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Two Exterior Algebras for Orthogonal and Symplectic Quantum Groups

AXEL SCHÜLER⋆
Department of Mathematics, University of Leipzig, Augustusplatz 10, 04109 Leipzig, Germany. e-mail: schueler@mathematik.uni-leipzig.de

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Abstract. Let Γ be one of the N^2-dimensional bicovariant first-order differential calculi on the quantum groups $\mathrm{O}_q(N)$ or $\mathrm{Sp}_q(N)$, where q is not a root of unity. We show that the second antisymmetrizer exterior algebra ${}_s\Gamma^\wedge$ is the quotient of the universal exterior algebra ${}_u\Gamma^\wedge$ by the principal ideal generated by $\theta\wedge\theta$. Here θ denotes the unique up to scalars bi-invariant 1-form. Moreover, $\theta\wedge\theta$ is central in ${}_u\Gamma^\wedge$ and ${}_u\Gamma^\wedge$ is an inner differential calculus.

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1. Introduction

More than a decade ago, Woronowicz provided a general framework for covariant differential calculus over arbitrary Hopf algebras, [12]. Since then, a theory of covariant differential calculus on Hopf algebras has been developed. (for an overview, see [6, Chapter 14]). In his paper, Woronowicz also introduced the concept of higher-order forms which is based on a braiding $\sigma\colon \Gamma\otimes_{\mathcal{A}}\Gamma \to \Gamma\otimes_{\mathcal{A}}\Gamma$. The braiding σ naturally generalizes the classical flip automorphism. It turns out that Woronowicz's external algebra ${}_w\Gamma^\wedge$ is not simply a bicovariant bimodule but a differential graded Hopf algebra [1], [6, Theorem 14.17]. However there are two other concepts of exterior algebras which are also differential graded Hopf algebras, [2, 7], [6, Theorem 14.18]. The 'second antisymmetrizer' exterior algebra ${}_s\Gamma^\wedge$ is also constructed using the braiding; but it involves only the antisymmetrizer $I-\sigma$ of second degree while Woronowicz's construction uses antisymmetrizers of all degrees. The universal exterior algebra ${}_u\Gamma^\wedge$ can be characterized by the following universal property. Each (left-covariant) differential calculus which contains a given first-order calculus Γ as its first-order part is a quotient of ${}_u\Gamma^\wedge$. It seems natural to enquire about the relation between these three concepts. For the quantum groups $\mathrm{GL}_q(N)$ and $\mathrm{SL}_q(N)$ and their standard bicovariant first-order differential calculi (abbreviated FODC) this problem was completely solved in [9].

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In this paper, we consider the quantum groups $\mathrm{O}_q(N)$ and $\mathrm{Sp}_q(N)$ together with their standard bicovariant FODC. The main result is stated in Theorem 1. Suppose that q is not a root of unity and let θ be the unique up to scalars bi-invariant 1-form in Γ. Then ${}_u\Gamma^\wedge/(\theta^2)$ and ${}_s\Gamma^\wedge$ are isomorphic differential graded Hopf algebras. Further, θ^2 is central in ${}_u\Gamma^\wedge$ and ${}_u\Gamma^\wedge$ is an inner differential calculus i.e. $\mathrm{d}\rho = \theta \wedge \rho - (-1)^n \rho \wedge \theta$ for $\rho \in {}_u\Gamma^{\wedge n}$. It is somehow astonishing that the left-invariant parts of ${}_u\Gamma^{\wedge 2}$ and ${}_s\Gamma^{\wedge 2}$ differ only in the single element θ^2.

This paper is organized as follows. Section 2 contains general notions and facts about bicovariant bimodules and bicovariant differential calculi over Hopf algebras. In Section 3, we recall the necessary facts about morphisms of corepresentations for orthogonal and symplectic quantum groups. We give a brief introduction into the graphical calculus with morphisms. The construction of bicovariant FODC on orthogonal and symplectic quantum groups is reviewed. The main result is stated in Theorem 1. In Section 4, a very useful criterion for the size of the space of left-invariant 2-forms of ${}_u\Gamma^\wedge$ in terms of the quantum Lie algebra is given. This criterion applies to arbitrary left-covariant differential calculi. We show that $\Gamma \otimes_{\mathcal{A}} \Gamma$ is the direct sum of 9 bicovariant subbimodules. Every bicovariant subbimodule of $\Gamma \otimes_{\mathcal{A}} \Gamma$ which contains $\theta \otimes_{\mathcal{A}} \theta$, already contains the kernel of $I - \sigma$. Section 5 exclusively deals with the universal differential calculus. The outcome of the very technical calculations is that $\theta \wedge \theta$ is non-zero and the unique up to scalars bi-invariant 2-form in ${}_u\Gamma^\wedge$.

We close the introduction by fixing assumptions and notations that are used in the sequel. All vector spaces, algebras, bialgebras, etc., are meant to be $\mathbb{C}$-vector spaces, unital $\mathbb{C}$-algebras, $\mathbb{C}$-bialgebras, etc. The linear span of a set $\{a_i \mid i \in K\}$ is denoted by $\langle a_i \mid i \in K \rangle$. $\mathcal{A}$ always denotes a Hopf algebra. We write $\mathcal{A}^\circ$ for the dual Hopf algebra. All modules, comodules, and bimodules are assumed to be $\mathcal{A}$-modules, $\mathcal{A}$-comodules, and $\mathcal{A}$-bimodules if nothing else is specified. Denote the comultiplication, the counit, and the antipode by Δ, ε, and by S, respectively. We use the notions 'right comodule' and 'corepresentation' of $\mathcal{A}$ as synonyms. By fixing a basis in the underlying vector space, we identify corepresentations and the corresponding matrices. Let v (resp. f) be a corepresentation (resp. a representation) of $\mathcal{A}$. As usual v^{c} (resp. f^{c}) denotes the contragredient corepresentation (resp. contragredient representation) of v (resp. of f). The space of intertwiners of corepresentations v and w is $\mathrm{Mor}(v, w)$. We write $\mathrm{Mor}(v)$ for $\mathrm{Mor}(v, v)$. By $\mathrm{End}(V)$ and $V \otimes W$ we always mean $\mathrm{End}_{\mathbb{C}}(V)$ and $V \otimes_{\mathbb{C}} W$, respectively. If A is a linear mapping, $A^\top$ denotes the transpose of A and $\mathrm{tr} A$ the trace of A. Lower indices of A always refer to the components of a tensor product where A acts ('leg numbering'). The unit matrix is denoted by I. Unless it is explicitly stated otherwise, we use Einstein convention to sum over repeated indices. Set $\tilde{a} = a - \varepsilon(a)1$ for $a \in \mathcal{A}$ and $\tilde{\mathcal{A}} = \ker \varepsilon = \{\tilde{a} \mid a \in \mathcal{A}\}$. We use Sweedler's notation for the coproduct $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ and for right comodules $\Delta_R(\rho) = \sum \rho_{(0)} \otimes \rho_{(1)}$. The mapping $\mathrm{Ad}_R \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by $\mathrm{Ad}_R a = \sum a_{(2)} \otimes S a_{(1)} a_{(3)}$ is called the *right adjoint*

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coaction of $\mathcal{A}$ on itself. The mapping $b \triangleleft a := Sa_{(1)}ba_{(2)}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, is called the *right adjoint action* of $\mathcal{A}$ on $\mathcal{B}$, where $\mathcal{B}$ is an $\mathcal{A}$-bimodule.

2. Preliminaries

In the next three subsections, we shall use the general framework of bicovariant differential calculus developed by Woronowicz [12], see also [6, Chapter 14]. We collect the main notions and facts needed in what follows.

2.1. BICOVARIANT BIMODULES

A *bicovariant bimodule* over $\mathcal{A}$ is a bimodule Γ together with linear mappings $\Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$ such that $(\Gamma, \Delta_L, \Delta_R)$ is a bicomodule and

$$\Delta_L(a\omega b) = \Delta(a)\Delta_L(\omega)\Delta(b), \quad \text{and} \quad \Delta_R(a\omega b) = \Delta(a)\Delta_R(\omega)\Delta(b),$$

for $a, b \in \mathcal{A}$ and $\omega \in \Gamma$. Let Γ be a bicovariant bimodule over $\mathcal{A}$. We call the elements of the vector space

$$\Gamma_L = \{\omega \mid \Delta_L(\omega) = 1 \otimes \omega\} \quad (\text{resp. } \Gamma_R = \{\omega \mid \Delta_R(\omega) = \omega \otimes 1\})$$

left-invariant (resp. *right-invariant*). The elements of $\Gamma_I = \Gamma_L \cap \Gamma_R$ are called *bi-invariant*. The structure of bicovariant bimodules has been completely characterized by Theorems 2.3 and 2.4 in [12]. We recall the corresponding result: Let $(\Gamma, \Delta_L, \Delta_R)$ be a bicovariant bimodule over $\mathcal{A}$ and let $\{\omega_i \mid i \in K\}$ be a finite linear basis of Γ_L. Then there exist matrices $v = (v^i_j)$ and $f = (f^i_j)$ of elements $v^i_j \in \mathcal{A}$ and of functionals f^i_j on $\mathcal{A}$, $i, j \in K$ such that v is a matrix corepresentation, f is matrix representation of $\mathcal{A}$, and

$$\omega_i \triangleleft a = f^i_n(a)\omega_n, \tag{1}$$

$$\Delta_R(\omega_i) = \omega_n \otimes v^n_i, \tag{2}$$

for $a \in \mathcal{A}$, $i \in K$. Conversely, if the corepresentation v and the representation f satisfy certain compatibility condition, then there exists a unique bicovariant bimodule Γ with (1) and (2) and $\{\omega_i \mid i \in K\}$ is a basis of Γ_L. In this situation we simply write $\Gamma = (v, f)$.

2.2 BICOVARIANT FIRST ORDER DIFFERENTIAL CALCULI

A *first-order differential calculus* over $\mathcal{A}$ (FODC for short) is an $\mathcal{A}$-bimodule Γ with a linear mapping $\mathrm{d} : \mathcal{A} \to \Gamma$ that satisfies the Leibniz rule $\mathrm{d}(ab) = \mathrm{d}a \cdot b + a \cdot \mathrm{d}b$ for $a, b \in \mathcal{A}$, and Γ is the linear span of elements $a\mathrm{d}b$ with $a, b \in \mathcal{A}$.

