x_j^* A^s x_j + b\right]^{-1/r} \equiv g\left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)$$

where $g(t) = t^{1/s}[at+b]^{-1/r}$. Here $g(t)$ is easily seen to have a maximum at $t = rb[a(s-r)]^{-1}$ which gives the upper bound of (6).

Strict equality holds if and only if it holds in (4) at

$$\sum_{j=1}^{m} w_j x_j^* A^s x_j = rb[a(s-r)]^{-1}. \tag{7}$$

Given the condition for strict equality in (3), strict equality in (7) is easily seen to hold if and only if it holds in (6). □

Note that the above proof is similar to the proof of Theorem 2 from [**3**]. In a similar manner to the proof of Theorem 3 in [**3**], we can prove the following.

THEOREM 3. *Let r, s, A, B, a, b, w_i, x_i, $i = 1, \dots, m$ be defined as in Theorem 1. If $\sum_{i=1}^{m} w_i x_i^* B x_i > 0$, then*

$$\left(\sum_{i=1}^{m} w_i x_i^* A^s x_i\right)^{1/s} - \left(\sum_{i=1}^{m} w_i x_i^* B x_i\right)^{1/r} \leqslant \Delta, \tag{8}$$

where

$$\Delta = [\theta\lambda_1^s + (1-\theta)\lambda_k^s]^{1/s} - [\theta\lambda_1^r + (1-\theta)\lambda_k^r]^{1/r}. \tag{9}$$

θ is defined as follows. Let

$$h(y) = y^{1/s} - (ay+b)^{1/r}.$$

Let J denote the open interval joining λ_k^s to λ_1^s. Let $\overline{J}$ be the closure of J. There is a unique $y^* \in \overline{J}$ where $h(y)$ attains its maximum in $\overline{J}$. This y^* lies in J. Set

$$\theta = \frac{y^* - \lambda_k^s}{\lambda_1^s - \lambda_k^s}.$$

Strict equality holds if and only if $x_j = u_{j1} + u_{jk}$ where $Au_{j1} = \lambda_1 u_{j1}$, $Au_{jk} = \lambda_k u_{jk}$ and $\sum_{j=1}^{m} w_j u_{j1}^* u_{j1} = \theta$. If $s \geqslant 1$, then y^* is the unique solution of $h'(y) = 0$ in J.

Since $A^{(r)}$ satisfies the conditions for B in Lemma 1, (3), (4) and (8) yield

$$\sum_{j=1}^{m} w_j x_j^* A^{(r)} x_j - a \sum_{j=1}^{m} w_j x_j^* A^s x_j - b \leqslant 0, \tag{10}$$

where a and b are defined by (1);

$$\left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)^{1/s} \left(\sum_{j=1}^{m} w_j x_j^* A^{(r)} x_j\right)^{-1/r} \leqslant \Omega, \tag{11}$$

where Ω is given by (5), and

$$\left(\sum_{j=1}^{m} w_j x_j^* A^s x_j\right)^{1/s} - \left(\sum_{j=1}^{m} w_j x_j^* A^{(r)} x_j\right)^{1/r} \leqslant \triangle, \tag{12}$$

where $\triangle$ is given by (9).

REMARK: (8) required $\sum_{j=1}^{m} w_j x_j^* B x_j > 0$. Since $\sum_{j=1}^{m} w_j x_j^* A^{(r)} x_j \geqslant 0$ for all x_j, we may regard (12) as being satisfied with strict inequality when $\sum_{j=1}^{m} w_j x_j^* A^{(r)} x_j = 0$. (A similar remark will be applicable to $\sum_{j=1}^{m} w_j x_j^* A^{+} x_j = 0$ in equation (18).)

SPECIAL CASES

Our results generalise a number of well-known inequalities. Thus, if $k = n$, that is, A is positive definite, $A^{+} = A^{-1}$ and $B = A^{(r)} = A^r$, and our results yield well-known inequalities for positive definite matrices.

Interesting special cases are, for $r = -1$, $s = 1$,

$$\sum_{j=1}^{m} w_j x_j^* A x_j + \lambda_1 \lambda_k \sum_{j=1}^{n} w_j x_j^* B x_j \leqslant \lambda_1 + \lambda_k; \tag{13}$$

$$\left(\sum_{j=1}^{m} w_j x_j^* A x_j\right)\left(\sum_{j=1}^{m} w_j x_j^* B x_j\right) \leqslant (\lambda_1 + \lambda_k)^2/(4\lambda_1\lambda_k); \tag{14}$$

$$\left(\sum_{j=1}^{m} w_j x_j^* A x_j\right) - \left(\sum_{j=1}^{m} w_j x_j^* B x_j\right)^{-1} \leqslant \left(\sqrt{\lambda_1} - \sqrt{\lambda_k}\right)^2. \tag{15}$$

REMARK: In our results for (4) we have the condition

$$\sum_{j=1}^{m} w_j x_j^* B x_j \geqslant 0.$$

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It is obvious that (14) is valid without this condition and thus is a generalisation of a result from [**2**].

Note that (14) as well as Theorem 3 are generalisations of an inequality of Ky Fan [**1**].

