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The entropy of polynomial diffeomorphisms of $\mathbf{C}^2$

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In this note we answer a question raised by Friedland and Milnor in [**FM**] concerning the topological entropy of polynomial diffeomorphisms of $\mathbf{C}^2$

Friedland and Milnor prove that a polynomial diffeomorphism is conjugate to a diffeomorphism of one of three types affine, elementary or cyclically reduced The first two families of maps are very simple from a dynamical point of view The third family contains diffeomorphisms which are dynamically very interesting The Hénon map is an example of a cyclically reduced diffeomorphism of degree 2

Topological entropy is most naturally defined for maps of compact spaces Since $\mathbf{C}^2$ is not compact Friedland and Milnor consider the map g, the extension of g to the one point compactification of $\mathbf{C}^2$ They prove that if g is a cyclically reduced diffeomorphism of (algebraic) degree d then the inequality $h(g) \leq \log d$ holds They raise the question of whether the inequality can be replaced by an equality We show that it can

THEOREM *If g is cyclically reduced then $h(g) = \log d$*

The Hénon map has been intensively studied as a map from $\mathbf{R}^2$ to itself and yet many important problems remain In particular the dependence of the entropy of g on the parameter values determining g is quite mysterious The above result suggests that the dynamics of the Hénon map when considered as a diffeomorphism of $\mathbf{C}^2$ may be simpler than when considered as a diffeomorphism of $\mathbf{R}^2$

Let $\mathrm{Per}_n(g)$ be the set of periodic points of period n Let

$$H(g) = \limsup_{n\to\infty} \frac{1}{n} \log^+ |\mathrm{Per}_n(g)|$$

COROLLARY $H(g) = \log d$

Proof of Corollary This follows by combining the above theorem with the result of [**FM**] that $h(g) \leq H(g) \leq \log d$

Friedland and Milnor show that every cyclically reduced polynomial diffeomorphism is conjugate to a composition of generalized Hénon maps of the form $g(x, y) = (y, p(y) - \delta x)$ where p is a polynomial and δ is a nonzero complex number The degree of g is the degree of p The degree of a composition of generalized Hénon

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maps is the product of the degrees of the factors We begin by proving some basic facts about these maps The proof of the theorem follows A version of Lemma 2 first appears in [DN]

LEMMA 1 (see [FM] Lemma 3 4) *For every generalized Hénon map* g $(x, y) \mapsto (y, z) = (y, p(y) - \delta x)$ *there exists a constant* κ *so that* $|y| > \kappa$ *implies that either* $|z| > |y|$ *or* $|x| > |y|$ *or both*

We fix the following notation Let $g = g_1 \circ g_2 \circ \quad g_n$ be a composition of generalized Hénon maps Let d be the degree of g Choose κ large enough so that Lemma 1 holds for each g_i Let

$$V^- = \{(x, y) \quad |y| > \kappa \text{ and } |y| > |x|\}$$
$$V^+ = \{(x, y) \quad |x| > \kappa \text{ and } |x| > |y|\}$$
$$V = \{(x, y) \quad |x| \le \kappa \text{ and } |y| \le \kappa\}$$

LEMMA 2
(1) $g(V^-) \subset V^-$
(2) $g(V^- \cup V) \subset V^- \cup V$
(3) $g^{-1}(V^+) \subset V^+$
(4) $g^{-1}(V^+ \cup V) \subset V^+ \cup V$

Proof It suffices to prove each assertion when g $(x, y) \mapsto (y, z)$ is itself a generalized Hénon map

(1) Let (x, y) be an element of V^- then $|y| > \kappa$ and $|y| > |x|$ By Lemma 1 $|z| > |y|$ and, since $|y| > \kappa$, we conclude that $|z| > \kappa$ This implies that $g(x, y) = (y, z)$ is in V^-
(2) By (1) it suffices to consider the case when (x, y) is an element of V We will show that $g(x, y) = (y, z)$ is in $V \cup V^-$ Consider two case If $|z| \le \kappa$ then, since $|y| \le \kappa$, (y, z) is in V If $|z| > \kappa$ then, since $|y| < \kappa$, we conclude that $|z| > |y|$ so (y, z) is in V^-
(3) Let (y, z) be an element of V^+ we want to show that $g^{-1}(y, z) = (x, y)$ is in V^+ Since $|y| > \kappa$ and $|y| > |z|$ Lemma 1 gives $|x| > |y|$ and, since $|y| > \kappa$ and $|x| > |y|$, we conclude that $|x| > \kappa$ This implies that (x, y) is in V^+
(4) By (3) it suffices to consider the case when (y, z) is an element of V We will show that (x, y) is in $V^+ \cup V$ If $x \le \kappa$ then since $|y| \le \kappa$ we conclude that (x, y) is in V If $x > \kappa$ then, since $|y| < \kappa$, we conclude that $|x| > |y|$ and (x, y) is in V^+

Notation Let $D_r \subset \mathbf{C}$ be the disk of radius r centered at the origin Let ι $\mathbf{C} \to \mathbf{C}^2$ be defined by $\iota(z) = (0, z)$ Let π $\mathbf{C}^2 \to \mathbf{C}$ be defined by $\pi(x, y) = y$