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A FODC Γ is called *bicovariant* if there exist linear mappings $\Delta_L: \Gamma \to \mathcal{A} \otimes \Gamma$ and $\Delta_R: \Gamma \to \Gamma \otimes \mathcal{A}$ such that

$$\Delta_L(a\mathrm{d}b) = \Delta(a)(\mathrm{id} \otimes \mathrm{d})\Delta(b),$$
$$\Delta_R(a\mathrm{d}b) = \Delta(a)(\mathrm{d} \otimes \mathrm{id})\Delta(b)$$

for all $a, b \in \mathcal{A}$. It turns out that $(\Gamma, \Delta_L, \Delta_R)$ is a bicovariant bimodule. A bicovariant FODC is called *inner* if there exists a bi-invariant 1-form $\theta \in \Gamma$ such that $\mathrm{d}a = \theta a - a\theta$, $a \in \mathcal{A}$. By the dimension of a bicovariant FODC we mean the dimension of the vector space Γ_L of left-invariant 1-forms. Let Γ be a bicovariant FODC over $\mathcal{A}$. Then the set $\mathcal{R}_\Gamma = \{a \in \widetilde{\mathcal{A}} \mid \omega(a) = 0\}$ is an Ad_R-invariant right ideal of $\widetilde{\mathcal{A}}$. Here $\omega: \mathcal{A} \to \Gamma_L$ is the mapping

$$\omega(a) = Sa_{(1)}\mathrm{d}a_{(2)}. \tag{3}$$

Conversely, for any Ad_R-invariant right ideal $\mathcal{R}$ of $\widetilde{\mathcal{A}}$, there exists a bicovariant FODC Γ such that $\mathcal{R}_\Gamma = \mathcal{R}$ (cf. [6, Proposition 14.7]).

The linear space

$$\mathcal{X}_\Gamma = \{X \in \mathcal{A}^\circ \mid X(1) = 0 \quad \text{and } X(p) = 0 \text{ for all } p \in \mathcal{R}_\Gamma\}$$

is called the *quantum Lie algebra* of Γ. We recall the main property. The space $\mathcal{X}_\Gamma$ is an ad_R-invariant subspace of the dual Hopf algebra $\mathcal{A}^\circ$ satisfying $\Delta(X) - 1 \otimes X \in \mathcal{X}_\Gamma \otimes \mathcal{A}^\circ$ for $X \in \mathcal{X}_\Gamma$, [6, Corollary 14.10].

2.3. HIGHER-ORDER DIFFERENTIAL CALCULI

In this subsection, we briefly repeat two concepts to construct higher-order differential calculi (DC for short) to a given bicovariant FODC Γ. Let $\Gamma = (v, f)$ be a bicovariant bimodule.

Obviously the tensor product $\Gamma^{\otimes k} = \Gamma \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Gamma$($k$ factors) is again a bicovariant bimodule. Define the tensor algebra $\Gamma^{\otimes} = \bigoplus_{k \geqslant 0} \Gamma^{\otimes k}$, $\Gamma^{\otimes 0} = \mathcal{A}$, over $\mathcal{A}$. It is also a bicovariant bimodule. Since bicovariant bimodules are free left $\mathcal{A}$-modules we always identify $(\Gamma \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Gamma)_L$ and $\Gamma_L \otimes \cdots \otimes \Gamma_L$. This justifies our notation $\omega_i \otimes \omega_j$ instead of $\omega_i \otimes_{\mathcal{A}} \omega_j$ for $\omega_i, \omega_j \in \Gamma_L$. There exists a unique isomorphism $\sigma: \Gamma \otimes_{\mathcal{A}} \Gamma \to \Gamma \otimes_{\mathcal{A}} \Gamma$, of bicovariant bimodules called the *braiding* with $\sigma(\omega \otimes \rho) = \rho_{(0)} \otimes (\omega \triangleleft \rho_{(1)})$, $\omega, \rho \in \Gamma_L$. Moreover, σ fulfils the braid equation

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)$$

in $\Gamma \otimes_{\mathcal{A}} \Gamma \otimes_{\mathcal{A}} \Gamma$. Let ${}_sJ$ denote the two-sided ideal in $\Gamma^{\otimes}$ generated by the kernel of A_2:

$$\Gamma \otimes_{\mathcal{A}} \Gamma \to \Gamma \otimes_{\mathcal{A}} \Gamma, \qquad A_2 = \mathrm{id} - \sigma.$$

We call ${}_s\Gamma^{\wedge} = \Gamma^{\otimes}/{}_sJ$ the *second antisymmetrizer exterior algebra* over Γ. Since σ is a morphism of bicomodules $(\Gamma \otimes_{\mathcal{A}} \Gamma)_L$ is invariant under σ. So there exist complex

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numbers σ^{ij}_{st} such that $\sigma(\omega_s\otimes\omega_t)=\sigma^{ij}_{st}\omega_i\otimes\omega_j$. By [12, (3.15)] we have

$$\sigma^{ij}_{st}=f^s_j(v^i_t). \tag{4}$$

Let $\mathscr{S}\colon\mathcal{A}\to\Gamma_L\otimes\Gamma_L$ be defined by $\mathscr{S}(a)=\omega(a_{(1)})\otimes\omega(a_{(2)})$. Let ${}_uJ$ denote the two-sided ideal of $\Gamma^{\otimes}$ generated by the vector space $\mathscr{S}(\mathcal{R}_\Gamma)$. Then ${}_u\Gamma^{\wedge}=\Gamma^{\otimes}/{}_uJ$ is called the *universal exterior algebra* over Γ. Both ${}_s\Gamma^{\wedge}$ and ${}_u\Gamma^{\wedge}$ are $\mathbb{N}_0$-graded algebras, bicovariant bimodules over $\mathcal{A}$ as well as differential graded Hopf algebras over $\mathcal{A}$. They are related by ${}_uJ\subseteq{}_sJ$. Their left-invariant subalgebras ${}_u\Gamma^{\wedge}_L$ and ${}_s\Gamma^{\wedge}_L$ are both quadratic algebras over the same vector space Γ_L.

3. Orthogonal and Symplectic Quantum Groups, their Standard FODC, and the Main Result

In this section we recall general facts about orthogonal and symplectic quantum groups. Throughout, $\mathcal{A}$ denotes one of the Hopf algebras $\mathcal{O}(\mathrm{O}_q(N))$ and $\mathcal{O}(\mathrm{Sp}_q(N))$ as defined in [3, Subsection 1.4]. We give a brief introduction into the graphical calculus with morphisms of corepresentations of $\mathcal{A}$ and we recall the construction of the standard bicovariant FODC over $\mathcal{A}$. At the end we state our main result.

As usual we set $\varepsilon=1$ in the orthogonal and $\varepsilon=-1$ in the symplectic case. Throughout the deformation parameter q is *not a root of unity*, and $N\geqslant 3$. We always use the abbreviations

$$\hat{q}=q-q^{-1},\qquad [2]_q=q+q^{-1},\qquad r=\varepsilon q^{N-\varepsilon},\quad\text{and}\quad x=1+\frac{r-r^{-1}}{q-q^{-1}}.$$

Recall that R denotes the complex invertible $N^2\times N^2$-matrix [3, (1.9)], $\hat{R}^{ab}_{st}=R^{ba}_{st}$, and $C=(C^i_j)$, $C^i_j=\varepsilon_i q^{\varrho_j}\delta_{ij'}$ defines the metric. Here we set $\varepsilon_i=1$ for $i\leqslant N/2$ and $\varepsilon_i=\varepsilon$ otherwise. The parameter ϱ_k are determined as follows. Let $k':=N+1-k$ for $k=1,\dots,N$. Then set $\varrho_k=(N+1-\varepsilon)/2-k$ for $k\leqslant N/2$ and $\varrho_{k'}=-\varrho_k$ for all $k=1,\dots,N$. The matrix K is given by $K^{ab}_{st}=C^a_bB^s_t$, where $B=C^{-1}=\varepsilon C$. We need the diagonal matrix $D=B^{\top}C$. Sometimes we use the notation $C^{ab}=C^a_b$, $C_{ab}=C^a_b$. Then $(C^{ab})\in\mathrm{End}(\mathbb{C},\mathbb{C}^N\otimes\mathbb{C}^N)$ and $(C_{ab})\in\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N,\mathbb{C})$. The N^2 generators of $\mathcal{A}$ are denoted by u^i_j, $i,j=1,\dots,N$, and we call $u=(u^i_j)$ the fundamental matrix corepresentation. The element $U=\sum_{i,j}D^j_iu^i_j$ is called the quantum trace. Note that

$$(C^{ab})\in\mathrm{Mor}(1,u\otimes u),\ (C_{ab})\in\mathrm{Mor}(u\otimes u,1),\ C^{\top}\in\mathrm{Mor}(u^{\mathrm{c}},u). \tag{5}$$

For $T=(T^{ab}_{st})\in\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)$ define the q-trace $\mathrm{tr}^1_qT\in\mathrm{End}(\mathbb{C}^N)$ by $(\mathrm{tr}^1_qT)^b_t=$

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$D_a^s T_{st}^{ab}$. We often use the following well known relations between $\hat{R}$, $\hat{R}^{-1}$, K, and D.

$$C_z^y \hat{R}_{st}^{yz} = r^{-1} C_t^s, \qquad \hat{R}_{yz}^{ab} C_z^y = r^{-1} C_b^a, \tag{6}$$

$$\hat{R} - \hat{R}^{-1} = \hat{q}(I - K), \tag{7}$$

$$x = \operatorname{tr} D, \tag{8}$$

$$\operatorname{tr}_q^1 \hat{R} = rI, \qquad \operatorname{tr}_q^1 I = xI, \qquad \operatorname{tr}_q^1 K = I. \tag{9}$$

The mapping $g_i \longmapsto \hat{R}_{i,i+1}$, $e_i \longmapsto K_{i,i+1}$ defines a representation of the Birman–Wenzl–Murakami algebra $\mathrm{C}(q, r)$, [11]. We shall give a brief introduction into the graphical calculus with morphisms, see also [8, Fig. 1 and Fig. 6]. The calculus is justified in [10]. Using the graphical calculus formulas and proofs become more transparent. In order to distinguish the places for the corepresentation u and u^{c} we use arrows in the graph (Figure 1). A vertex stands for u, resp. u^{c}, if the corresponding edge is downward directed, resp. upward directed. Since for orthogonal and symplectic quantum groups u and u^{c} are isomorphic, it appears that one edge has *two* directions. For instance, the intertwiner $C^{\top} \in \mathrm{Mor}(u, u^{\mathrm{c}})$ is represented by a vertical edge downward directed at the bottom and upward directed at the top. Removing a curl by rotating part of the diagram clockwise (resp. anti-clockwise) acquires a factor r (resp. r^{-1}) (First Reidemeister move). A closed loop gives the factor x.

The matrix $\hat{R}$ has the spectral decomposition

$$\hat{R} = q\hat{P}^{+} - q^{-1}\hat{P}^{-} + r^{-1}\hat{P}^{0},$$

where $\hat{P}^{\tau}$, $\tau \in \{+, -, 0\}$, is idempotent.