Of special interest is the case where B is the generalised inverse of A. In this case (13), (14) and (15) become

$$\sum_{j=1}^{m} w_j x_j^* A x_j + \lambda_1\lambda_k \sum_{j=1}^{m} w_j x_j^* A^+ x_j \leqslant \lambda_1 + \lambda_k; \tag{16}$$

$$\left(\sum_{j=1}^{m} w_j x_j^* A x_j\right)\left(\sum_{j=1}^{m} w_j x_j^* A^+ x_j\right) \leqslant (\lambda_1 + \lambda_k)^2/4\lambda_1\lambda_2; \tag{17}$$

$$\left(\sum_{j=1}^{m} w_j x_j^* A x_j\right) - \left(\sum_{j=1}^{n} w_j x_j^* A^+ x_j\right)^{-1} \leqslant \left(\sqrt{\lambda_1} - \sqrt{\lambda_k}\right)^2. \tag{18}$$

As noted (18) may be regarded as being satisfied with a strict inequality when

$$\sum_{j=1}^{m} w_j x_j^* A^+ x_j = 0.$$

Hölder's Inequality.

The following result proved in [**3**] will be used to establish a generalisation of the converse of a matrix form of Hölder's inequality.

Lemma 2. [**3**] *Let r, s, p and q be real numbers such that $r > 0$, $s > 0$, $0 < p < 1$ and $p^{-1} + q^{-1} = 1$, $\alpha = ps - qr$. Let A be defined as in Lemma 1 and let C be a Hermitian matrix that satisfies*

$$A^{-rq} C A^{-rq} = A^{-rq},\ A^{-rq} C = C A^{-rq}$$

and has $n - k$ non-positive roots. Then for any x, we have

$$\lambda_k^{-\alpha/q}(\gamma^{\alpha/p} - 1) x^* A^{sp} x + \lambda_1^{\alpha/p}(\gamma^{\alpha/q} - 1) x^* C x \geqslant (\gamma^\alpha - 1) x^* A^{r+s} x \tag{19}$$

where $\gamma = \lambda_1/\lambda_k$.

In a manner similar to the proof of Theorem 1, Lemma 2 yields the following:

Theorem 4. *Let r, s, p, q, α, γ, A and C be as defined in Lemma 2 and let w_j, x_j, $j = 1, \ldots, m$ be defined as in Theorem 1. Then*

$$\lambda_k^{-\alpha/q}(\gamma^{\alpha/p} - 1) \sum_{j=1}^{m} w_j x_j^* A^{sp} x_j + \lambda_1^{\alpha/p}(\gamma^{\alpha/q} - 1) \sum_{j=1}^{m} w_j x_j^* C x_j \geqslant$$

$$(\gamma^\alpha - 1) \sum_{j=1}^{m} w_j x_j^* A^{r+s} x_j. \tag{20}$$

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Analogous to the proof of Theorem 5 in [**3**], we can establish the following:

Theorem 5. *Let r, s, p, q, A, C, α, w_j, x_j, $j = 1, \ldots, m$ be defined as in Theorem 4. If $\sum_{j=1}^{m} w_j x_j C x_j > 0$, then*

$$\sum_{j=1}^{m} w_j x_j^* A^{r+s} x_j \leqslant K \left(\sum_{j=1}^{m} w_j x_j^* A^{sp} x_j \right)^{1/p} \left(\sum_{j=1}^{m} w_j x_j^* C x_j \right)^{1/q}, \tag{21}$$

where

$$K = p^{1/p} |q|^{1/q} \gamma^{\alpha/(pq)} \left(\gamma^{\alpha/p} - 1\right)^{1/p} \left(1 - \gamma^{\alpha/q}\right)^{1/q} / \left(\gamma^{\alpha} - 1\right). \tag{22}$$

Since $A^{(rq)}$ satisfies the conditions for C in Lemma 2, (19), (20) and (21) give

$$\lambda_k^{-\alpha/q} \left(\gamma^{\alpha/p} - 1\right) x^* A^{sp} x + \lambda_1^{\alpha/p} \left(\gamma^{\alpha/q} - 1\right) x^* A^{(rq)} x \geqslant (\gamma^{\alpha} - 1) x^* A^{r+s} x; \tag{23}$$

$$\lambda_k^{-\alpha/q} \left(\gamma^{\alpha/p} - 1\right) \sum_{j=1}^{m} w_j x_j^* A^{sp} x_j + \lambda_1^{\alpha/p} \left(\gamma^{\alpha/q} - 1\right) \sum_{j=1}^{m} w_j x_j^* A^{(rq)} x_j \geqslant (\gamma^{\alpha} - 1) \sum_{j=1}^{m} w_j x_j^* A^{r+s} x_j; \tag{24}$$

$$\sum_{j=1}^{m} w_j x_j^* A^{r+s} x_j \leqslant K \left(\sum_{j=1}^{m} w_j x_j^* A^{sp} x_j \right)^{1/p} \left(\sum_{j=1}^{m} w_j x_j^* A^{(rq)} x_j \right)^{1/q}, \tag{25}$$

where K is given by (22).

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