We repeat the method of Jurčo [5] to construct bicovariant FODC over $\mathcal{A}$. For the more general construction of bicovariant FODC over coquasitriangular Hopf algebras see [6, Section 14.5]. Let $\ell^{\pm} = (\ell_j^{\pm i})$ be the $N \times N$-matrix of linear functionals $\ell_j^{\pm i}$ on $\mathcal{A}$ as defined in [3, Section 2]. Recall that $\ell^{\pm}$ is uniquely determined by $\ell_j^{\pm i}(u_n^m) = (\hat{R}^{\pm 1})_{nj}^{im}$ and the property that $\ell^{\pm}\colon \mathcal{A} \to \mathrm{End}(\mathbb{C}^N)$ is a unital algebra homomorphism. Note that $\ell_j^{\pm i}(Su_n^m) = (\hat{R}^{\mp})_{jn}^{mi}$. Define the bicovariant bimodules

$$\Gamma_{\pm} = (u \otimes u^{\mathrm{c}}, \varepsilon_{\pm} \otimes \ell^{-\mathrm{c}} \otimes \ell^{+}),$$

where $\varepsilon_{+} = \varepsilon$ and ε_{-} is the character on $\mathcal{A}$ given by $\varepsilon_{-}(u_j^i) = -\delta_{ij}$. The structure of $\Gamma_{\pm}$

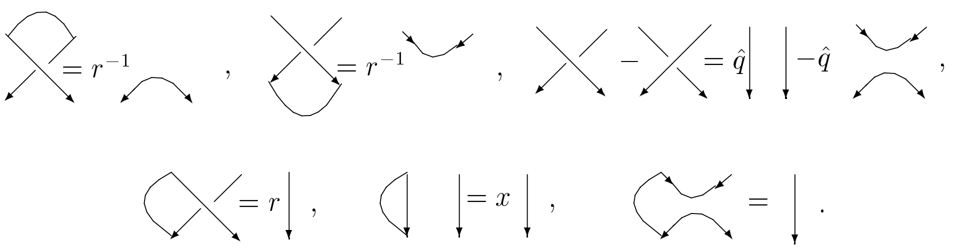

Figure 1. The graphical representation of (6), (7), and (9).

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can easily be described as follows. There exists a basis $\{\theta_{ij} \mid i, j = 1, \dots, N\}$ of $(\Gamma_\pm)_L$ such that the right adjoint action and the right coaction are given by

$$\theta_{ij} \triangleleft a = \varepsilon_\pm(a_{(1)}) S(\ell_i^{-m}) \ell_n^{+j}(a_{(2)}) \theta_{mn}, \quad a \in \mathcal{A},$$
$$\Delta_R \theta_{ij} = \theta_{mn} \otimes u_i^m (u^{\mathrm{c}})_j^n, \quad i, j = 1, \dots, N.$$

In particular

$$\theta_{ij} \triangleleft u_t^s = \pm \hat{R}_{iy}^{sm} \hat{R}_{tn}^{jy} \theta_{mn}, \qquad \theta \triangleleft u_t^s = \pm (\hat{R}^2)_{tn}^{sm} \theta_{mn}, \tag{10}$$

where $\theta = \sum_i \theta_{ii}$ is the unique up to scalars bi-invariant element. Defining $\mathrm{d}a = \theta a - a\theta$ for $a \in \mathcal{A}$, $(\Gamma_\pm, \mathrm{d})$ becomes a bicovariant FODC over $\mathcal{A}$. The basis $\{X_{ij}^\pm\}$ of the quantum Lie algebra $\mathcal{X}_\pm$ dual to $\{\theta_{ij}\}$ is given by

$$X_{ij}^\pm := \varepsilon_\pm \ell_j^i - \delta_{ij} := \varepsilon_\pm S(\ell_y^{-i}) \ell_j^{+y} - \delta_{ij}.$$

One easily checks that $X_0^\pm := (D^{-1})_i^j X_{ij}^\pm = \varepsilon_\pm (D^{-1})_i^j \ell_j^i - x$ is an ad_R-invariant element of $\mathcal{A}^\circ$. The braiding σ of $\Gamma_\pm$ can be obtained as follows. Inserting $v = u \otimes u^{\mathrm{c}}$ and $f_\pm = \varepsilon_\pm \otimes \ell^{-\mathrm{c}} \otimes \ell^+$ into equation (4) the braiding matrices of Γ_+ and Γ_- coincide

$$\sigma = \grave{R}_{23}^{-} \hat{R}_{12} \check{R}_{34}^{-1} \acute{R}_{23}, \tag{11}$$

where the matrices $\acute{R}$, $\check{R}$, and $\grave{R}^-$ are defined as follows. For a complex $N^2 \times N^2$-matrix T with $\hat{T} \in \mathrm{Mor}(u \otimes u)$ define the matrices $\check{T}_{st}^{ab} = \hat{T}_{ba}^{ts}$, $\acute{T}_{st}^{ab} = \hat{T}_{tb}^{sa}$, and $\grave{T} = (\acute{T})^{-1}$. Note that $\acute{T} \in \mathrm{Mor}(u^{\mathrm{c}} \otimes u, u \otimes u^{\mathrm{c}})$ and $\check{T} \in \mathrm{Mor}(u^{\mathrm{c}} \otimes u^{\mathrm{c}})$.

Now we can formulate our main result.

THEOREM 1. *Let $\mathcal{A}$ be one of the Hopf algebras $\mathcal{O}(\mathrm{O}_q(N))$ or $\mathcal{O}(\mathrm{Sp}_q(N))$, $N \geqslant 3$, and q not a root of unity. Let Γ be one of the bicovariant FODC $\Gamma_\pm$, and $2x + (q - q^{-1})(r - r^{-1}) \neq 0$ in case of Γ_-. Denote the unique up to scalars bi-invariant 1-form by θ.*

(i) *Then the quotient ${}_u\Gamma^\wedge/(\theta^2)$ and the second antisymmetrizer algebra ${}_s\Gamma^\wedge$ are isomorphic bicovariant bimodules.*
(ii) *The bi-invariant 2-form θ^2 is central in ${}_u\Gamma^\wedge$. The calculus ${}_u\Gamma^\wedge$ is inner, i.e.*

$$\mathrm{d}\rho = \theta \wedge \rho - (-1)^n \rho \wedge \theta, \quad \rho \in {}_u\Gamma^{\wedge n}. \tag{12}$$

Remark. Theorem 1 is true for the quantum group $\mathrm{SL}_q(2)$ and the $4D_\pm$ biocovariant FODC as well, [9, Theorem 3.3(iii)]. In cases $\mathrm{SL}_q(N)$ and $\mathrm{GL}_q(N)$, $N \geqslant 3$, we have ${}_u\Gamma^\wedge \cong_s \Gamma^\wedge$, [9, Theorem 3.3(ii)]. For the quantum super group $\mathrm{GL}_q(m \mid n)$ the relation ${}_u\Gamma^\wedge \cong_s \Gamma^\wedge$ was proved in [7, Section 5.3].

Remark. The isomorphism of bicovariant bimodules ${}_u\Gamma^\wedge/(\theta^2)$ and ${}_s\Gamma^\wedge$ implies its isomorphy as differential graded Hopf algebras.

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4. Proof of the Theorem

In the first part of this section, we study the duality of $\Gamma_L \otimes \Gamma_L$ and $\mathcal{X} \otimes \mathcal{X}$ in more detail. In the second part, we examine how $\Gamma \otimes_{\mathcal{A}} \Gamma$ splits into bicovariant sub-bimodules. We shall prove that the space of bi-invariant elements of $\Gamma \otimes_{\mathcal{A}} \Gamma$ generates the whole bimodule $\ker A_2$.

4.1. DUALITY

There is a useful criterion to describe the dimension of the space of left-invariant 2-forms of ${}_u\Gamma^\wedge$ in terms of the quantum Lie algebra.

LEMMA 2. *Let $\mathcal{A}$ be an arbitrary Hopf algebra, Γ a left-covariant FODC over $\mathcal{A}$ with quantum Lie algebra $\mathcal{X}$, and ${}_u\Gamma^\wedge$ the universal differential calculus over Γ. Then*

$$\dim_u \Gamma_L^{\wedge 2} = \dim\{T \in \mathcal{X} \otimes \mathcal{X} \mid \mu(T) \in \mathcal{X}\},$$

where $\mu: \mathcal{X} \otimes \mathcal{X} \to \mathcal{A}^\circ$ denotes the multiplication map.

Proof. We use the following simple lemma from linear algebra without proof. Let $B: V \times W \to \mathbb{C}$ be a nondegenerate linear pairing of finite-dimensional vector spaces and U a subspace of V. Then the induced pairing $\overline{B}: V/U \times U^\perp \to \mathbb{C}$ with $U^\perp = \{w \in W \mid B(u, w) = 0 \text{ for } u \in U\}$ is also nondegenerate. Applying this lemma to the non-degenerate pairing $\langle \cdot, \cdot \rangle: \Gamma_L \otimes \Gamma_L \times \mathcal{X} \otimes \mathcal{X} \to \mathbb{C}$, [12, p. 164], and $U = \mathcal{S}(\mathcal{R})$ we have $T = \alpha^{ij} X_i \otimes X_j \in U^\perp$ if and only if

$$0 = \langle \omega(p_{(1)}) \otimes \omega(p_{(2)}),\ \alpha^{ij} X_i \otimes X_j \rangle = \alpha^{ij} X_i(p_{(1)}) X_j(p_{(2)}) = \mu(T)(p)$$

for $p \in \mathcal{R}$. Hence, $T \in U^\perp$ if and only if $\mu(T) \in \mathcal{X}$. Consequently $U^\perp = \mu^{-1}(\mathcal{X})$, where $\mu^{-1}(\mathcal{X})$ denotes the pre-image of $\mathcal{X}$ under μ. Since the induced pairing is nondegenerate too and ${}_u\Gamma_L^{\wedge 2} = \Gamma_L \otimes \Gamma_L / \mathcal{S}(\mathcal{R})$ by definition, the assertion of the lemma is proved.

Remark. Suppose Γ to be bicovariant. Since for $f: V \to V$ linear, $(\ker f)^\perp = \operatorname{im} f^\top$, the pairing also factorizes to a nondegenerate pairing of ${}_s\Gamma_L^{\wedge 2} \times \mathcal{X} \wedge \mathcal{X}$, where $\mathcal{X} \wedge \mathcal{X} = A_2^\top(\mathcal{X} \otimes \mathcal{X})$ and $A_2^\top$ is the dual mapping to $A_2 \restriction (\Gamma_L \otimes \Gamma_L)$.

We proceed with a result for a dual pairing of a comodule and a module.

PROPOSITION 3. *Let V be a right $\mathcal{A}$-comodule, W a right $\mathcal{A}^\circ$-module, and $\langle , \rangle: V \times W \to \mathbb{C}$ a nondegenerate dual pairing of vector spaces. Moreover*

$$\langle v, w \cdot f \rangle = \langle v_{(0)} f(v_{(1)}), w \rangle,$$

for $v \in V$, $w \in W$, and $f \in \mathcal{A}^\circ$.

If $P \in \mathrm{Mor}(V)$ then $P^\top \in \mathrm{Mor}(W)$. If in addition $P^2 = P$, then the induced pairing $\mathrm{im} P \times \mathrm{im} P^\top \to \mathbb{C}$ is nondegenerate too.

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Proof. Since $P \in \mathrm{Mor}(V)$, $Pv_{(0)} \otimes v_{(1)} = (Pv)_{(0)} \otimes (Pv)_{(1)}$. For $v \in V$, $w \in W$, and $f \in \mathcal{A}^\circ$ we thus get

$$
\begin{aligned}
\langle v, P^\top w \cdot f\rangle &= \langle v_{(0)} f(v_{(1)}), P^\top w\rangle \\
&= \langle P v_{(0)} f(v_{(1)}), w\rangle \\
&= \langle (Pv)_{(0)} f((Pv)_{(1)}), w\rangle \\
&= \langle Pv, w \cdot f\rangle \\
&= \langle v, P^\top (w \cdot f)\rangle .
\end{aligned}
$$

Since the pairing is nondegenerate the first assertion follows.

Since P and $P^\top$ are morphisms, the corresponding subspaces are invariant. Let $v_0 \in \mathrm{im}P$, i.e. $v_0 = Pv_0$, and suppose $0 = \langle v_0, P^\top w\rangle$ for all $w \in W$. Then $0 = \langle Pv_0, P^\top w\rangle = \langle P^2 v_0, w\rangle = \langle v_0, w\rangle$. Since the pairing is nondegenerate, $v_0 = 0$. Similarly, one shows that $\mathrm{im}P$ separates the elements of $\mathrm{im}P^\top$.

COROLLARY 4. *Let $\mathcal{A}$ be an arbitrary Hopf algebra, Γ a bicovariant FODC over $\mathcal{A}$ with quantum Lie algebra $\mathcal{X}$, and $P \in \mathrm{Mor}(\Delta_R)$, $P^2 = P$. We restrict Δ_R to $(\Gamma \otimes_{\mathcal{A}} \Gamma)_L$ or a suitable quotient. Then $\mathrm{im}P$ is a Δ_R-invariant subspace of $(\Gamma \otimes_{\mathcal{A}} \Gamma)_L$ (${}_u\Gamma_L^{\wedge 2}$ resp. ${}_s\Gamma_L^{\wedge 2}$), and $\mathrm{im}P^\top$ is an ad_R-invariant subspace of $\mathcal{X} \otimes \mathcal{X}$ ($\mu^{-1}(\mathcal{X})$ resp. $\mathcal{X} \wedge \mathcal{X}$). The induced pairing $\mathrm{im}P \times \mathrm{im}P^\top \to \mathbb{C}$ is nondegenerate.*

Proof. (i) Since $\mathcal{S}(\mathcal{R})$ and $(\ker A_2)_L$ are Δ_R-invariant, and since $\mu^{-1}(\mathcal{X})$ and $A_2^\top(\mathcal{X} \otimes \mathcal{X})$ are ad_R-invariant, the mappings Δ_R and ad_R are well-defined on both quotients ${}_u\Gamma_L^{\wedge 2}$ and ${}_s\Gamma_L^{\wedge 2}$ resp. $\mu^{-1}(\mathcal{X} \otimes \mathcal{X})$ and $\mathcal{X} \wedge \mathcal{X}$.

It follows from [12, (5.17) and (5.21)] that for $\rho \in (\Gamma \otimes_{\mathcal{A}} \Gamma)_L$, $Y \in \mathcal{X} \otimes \mathcal{X}$, and $f \in \mathcal{A}^\circ$

$$
\langle \rho_{(0)} f(\rho_{(1)}),\ Y\rangle = \langle \rho,\ Y \triangleleft f\rangle .
$$

Thus Proposition 3 applies to our situation.

Let $\mathcal{B}$ be a right $\mathcal{A}^\circ$-module with respect to ad_R. For the space of invariants we use the notation $\mathcal{B}_0 = \{b \in \mathcal{B} \mid b \triangleleft f = \varepsilon(f) b, f \in \mathcal{A}^\circ\}$. Our next aim is to compare the bi-invariant components of $\Gamma \otimes_{\mathcal{A}} \Gamma$, ${}_s\Gamma^{\wedge 2}$, and ${}_u\Gamma^{\wedge 2}$ with the invariant subspaces $(\mathcal{X} \otimes \mathcal{X})_0$, $(\mathcal{X} \wedge \mathcal{X})_0$, and $\mu^{-1}(\mathcal{X})_0$, respectively.

LEMMA 5. *Let $\mathcal{A}$ be one of the Hopf algebras $\mathcal{O}(\mathrm{O}_q(N))$ or $\mathcal{O}(\mathrm{Sp}_q(N))$, $N \geqslant 3$, Γ one of the N^2-dimensional bicovariant FODC $\Gamma_\pm$ over $\mathcal{A}$ and let $\mathcal{X}$ be the corresponding quantum Lie algebra. Then we have*

(i) $\dim \Gamma_I^{\otimes 2} = 3$, $\quad \dim(\mathcal{X} \otimes \mathcal{X})_0 = 3$,
(ii) $\dim {}_s\Gamma_I^{\wedge 2} = 0$, $\quad \dim(\mathcal{X} \wedge \mathcal{X})_0 = 0$, $\quad \dim(\ker A_2)_I = 3$,
(iii) $\dim {}_u\Gamma_I^{\wedge 2} = 1$ $\quad \mu^{-1}(\mathcal{X})_0 = \langle T\rangle$, $\quad \dim(\mathcal{S}(\mathcal{R})_I) = 2$,

where $T = X_{ij} \otimes X_{mn} B^i_y \hat{R}^{jy}_{mz} C^n_z$.

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Proof. (i) It is well known that $\dim \mathrm{Mor}(u \otimes u) = 3$, and I, $\hat{R}$, and K form a linear basis of $\mathrm{Mor}(u \otimes u)$. Using (5) it is easy to see that the mapping $T \mapsto (B^s_z C^r_y T^{ab}_{yz})$ defines a linear isomorphism $\mathrm{Mor}(u \otimes u) \to \mathrm{Mor}(1, u \otimes u^c \otimes u \otimes u^c)$. Since

$$\rho = \alpha^{ijmn} \theta_{ij} \otimes \theta_{mn} \in \Gamma \otimes_{\mathcal{A}} \Gamma$$

is bi-invariant if and only if $(\alpha^{ijmn}) \in \mathrm{Mor}(1, u \otimes u^c \otimes u \otimes u^c)$, (i) is proved.

(ii) The elements

$$\theta \otimes \theta, \quad \eta = D^k_j \theta_{ik} \otimes \theta_{ji} \quad \text{and} \quad \xi = C^i_x \hat{R}^{ym}_{xn} B^y_j \theta_{ij} \otimes \theta_{mn}$$

form a basis of $\Gamma_I^{\otimes 2}$. Using the graphical calculus it is not difficult to check that σ acts as the identity on $\Gamma_I^{\otimes 2}$, see Figure 2.

Consequently, $\Gamma_I^{\otimes 2} \subseteq \ker A_2$ and ${}_s\Gamma_I^{\wedge 2} = \{0\}$. By Corollary 4 we obtain $(\mathcal{X} \wedge \mathcal{X})_0 = 0$.

(iii) By [3, (2.3)], $\hat{R}^{jy}_{vj} (\ell^{-c})^m_y \ell^{+j}_z = \ell^{+w}_j (\ell^{-c})^v_m \hat{R}^{jy}_{mz}$. Further by [3, Remark 21], $\ell^{\pm} C^{\top} \ell^{\pm\top} = C^{\top} 1$. Let

$$T_0 = \ell^i_j \ell^m_n B^i_y \hat{R}^{jy}_{mz} C^n_z = (\ell^{-c})^w_i \ell^{+w}_j (\ell^{-c})^v_m \ell^{+v}_n B^i_y \hat{R}^{jy}_{mz} C^n_z.$$

Using the above identities, (6) twice, and finally (8) one gets

$$\begin{aligned} T_0 &= S\ell^{-i}_w (\ell^{-c})^m_y \ell^{+j}_z \hat{R}^{wm}_{vj} \ell^{+v}_n B^i_y C^n_z \\ &= S\ell^{-i}_w S\ell^{-y}_m \hat{R}^{wm}_{vj} C^v_j B^i_y \\ &= r^{-1} S(\ell^{-y}_m \ell^{-i}_w C^w_m) B^i_y \\ &= r^{-1} C^i_y B^i_y 1 = r^{-1} x. \end{aligned}$$

Using $X^{\pm}_{ij} = \varepsilon_{\pm} \ell^i_j - \delta^i_j$, the above calculation, and again (6) and (8) it follows that

$$\begin{aligned} \mu(T) &= (\varepsilon_{\pm} \ell^i_j - \delta^i_j)(\varepsilon_{\pm} \ell^m_n - \delta^m_n) B^i_y \hat{R}^{jy}_{mz} C^n_z \\ &= r^{-1} x - \varepsilon_{\pm} B^i_y \hat{R}^{iy}_{mz} C^n_z \ell^m_n - \varepsilon_{\pm} B^i_y \hat{R}^{jy}_{mz} C^m_z \ell^i_j + B^i_y \hat{R}^{iy}_{mz} C^m_z \\ &= r^{-1} (x - \varepsilon_{\pm} B^m_z C^n_z \ell^m_n - \varepsilon_{\pm} B^i_y C^j_y \ell^i_j + x) \\ &= -2 r^{-1} X^{\pm}_0. \end{aligned}$$

Consequently, $T \in \mu^{-1}(\mathcal{X})$ and $\dim \mu^{-1}(\mathcal{X})_0 \geqslant 1$. By Corollary 4 applied to the projection $P \in \mathrm{Mor}(\Delta_R \lceil_u \Gamma_L^{\wedge 2})$ onto the space ${}_u\Gamma_I^{\wedge 2}$, the pairing ${}_u\Gamma_I^{\wedge 2} \times \mu^{-1}(\mathcal{X})_0 \to \mathbb{C}$ is nondegenerate. Since ${}_u\Gamma_I^{\wedge 2} = \Gamma_I^{\otimes 2} / \mathcal{S}(\mathcal{R})_I$, $\dim \Gamma_I^{\otimes 2} = 3$ by (i),

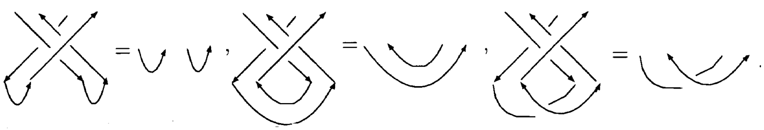

Figure 2. σ acts as the identity on $\theta \otimes \theta$, η, and ξ.

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and $\dim \mathscr{S}(\mathcal{R})_I \geqslant 2$ by the result of Section 5 we get $\dim \mu^{-1}(\mathcal{X})_0 = \dim_u \Gamma_I^{\wedge 2} = 1$. This completes the proof.

4.2 BICOVARIANT SUBBIMODULES

We shall describe a method to construct a class of bicovariant subbimodules of $\Gamma \otimes_{\mathcal{A}} \Gamma$. This method is also applicable to higher tensor products $\Gamma^{\otimes k}$, cf. [4, p. 1356]. In this subsection $\mathcal{A}$ is one of the Hopf algebras $\mathcal{O}(\mathrm{O}_q(N))$ or $\mathcal{O}(\mathrm{Sp}_q(N))$ and Γ always denotes one of the N^2-dimensional bicovariant bimodules $\Gamma_\pm$ over $\mathcal{A}$ with left-invariant basis $\{\theta_{ij} \mid i, j = 1, \ldots, N\}$. The canonical basis of $\mathbb{C}^N$ is $\{e_1, \ldots, e_N\}$.

LEMMA 6. (i) *Let* $\hat{P}$ *and* $\check{Q}$ *be idempotents in* $\mathrm{Mor}(u \otimes u)$ *and* $\mathrm{Mor}(u^c \otimes u^c)$*, respectively. We identify the underlying spaces of the right coaction on* $\Gamma_L \otimes \Gamma_L$ *and the equivalent matrix corepresentation* $u \otimes u^c \otimes u \otimes u^c$ *via* $\theta_{ij} \otimes \theta_{kl} \to e_i \otimes e_j \otimes e_k \otimes e_l$. *Then the subspace*

$$\grave{R}^{-}_{23} \hat{P}_{12} \check{Q}_{34} \acute{R}_{23} (\Gamma_L \otimes \Gamma_L) \tag{13}$$

of $\Gamma_L \otimes \Gamma_L$ *is the left-invariant basis of a bicovariant subbimodule of* $\Gamma \otimes_{\mathcal{A}} \Gamma$ *of dimension* $\mathrm{rk}(\hat{P})\,\mathrm{rk}(\check{Q})$.

(ii) $\Gamma \otimes_{\mathcal{A}} \Gamma$ *is the direct sum of* 9 *bicovariant subbimodules* $\Lambda^{\tau\nu}$, $\tau, \nu \in \{+, -, 0\}$, *generated by the left-invariant elements*

$$\grave{R}^{-}_{23} \hat{P}^{\tau}_{12} \check{P}^{\nu}_{34} \acute{R}_{23} (\Gamma_L \otimes \Gamma_L).$$

Moreover we have the following identity of bicovariant bimodules

$$\ker A_2 = \Lambda^{++} \oplus \Lambda^{--} \oplus \Lambda^{00}.$$

Proof. (i) Since all four in (13) appearing mappings are morphisms of corepresentations one easily checks that $T := \grave{R}^{-}_{23} \hat{P}_{12} \check{Q}_{34} \acute{R}_{23} \in \mathrm{Mor}(u \otimes u^c \otimes u \otimes u^c)$. Hence the space is closed under the right coaction. Now we compute the right adjoint action. Set $\theta^{PQ}_{mnkl} = T(\theta_{mn} \otimes \theta_{kl})$. By (10)

$$\begin{aligned} \theta^{PQ}_{mnkl} \triangleleft u^i_j &= T^{vwcd}_{mnkl} \hat{R}^{ia}_{vy} \hat{R}^{wy}_{eb} \hat{R}^{es}_{cz} \hat{R}^{dz}_{jt} \theta_{ab} \otimes \theta_{st} \\ &= (\hat{R}_{12} \acute{R}_{23} \hat{R}_{34} \acute{R}_{45} T_{1234})^{iabst}_{mnklj} \theta_{ab} \otimes \theta_{st} \\ &= (T_{1234} \hat{R}_{12} \acute{R}_{23} \hat{R}_{34} \acute{R}_{45})^{iabst}_{mnklj} \theta_{ab} \otimes \theta_{st} \\ &= \theta^{PQ}_{vwst} (\hat{R}_{12} \acute{R}_{23} \hat{R}_{34} \acute{R}_{45})^{ivwst}_{mnklj}. \end{aligned}$$

The second last equation becomes evident taking a look at the graphical presentation of these equations, see Figure 3.

Consequently, $T(\Gamma_L \otimes \Gamma_L)$ is closed under the right adjoint action. Hence $\mathcal{A}T(\Gamma_L \otimes \Gamma_L)$ is a bicovariant subbimodule.

(ii) The first part follows from (i) and the fact that $(\hat{P}^+ + \hat{P}^- + \hat{P}^0)_{12}$ $(\check{P}^+ + \check{P}^- + \check{P}^0)_{34}$ is the identity of $(\mathbb{C}^N)^{\otimes 4}$. In addition $\hat{P}^\tau$ and $\hat{P}^\nu$ as well as $\check{P}^\tau$

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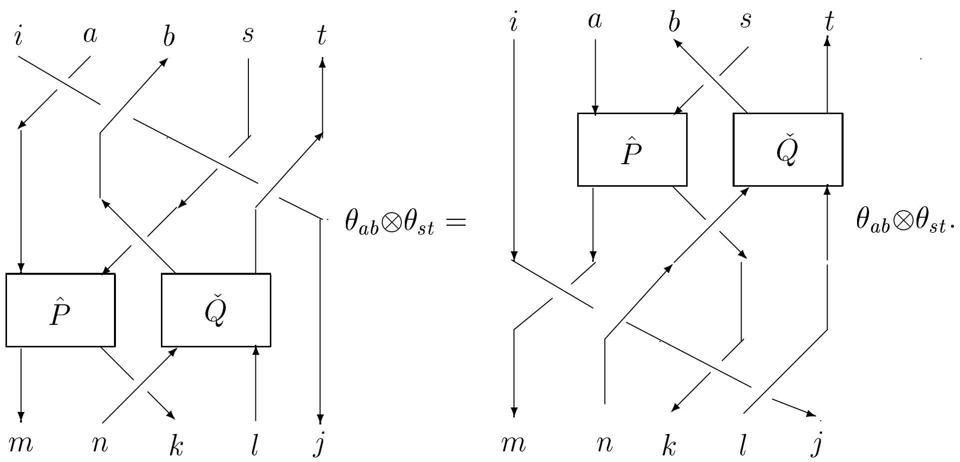

Figure 3. The linear span $\langle\theta^{PQ}\rangle$ is closed under the right adjoint action.

and $\check P^{\nu}$ are pairwise orthogonal idempotents, respectively. Hence the sum is direct. To the second part. Let λ_τ denote the eigenvalue of $\hat R$ with respect to the idempotent $\hat P^{\tau}$, $\hat R\hat P^{\tau} = \lambda_\tau \hat P^{\tau}$, namely $\lambda_+ = q$, $\lambda_- = -q^{-1}$, and $\lambda_0 = r^{-1}$. Note that $\check R\check P^{\tau} = \lambda_\tau \check P^{\tau}$ as well. Put $\rho = \grave R^{-}_{23}\hat P^{\tau}_{12}\check P^{\nu}_{34}\acute R_{23}(\theta_{ijkl})$. Then by (11)

$$\begin{aligned}\sigma(\rho) &= \grave R^{-}_{23}\hat R_{12}\check R^{-1}_{34}\acute R_{23}\cdot \grave R^{-}_{23}\hat P^{\tau}_{12}\check P^{\nu}_{34}\acute R_{23}(\theta_{ijkl})\\ &= \lambda_\tau\lambda_\nu^{-1}\grave R^{-}_{23}\hat P^{\tau}_{12}\check P^{\nu}_{34}\acute R_{23}(\theta_{ijkl})\\ &= \lambda_\tau\lambda_\nu^{-1}\rho.\end{aligned}$$

Since q is not a root of unity, $\lambda_\tau \neq \lambda_\nu$ for $\tau\neq\nu$. Hence $\rho\in(\ker A_2)_L$ if and only if $\tau=\nu$. We thus get $(\ker A_2)_L = \Lambda_L^{++}\oplus\Lambda_L^{--}\oplus\Lambda_L^{00}$ as linear spaces. By (i) each space on the right hand side generates a bicovariant subbimodule. This completes the proof.

To simplify notations we choose a new basis of $(\Gamma\otimes_{\mathcal A}\Gamma)_L$

$$\overline{\theta}_{vwst} = \grave R^{-yz}_{ws}\theta_{vy}\otimes\theta_{zt},\qquad \theta_{vwst} = \acute R^{yz}_{ws}\overline{\theta}_{vyzt}. \tag{14}$$

The right coaction now reads $u\otimes u\otimes u^{c}\otimes u^{c}$ and the braiding in the new basis is $\overline{\sigma} = \hat R_{12}\check R^{-1}_{34}$. We simply write Λ^{τ} instead of $\Lambda^{\tau\tau}$, $\tau\in\{+,-,0\}$. Since the corresponding $\hat P^{\tau}$ subcorepresentation of $u\otimes u$ is irreducible, by Schur's lemma Λ^{τ} has a *unique* up to scalars bi-invariant element η^{τ}, see Figure 4.

The relations with the old basis of $\Lambda_I^{\otimes 2}$ are $\eta = r(\eta^0+\eta^++\eta^-)$, $\xi = x\eta^0$, and

$$\theta\otimes\theta = q\eta^+ - q^{-1}\eta^- + r^{-1}\eta^0. \tag{15}$$

The next lemma is the key step in our proof.

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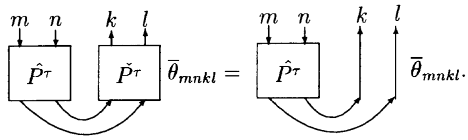
$\eta^\tau = (\hat{P}^\tau)^{mn}_{lk}\overline{\theta}_{mnkl} =$ 

Figure 4. The bi-invariant elements η^+, η^-, and η^0.

LEMMA 7. *Let Γ be one of the bicovariant FODC $\Gamma_\pm$ over $\mathcal{A}$ and Λ^τ, $\tau \in \{+,-,0\}$, the above defined bicovariant subbimodule of* $\ker A_2$. *Then Λ^τ is generated by the single element η^τ. More precisely, $\eta^\tau \triangleleft \mathcal{A} = \Lambda^\tau_L$.*

Proof. By Lemma 2 the canonical left-invariant basis of Λ^τ is

$$\theta^\tau_{mnkl} = \grave{R}^-_{23}\hat{P}_{12}\check{Q}_{34}\acute{R}_{23}(\theta_{mn}\otimes\theta_{kl}) = \grave{R}^{-ws}_{yz}(\hat{P}^\tau)^{vy}_{ma}(\check{P}^\tau)^{zt}_{bl}\acute{R}^{ab}_{nk}\theta_{vw}\otimes\theta_{st}.$$

The proof is in two steps. First we compute $\eta^\tau \triangleleft u^i_j$ and obtain elements

$$\eta^\tau_{ij} = B^i_x(\hat{P}^\tau)^{mn}_{xk}(\check{P}^\tau)^{vw}_{ky}C^j_y\overline{\theta}_{mnvw},$$
$$\xi^\tau_{ij} = (\hat{P}^\tau)^{mn}_{yk}(\check{P}^\tau)^{vw}_{kz}\acute{R}^{yz}_{ij}\overline{\theta}_{mnvw}.$$

The graphical presentation of η^τ_{ij} and ξ^τ_{ij} is as in Figure 5.

First we will show that

$$\eta^\tau \triangleleft \widetilde{u}^i_j = \hat{q}(\lambda_\tau^2+1)\xi^\tau_{ij} - r^{-1}\hat{q}(1+\lambda_\tau^{-2})\eta^\tau_{ij}, \tag{16}$$

$$\eta^\tau \triangleleft \widetilde{U} = \alpha_\tau\eta^\tau, \quad \alpha_\tau\hat{q}(\lambda_\tau+\lambda_\tau^{-1})(\lambda_\tau r - \lambda_\tau^{-1}r^{-1}). \tag{17}$$

By (14) and (10) one has

$$\begin{aligned}\overline{\theta}_{mnkl} \triangleleft u^i_j &= \grave{R}^{-ab}_{nk}(\theta_{ma} \triangleleft u^i_c)(\theta_{bl} \triangleleft u^c_j)\\ &= \acute{R}^{-ab}_{nk}\hat{R}^{iv}_{my}\acute{R}^{ay}_{cd}\hat{R}^{cp}_{bz}\acute{R}^{lz}_{jt}\theta_{vd}\otimes\theta_{pt}\\ &= (\acute{R}_{34}\hat{R}_{12}\acute{R}_{23}\hat{R}_{34}\acute{R}_{45}\grave{R}^-_{23})^{ivwst}_{mnklj}\overline{\theta}_{vwst}.\end{aligned} \tag{18}$$

The graphical presentation of (18) is Figure 6. Now we explain Figure 7.

In the first step we replaced the crossing in the dash box using (7). In the second step we did the same with the $\hat{R}$-matrix in the first dash box. This gives the first three terms in the next line. Moreover the dash box in the second summand is multiplied by $\check{P}^\tau$ and gives $\lambda_\tau I$ (no crossing). Similarly, a second crossing in the same term gives another λ_τ. With the third summand we are dealing in the same way; in addition the curl gives the factor r^{-1}. Since $\widetilde{u}^i_j = u^i_j - \delta_{ij}1$, (16) follows immediately.

Note that for $\tau = 0$

$$\eta^0 \triangleleft u^i_j = \delta_{ij}\eta^0 \tag{19}$$

is obvious from the first line in Figure 7 since $\hat{P}^0 = x^{-1}K$ and no crossing appears

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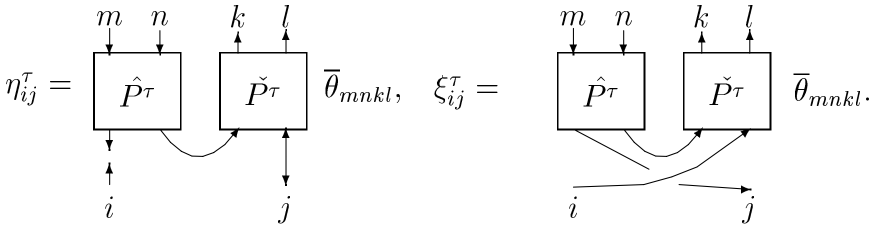

Figure 5. The elements η_{ij}^{τ} and ξ_{ij}^{τ}.

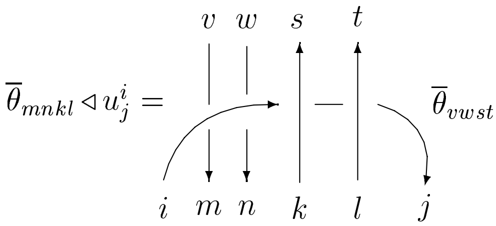

Figure 6. The right adjoint action of u_j^i on $\overline{\theta}_{mnkl}$.

there. Moreover $\eta_{ij}^0 = \delta_{ij}\eta^0$ and $\xi_{ij}^0 = r\delta_{ij}\eta^0$ and (16) and (17) are valid. Since Λ^0 is one-dimensional there is nothing to prove. Now we fix $\tau \in \{+, -\}$. We shall eliminate η_{ij}^{τ} from (16). Multiplying (16) by D_i^j and using $D_i^j\eta_{ij}^{\tau} = \eta^{\tau}$, $D_i^j\xi_{ij}^{\tau} = r\eta^{\tau}$ gives (17). Since q is not a root of unity, $T_{\tau} = \hat{q}(\lambda_{\tau}^2 + 1)\hat{R}^{-1} - r^{-1}q(1 + \lambda_{\tau}^{-2})$ is invertible with inverse

$$
\begin{aligned}
T_+^{-1} &= \frac{1}{\hat{q}[2]_q}\left(\frac{1}{1 - q^{-1}r^{-1}}\hat{P}^+ + \frac{1}{-q^2 - q^{-1}r^{-1}}\hat{P}^- + \frac{1}{rq - r^{-1}q^{-1}}\hat{P}^0\right),\\
T_-^{-1} &= \frac{1}{\hat{q}[2]_q}\left(\frac{1}{q^{-2} - qr^{-1}}\hat{P}^+ + \frac{1}{-1 - qr^{-1}}\hat{P}^- + \frac{1}{q^{-1}r - qr^{-1}}\hat{P}^0\right).
\end{aligned}
$$

Set $(S_{\tau})_{st}^{ij} = B_i^y(T_{\tau}^{-1})_{zt}^{yj}C_z^s$ and multiply (16) by $(S_{\tau})_{st}^{ij}$. Then we obtain $(S_{\tau})_{st}^{ij}\eta^{\tau} \triangleleft \widetilde{u}_j^i = \eta_{st}^{\tau}$. Consequently, $\eta_{st}^{\tau} \in \Lambda^{\tau}$ for $s, t = 1, \dots, N$ and $\tau \in \{+, -\}$.

In the second step we again compute the right adjoint action of u_j^i but on elements η_{st}^{τ}. We obtain elements

$$
\begin{aligned}
\xi_{sijt}^{\tau} &= (\hat{P}^{\tau})_{sy}^{mn}(\check{P}^{\tau})_{zt}^{vw}\acute{R}_{ij}^{yz}\overline{\theta}_{mnvw},\\
\eta_{sijt}^{\tau} &= B_y^sB_z^i(\hat{P}^{\tau})_{yz}^{mn}(\check{P}^{\tau})_{dc}^{vw}C_d^jC_c^t\overline{\theta}_{mnvw}.
\end{aligned}
$$

Obviously $\xi_{sijt}^{\tau} = \eta_{syzt}^{\tau}\acute{R}_{ij}^{yz}$. Graphically they are represented in Figure 8.

From (18), we obtain Figure 9.

Replacing one crossing $\hat{R}$ by $\hat{R}^{-1} + \hat{q}I - \hat{q}K$ similarly to the graphical calculations in the first part of the proof one can show that

$$
\zeta_{sijt}^{\tau} := \eta_{uv}^{\tau} \triangleleft u_b^a\check{R}_{si}^{au}\acute{R}_{jt}^{vb} - \delta_{ij}\eta_{st}^{\tau} = \hat{q}\xi_{sijt}^{\tau} - \hat{q}r^{-1}\eta_{sijt}^{\tau}.
$$

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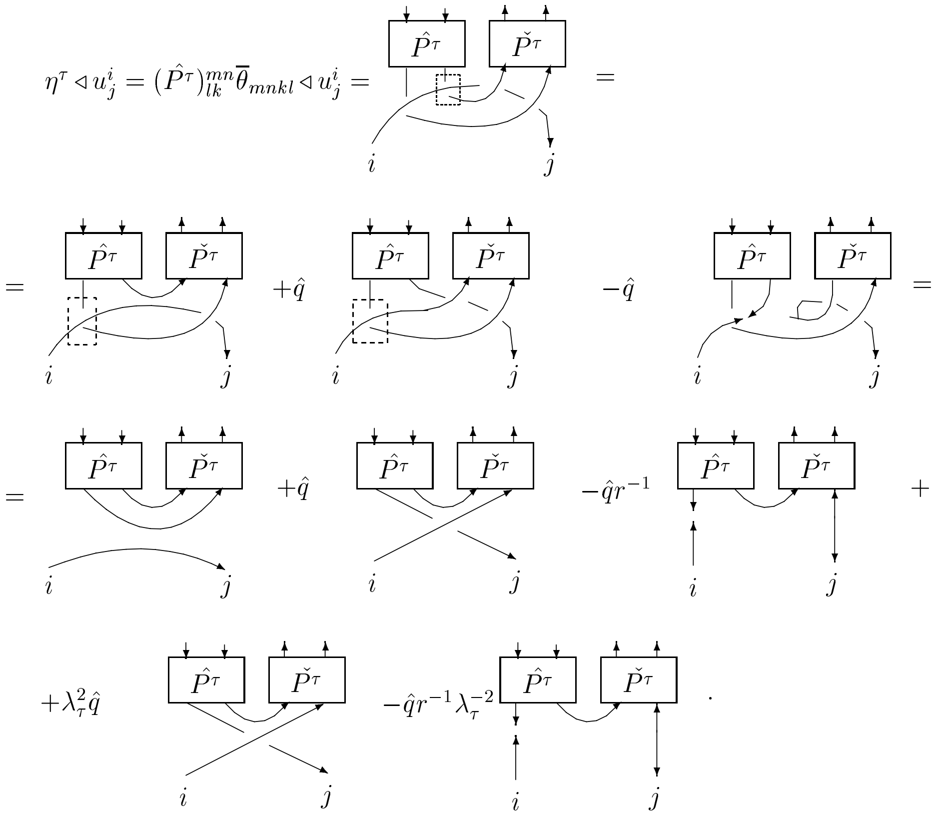

Figure 7. The proof of (16).

Since q is not a root of unity, $T = \hat{q}(\hat{R}^{-1} - r^{-1}I)$ is invertible with inverse $T^{-1} = \hat{q}^{-1}((q^{-1} - r^{-1})^{-1}\hat{P}^{+} - (q + r^{-1})^{-1}\hat{P}^{-})$. Therefore

$$B^y_i (T^{-1})^{yj}_{zl} C^k_z \zeta^\tau_{sijt} = \eta^\tau_{sklt}$$

belongs to Λ^τ_L. Finally we have $B^m_v B^a_y C^b_z C^l_t \eta^\tau_{vyzt} \acute{R}^{ab}_{nk} = \theta^\tau_{mnkl}$ which completes the proof.

Now we are ready to complete the proof of the theorem. By Lemma 5 (iii) both η^{+} and η^{-} belong to $\mathcal{S}(\mathcal{R})$ (see Section 5) and $\dim_u \Gamma^{\wedge 2}_I = 1$. Hence $\eta^0 \notin \mathcal{S}(\mathcal{R})_0$. Since $\theta \otimes \theta \equiv r^{-1}\eta^0 \bmod \mathcal{S}(\mathcal{R})$ by (15) and $a\eta^0 = \eta^0 a$, $a \in \mathcal{A}$, by (19) we get

$$a\theta \wedge \theta = \theta \wedge \theta a, \quad a \in \mathcal{A}. \tag{20}$$

We prove (12) by induction over the degree n of $\rho \in {}_u\Gamma^{\wedge n}$. For $n = 0$ it is true by the definition of the FODC. Suppose it is true for $n-1$. Since there exist $\alpha_i \in {}_u\Gamma^{\wedge n-1}$ and

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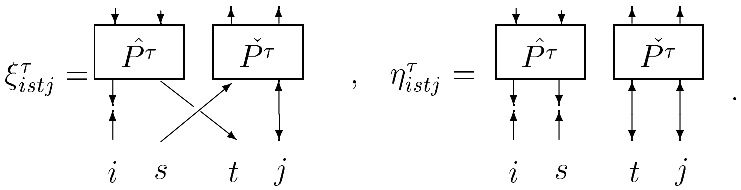

Figure 8. The elements ξ^{τ}_{istj} and η^{τ}_{istj}.

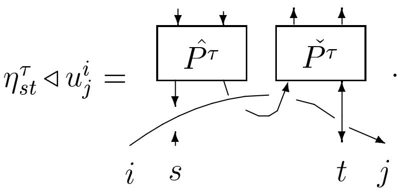

Figure 9. The right adjoint action of u^i_j on η^{τ}_{st}.

$b_i \in \mathcal{A}$ such that $\rho = \alpha_i \mathrm{d} b_i$, we obtain by induction assumption and by (20)

$$\begin{aligned}
\mathrm{d}\rho = \mathrm{d}\alpha_i \mathrm{d} b_i &= \theta\alpha_i \mathrm{d} b_i - (-1)^{n-1}\alpha_i\theta(\theta b_i - b_i\theta)\\
&= \theta\rho - (-1)^n(\alpha_i\theta b_i\theta - \alpha_i b_i\theta^2)\\
&= \theta\rho - (-1)^n\alpha_i \mathrm{d} b_i\theta\\
&= \theta\rho - (-1)^n\rho\theta.
\end{aligned}$$

Using $\mathrm{d}^2\rho = 0$ and (12) twice gives $\theta^2\rho = \rho\theta^2$, and θ^2 is central in ${}_u\Gamma^{\wedge 2}$. This completes the proof of (ii).

By Lemma 5 (iii) and Lemma 7

$$\mathcal{S}(\mathcal{R}) \supseteq \mathcal{S}(\mathcal{R})_0 \triangleleft \mathcal{A} = \Lambda^+_L \oplus \Lambda^-_L.$$

Since $\mathcal{S}(\mathcal{R}) \subseteq \ker A_2$ by universality of ${}_u\Gamma^{\wedge}$ and $\eta^0 \in \ker A_2$, we conclude with Lemma 6 (ii) that the above inclusion is not strict,

$$\mathcal{S}(\mathcal{R}) = \Lambda^+_L \oplus \Lambda^-_L \quad \text{and} \quad (\ker A_2)_L = \mathcal{S}(\mathcal{R}) \oplus \Lambda^0_L.$$

Since both ${}_u\Gamma^{\wedge}_L$ and ${}_s\Gamma^{\wedge}_L$ are quadratic algebras,

$${}_u\Gamma^{\wedge}_L/(\eta^0) \cong \Gamma^{\otimes}_L/(\mathcal{S}(\mathcal{R}) \oplus \langle\eta^0\rangle) \cong \Gamma^{\otimes}_L/(\ker A_2)_L \cong {}_s\,\Gamma^{\wedge}_L.$$

Since both ${}_u\Gamma^{\wedge}_L/(\eta^0)$ and ${}_s\Gamma^{\wedge}$ are *free* left $\mathcal{A}$-modules it follows ${}_u\Gamma^{\wedge}/(\eta^0) \cong {}_s\,\Gamma^{\wedge}$. Noting that $\theta^2 = r^{-1}\eta^0$ in ${}_u\Gamma^{\wedge}$ completes the proof of the theorem.

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5. The Bi-invariant 2-form of the Universal Differential Calculus

In this section we will complete the proof of Lemma 5 (iii) and show that both bi-invariant elements η^+ and η^- belong to $\mathcal{S}(\mathcal{R})$. We give different proofs for the cases Γ_+ and Γ_-. The first proof for Γ_+ is self-contained and much easier than the second one. In the later one we take results from [8] and make use of a computer algebra program to simplify long terms. For q *transcendental* however the first proof works for Γ_- too.

We recall some identities which are easily proved using (3) and the Leibniz rule. Equations (22), [6, formula (14.3)], and (24), [6, Lemma 14.15], are valid for arbitrary left-covariant FODC while (21) and (23) in addition require $\mathrm{d}a = \theta a - a\theta$. For $a, b \in \mathcal{A}$ and $p \in \mathcal{R}$ we have

$$\omega(a) = \theta \triangleleft a + \varepsilon(a)\theta, \qquad \theta \triangleleft p = 0, \tag{21}$$

$$\omega(ab) = \omega(a) \triangleleft b + \varepsilon(a)\omega(b), \tag{22}$$

$$\begin{aligned} &\mathcal{S}(a) = (\theta\otimes\theta) \triangleleft a - \theta \otimes (\theta \triangleleft a) - (\theta \triangleleft a) \otimes \theta + \varepsilon(a)\theta \otimes \theta, \\ &\mathcal{S}(p) = (\theta\otimes\theta) \triangleleft p, \end{aligned} \tag{23}$$

$$\begin{aligned} &\mathcal{S}(\widetilde{a}b) = \mathcal{S}(a) \triangleleft b + \omega(a) \triangleleft b_{(1)} \otimes \omega(b_{(2)}) + \omega(b_{(1)}) \otimes (\omega(a) \triangleleft b_{(2)}), \\ &\mathcal{S}(pb) = \mathcal{S}(p) \triangleleft b. \end{aligned} \tag{24}$$

We abbreviate $\hat{r} = r - r^{-1}$. In what follows we do *not sum* over signs τ and ν.

Part 1. $\Gamma = \Gamma_+$. First we show $Q := \widetilde{U} \cdot \widetilde{U} - \hat{q}\hat{r}\widetilde{U} \in \mathcal{R}$. By (10) and (9) we obtain $\theta \triangleleft U = \mathrm{tr}^1_q(\hat{R}^2)^m_n \theta_{mn} = (\hat{q}\hat{r} + x)\theta$. Using (21), (22), and $\varepsilon(U) = x$ we have $\omega(Q) = (\theta \triangleleft \widetilde{U}) \triangleleft \widetilde{U} - \hat{q}\hat{r}\theta \triangleleft \widetilde{U} = 0$. In addition $\varepsilon(Q) = 0$; hence $Q \in \mathcal{R}$. Next we compute $\mathcal{S}(Q)$. Since $Q \in \mathcal{R}$, by (23) we have $\mathcal{S}(Q) = (\theta \otimes \theta) \triangleleft Q$. Using (15) and (17) we get

$$\begin{aligned} \mathcal{S}(Q) = (\theta\otimes\theta) \triangleleft Q &= (q\eta^+ - q^{-1}\eta^-) \triangleleft (\widetilde{U}\widetilde{U} - \hat{q}\hat{r}\widetilde{U}) \\ &= q\alpha_+^2\eta^+ - q^{-1}\alpha_-^2\eta^- - \hat{q}\hat{r}(q\alpha_+\eta^+ - q^{-1}\alpha_-\eta^-) \\ &= q\alpha_+(\alpha_+ - \hat{q}\hat{r})\eta^+ - q^{-1}\alpha_-(\alpha_- - \hat{q}\hat{r})\eta^-. \end{aligned} \tag{25}$$

Since $\mathcal{R}$ is a right ideal $Q\widetilde{U} \in \mathcal{R}$. By (24) and (25)

$$\mathcal{S}(Q\widetilde{U}) = q\alpha_+^2(\alpha_+ - \hat{q}\hat{r})\eta^+ - q^{-1}\alpha_-^2(\alpha_- - \hat{q}\hat{r})\eta^-. \tag{26}$$

Solving this linear system (25) and (26) in η^+ and η^- we have to consider its coefficient determinant

$$\begin{aligned} \det &= \alpha_+\alpha_-(\alpha_+ - \alpha_-)(\alpha_+ - \hat{q}\hat{r})(\alpha_- - \hat{q}\hat{r}) \\ &= (r + r^{-1})\hat{q}^6[2]_q^3(qr - q^{-1}r^{-1})(q^{-1}r - qr^{-1}) \times \\ &\quad \times (q^2r - q^{-2}r^{-1})(q^{-2}r - q^2r^{-1}). \end{aligned}$$

Since q is not a root of unity, $\det \neq 0$. Hence both η^+ and η^- belong to $\mathcal{S}(\mathcal{R})$.

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Part 2. $\Gamma = \Gamma_-$. We denote the critical value by c, $c = \hat{q}q^2r(2x + \hat{q}\hat{r}) = (q^4+1)r^2 + 2q(q^2-1)r - (q^4+1)$. We recall some of the defining constants for Γ_- from [8, p. 656].

$$\begin{aligned}
\mu^+ &= \frac{\hat{r}(-q^2r + q^{-2}r^{-1} - \hat{q})}{\hat{q}\hat{r} + 2x},\\
\mu^- &= \frac{\hat{r}(-q^{-2}r + q^2r^{-1} - \hat{q})}{\hat{q}\hat{r} + 2x},\\
\mu &:= \mu^+ + \mu^- - 2x = -\hat{q}\hat{r} - 2x,
\end{aligned}$$

The idempotents $\hat{P}^\nu$, $\nu \in \{+,-\}$, and their q-traces are as follows

$$\hat{P}^\nu = (\lambda_\nu + \lambda_\nu^{-1})^{-1}(\lambda_\nu^{-1}I + \hat{R} + \hat{q}(1 - r\lambda_\nu)^{-1}K), \tag{27}$$

$$\mathrm{tr}_q^1(\hat{P}^\nu) = \frac{\hat{r}(\lambda_\nu^2 r - \lambda_\nu^{-1})}{\hat{q}(\lambda_\nu + \lambda_\nu^{-1})(\lambda_\nu r - 1)} I =: t_\nu I. \tag{28}$$

There are two Ad_R-invariant quadratic elements in $\mathcal{A}$, namely $V_\nu = D_a^b D_i^j (\hat{P}^\nu)^{ai}_{yz} u_b^y u_j^z$. One has $\varepsilon(V_\nu) = xt_\nu$. Note that $\widetilde{W_\nu} \in \mathcal{R}$, where $W_\nu = V_\nu - \mu^\nu U$, [8, p. 656 eq. (3)]. Suppose $a \in \mathcal{A}$ is Ad_R-invariant and $\rho \in \Lambda_I$, where Λ is a bicovariant bimodule. Then one has $\rho \triangleleft a \in \Lambda_I$. Namely,

$$\Delta_R(\rho \triangleleft a) = (Sa_{(2)} \otimes Sa_{(1)})(\rho \otimes 1)(a_{(3)} \otimes a_{(4)}) = \rho \triangleleft a_{(2)} \otimes Sa_{(1)}a_{(3)} = \rho \triangleleft a \otimes 1.$$

Applying this fact to Λ^τ and V_ν, and noting that , $\tau \in \{+,-\}$, is the only bi-invariant element of Λ^τ (up to scalars), there exist complex numbers $c_{\tau\nu}$, $\tau, \nu \in \{+,-\}$, defined by $\eta^\tau \triangleleft \widetilde{V_\nu} = c_{\tau\nu}\eta^\tau$. We shall determine these constants. By the definition of V_ν and (16)

$$\begin{aligned}
\eta^\tau \triangleleft V_\nu &= \eta^\tau \triangleleft u_j^i u_t^s (\hat{P}^\nu)^{jt}_{ab} D_i^a D_s^b\\
&= \left(\delta_{ij}\eta^\tau + \hat{q}(\lambda_\tau^2 + 1)\xi_{ij}^\tau - r^{-1}\hat{q}(1 + \lambda_\tau^{-2})\eta_{ij}^\tau\right) \triangleleft u_t^s (\hat{P}^\tau)^{jt}_{ab} D_i^a D_s^b.
\end{aligned}$$

We carry out the calculations for the first term. By (16) and (28) we have

$$\delta_{ij}\,\eta^\tau \triangleleft (u_t^s (\hat{P}^\tau)^{jt}_{ab} D_i^a D_s^b) = t_\nu \eta^\tau \triangleleft U = t_\nu(x + \alpha_\tau)\eta^\tau.$$

Using graphical calculations we obtain for the other two terms

$$\begin{aligned}
\xi_{ij}^\tau \triangleleft (u_t^s (\hat{P}^\tau)^{jt}_{ab} D_i^a D_s^b) &= (\hat{q}\delta_{\tau,\nu} r^2 \lambda_\tau^2 + r\lambda_\nu^2 t_\nu - \hat{q}\lambda_\nu \lambda_\tau^{-1} e_{\tau\nu})\eta^\tau\\
\eta_{ij}^\tau \triangleleft (u_t^s (\hat{P}^\tau)^{jt}_{ab} D_i^a D_s^b) &= (\hat{q}r\lambda_\nu^{-1}\lambda_\tau e_{\tau\nu} + \lambda_\nu^{-2} t_\nu - \hat{q}r^{-1}\lambda_\nu^{-2}\delta_{\tau,\nu})\eta^\tau,
\end{aligned}$$

where

$$e_{\tau\nu} = (\lambda_\nu + \lambda_\nu^{-1})^{-1}(\lambda_\tau^{-1} + \hat{q}(1 - r\lambda_\nu)^{-1})$$

is obtained from (27) and the picture in Figure 10

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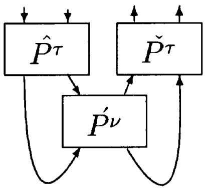

$$= e_{\tau\nu}\eta^{\tau}.$$

Figure 10. The definition of $e_{\tau\nu}$.

Finally we obtain

$$\begin{aligned}
c_{++} &= c^{-1}(q^2+1)(q^2r^2-1)r^{-2}q^{-6}\big((q^{12}+q^4)r^4+\\
&\quad+(2q^{11}-2q^9+2q^5-2)r^3+(-q^{12}+q^8-4q^6+q^4-1)r^2+\\
&\quad+(-2q^9+2q^7-2q^3+2q)r+q^8+1\big)\\
c_{+-} &= c^{-1}[2]_q 2(q-q^{-1}r^{-2})(r+q^3)(r-q)\big((q^2-1+q^{-2})r^2+\\
&\quad+\hat{q}r-q^2+1-q^{-2}\big)\\
c_{-+} &= c^{-1}[2]_q(2(q^{-1}-qr^{-2})(qr+1)(q^3r-1)\big((q^2-1+q^{-2})r^2+\\
&\quad+\hat{q}r-q^2+1-q^{-2}\big)\\
c_{--} &= c^{-1}(q^2+1)(r^2-q^2)r^{-2}q^{-6}\big((q^8+1)r^4+\\
&\quad+(2q^9-2q^7+2q^3-2q)r^3+(-q^{12}+q^8-4q^6+q^4-1)r^2+\\
&\quad+(-2q^{11}+2q^9-2q^5+2q^3)r+q^{12}+q^4\big)
\end{aligned}$$

Now we are able to compute four bi-invariant elements of $\mathscr{S}(\mathcal{R})$. Using (15) and (17) we have

$$\begin{aligned}
\mathscr{S}(W_v) &= (\theta\otimes\theta)\triangleleft(\widetilde{V}_v-\mu^v\widetilde{U})\\
&= q(c_{+v}-\mu^v\alpha_+)\eta^+-q^{-1}(c_{-v}-\mu^v\alpha_-)\eta^-.
\end{aligned}\tag{29}$$

Similarly to Part 1 we get

$$\mathscr{S}(W_v\widetilde{U}) = q\alpha_+(c_{+v}-\mu^v\alpha_+)\eta^+-q^{-1}\alpha_-(c_{-v}-\mu^v\alpha_-)\eta^-.\tag{30}$$

Consider the 4×2–coefficient matrix $T=(T_{ij})$ for the linear system of Equations (29) and (30), $v\in\{+,-\}$. The two columns are

$$\begin{aligned}
&q\big(c_{++}-\mu^+\alpha_+,\, c_{+-}-\mu^-\alpha_+,\, \alpha_+(c_{++}-\mu^+\alpha_+),\, \alpha_+(c_{+-}-\mu^-\alpha_+)\big)^{\top},\\
&-q^{-1}(c_{-+}-\mu^+\alpha_-,\, c_{--}-\mu^-\alpha_-,\, \alpha_-(c_{-+}-\mu^+\alpha_-),\, \alpha_-(c_{--}-\mu^-\alpha_-)\big)^{\top}.
\end{aligned}$$

We distinguish three cases: The first column is zero, the second column is zero, and no column vanishes, respectively.

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Case 1. $c_{++} - \mu^+\alpha_+ = c_{+-} - \mu^-\alpha_+ = 0$. We obtain the following two equations

$$
\begin{aligned}
0 &= (q^2+1)(qr-1)(qr+1)\times \\
&\quad \times (q^{12}r^4 + q^4r^4 + 2q^{11}r^3 - 2q^9r^3 + 2q^5r^3 - 2q^3r^3 - q^{12}r^2 + \\
&\quad + q^8r^2 - 4q^6r^2 + q^4r^2 - r^2 - 2q^9r + 2q^7r - 2q^3r + 2qr + q^8 + 1), \\
0 &= 2(qr-1)(qr+1)(r+q^3)(r-q)\times \\
&\quad \times (q^4r^2 - q^2r^2 + r^2 + q^3r - qr - q^4 + q^2 - 1)(q^2+1).
\end{aligned}
$$

Since q is not a root of unity we have

$$
\begin{aligned}
d_1 &= q^{12}r^4 + q^4r^4 + 2q^{11}r^3 - 2q^9r^3 + 2q^5r^3 - 2q^3r^3 - q^{12}r^2 + \\
&\quad + q^8r^2 - 4q^6r^2 + q^4r^2 - r^2 - 2q^9r + 2q^7r - 2q^3r + 2qr + q^8 + 1 = 0, \\
d_2 &= q^4r^2 - q^2r^2 + r^2 + q^3r - qr - q^4 + q^2 - 1 = 0.
\end{aligned}
$$

Using the Euclidean algorithm we eliminate powers of r. We end up with polynomials

$$
\begin{aligned}
a &= (q^4+1)(q^6-q^2+1)(q^8+1)q^4r^3 + (q^4-q^2+1)(3q^{14} - 2q^{10} + \\
&\quad + 2q^8 - q^6 + 2q^2 - 2)q^3r^2 + (q^{20} - 2q^{18} + 2q^{14} - 4q^{12} - q^{10} + \\
&\quad + 5q^8 - 6q^6 + q^4 + q^2 - 1)r + (q^4-q^2+1)(q^{16} - 4q^{14} + 2q^{12} + \\
&\quad + 2q^{10} - 5q^8 + 2q^6 + 2q^4 - 4q^2 + 2)q
\end{aligned}
$$

and

$$
\begin{aligned}
b &= (-q^4-1)(q^6-q^2+1)(q^4-q^2+1)^4(q-1)^6(q+1)^6qr + (-2q^{12} + \\
&\quad + 3q^{10} - 3q^8 + q^6 + q^4 - 2q^2 + 1)(q^4-q^2+1)^3(q-1)^6(q+1)^6q^2
\end{aligned}
$$

such that $ad_2 + bd_1 = -(q^6+q^3+1)(q^6-q^3+1)(q-1)^6(q+1)^6q$ (there is no r left). Since $d_1 = d_2 = 0$, q is a root of unity which contradicts our assumption. Hence Case 1 is impossible.

Case 2. $c_{-+} - \mu^+\alpha_- = c_{--} - \mu^-\alpha_- = 0$. Similarly to Case 1 we have

$$
\begin{aligned}
d_3 &= q^4r^2 - q^2r^2 + r^2 + q^3r - qr - q^4 + q^2 - 1 = 0, \\
d_4 &= q^8r^4 + r^4 + 2q^9r^3 - 2q^7r^3 + 2q^3r^3 - 2qr^3 - q^{12}r^2 + q^8r^2 - 4q^6r^2 + \\
&\quad + q^4r^2 - r^2 - 2q^{11}r + 2q^9r - 2q^5r + 2q^3r + q^{12} + q^4 = 0.
\end{aligned}
$$

Again there exist polynomials a and b in q and r such that $ad_3 + bd_4 = (q^6+q^3+1)(q^6-q^3+1)(q^4-q^2+1)(q-1)^{12}(q+1)^{12}q^2$. This contradicts our assumption that q is not a root of unity. Hence, the only possibility is

Case 3. We will show, that T has rank 2. Suppose to the contrary that T has at least rank 1. Then the 2×2-matrices built from the first and third rows, respectively, from the second and fourth rows, both have zero determinant. Since $\alpha_+ - \alpha_- \neq 0$ this is equivalent to $(c_{++} - \mu^+\alpha_+)(c_{-+} - \mu^+\alpha_-) = 0$ and $(c_{+-} - \mu^-\alpha_+)(c_{--} - \mu^-\alpha_-) = 0$.

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Since, moreover, the matrix built from the first two rows has zero determinant, we conclude $c_{++} - \mu^+\alpha_+ = c_{+-} - \mu^-\alpha_+ = 0$ or $c_{--} - \mu^-\alpha_- = 0$. But this is impossible by Cases 1 and 2. Hence, T has rank 2; both η^+ and η^- belong to $\mathscr{S}(\mathcal{R})$.

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