LEMMA 3 *The set* V^- *is homotopy equivalent to* $\mathbf{S}^1$ *the map* ι $\partial D_{2\kappa} \to V^-$ *is a homotopy equivalence The topological degree of the map induced by* g *on* V^- *is the algebraic degree of* g

Proof Let $\mathbf{C}_y$ be the y-axis Let $\phi_t(x, y) = ((1-t)x, y)$ for $t \in [0, 1]$ Now $\phi_t(V^-) \subset V^-$, ϕ_0 is the identity on V^- and ϕ_1 is the projection from V^- to $V^- \cap \mathbf{C}_y$ Thus ϕ provides a retraction from V^- to $V^- \cap \mathbf{C}_y$ The set $V^- \cap \mathbf{C}_y$ is the image of ι $\mathbf{C} - D_\kappa$ Both ι $\mathbf{C} - D_{2\kappa} \to V^-$ and π $V^- \to \mathbf{C} - D_\kappa$ are homotopy equivalences

To prove the last assertion it suffices to consider a single generalized Hénon map g_i $(x, y) \mapsto (y, p(y) - \delta x)$ If we can prove it for a single such map it will follow for a composition of generalized Hénon maps because both the algebraic and homological degrees of generalized Hénon multiply under composition We compute the degree of the map from V^- to itself by computing the degree of the map $\pi \circ g \circ \iota$ This is an equivalent problem because π and ι are homotopy equivalences This map is given by $y \mapsto p(y)$ Let d_i be the algebraic degree of g_i then d_i is the degree of p_i If L is sufficiently large then the topological degree of the map on $\mathbf{C} - D_L$ induced by p is the degree of the polynomial p The inclusion $\mathbf{C} - D_k \subset \mathbf{C} - D_L$ is a homotopy equivalence Thus p has degree d_i on $\mathbf{C} - D_\kappa$

LEMMA 4 *Let* f $(D, \partial D) \to (V^- \cup V, V^-)$ *be a holomorphic map Let* deg (f) *denote the topological degree of* f $\partial D \to V^-$ *Then* area $(f(D) \cap V) \geq$ area (D_κ) deg (f)

Proof The projection map π sends v to D_κ The induced map from $f(D) \cap V$ to D_κ is a proper map and therefore a branched cover We see that the covering degree is deg (f) by noting that $\pi f(\partial D)$ wraps deg (f) times around D_κ Let U be the set obtained from D_κ by removing the critical points of the projection and removing arcs connecting the critical points to the boundary of D_κ The area of U is the same as the area of D_κ and $f(D) \cap \pi^{-1} U$ consists of deg (f) components each mapped bijectively onto U by π Now π does not increase lengths and hence does not increase area so the area of each component is at least area $(U) =$ area (D_κ) Thus the area of $f(D) \cap V$ is at least area (D_κ) deg (f)

Proof of Theorem 1 Let $K^+ \subset \mathbf{C}^2$ be the set of points with bounded forward orbits and let K^- be the set of points with bounded backwards orbits Let $K = K^+ \cap K^-$ When g is cyclically reduced an argument from **[FM]** Lemma 3 5 proves that $K^+ \subset V \cup V^-$, $K^- \subset V \cup V^+$ hence $K \subset V$ The same argument shows that all points outside of K are wandering The set K is compact and is in fact the maximal compact invariant subset of $\mathbf{C}^2$

Friedland and Milnor give log d as an upper bound for the entropy of $h(g)$ The inequality $h(g) \geq h(g|K)$ is a basic property of entropy It suffices to prove the lower bound $h(g|K) \geq \log(d)$

Lemmas 3 and 4 imply that the area of $g^n \iota(D_{2\kappa}) \cap V$ is at least constant d^n Thus the volume growth, as defined in [Y], of the submanifold $\iota(D_{2\kappa})$ is at least log d We wish to apply the result of Yomdin ([Y], see also [G]) which says that, for C^∞ maps of compact manifolds, volume growth of submanifolds is a lower bound for entropy We cannot apply this theorem directly to $\mathbf{C}^2$ because it is not compact We cannot apply this theorem directly to K because it is not a manifold and we do not have information on the area of $g^n \iota(D_{2\kappa}) \cap K$ We proceed by an indirect course, we approximate the set K^+ by manifolds with boundary V_n defined below

Let $d_n(x, y) = \max_{i=0 \ldots n-1} d(g^i(x), g^i(y))$ For X a compact subset of $\mathbf{C}^2$ we denote by $M(n, \varepsilon, X)$ the minimum number of ε-balls in the d_n metric needed to cover X Let $v(n)$ be the area of $g^n \iota(D_{2\kappa}) \cap V$ Let $V_n = V \cap g^{-n}(V)$ Let $v^0(n, \varepsilon)$ be the maximum of the area of $g^n \iota(S')$ where S' is $\iota^{-1}(S)$ for S an ε-ball of V_n in the d_n

metric If we choose a minimal covering of V_n by ε-balls S_i then the area of $g^n\iota(D_{2\kappa})\cap V$ is bounded above by the sum of the areas of $g^n\iota(S_i')$ The sum of areas is bounded above by the number of balls times the maximum area This gives

$$v(n)\le M(n,\varepsilon,V_n)v^0(n,\varepsilon)$$

Taking limits gives

$$\limsup_{n\to\infty}\frac{1}{n}\log v(n)\le\limsup_{n\to\infty}\frac{1}{n}\log M(n,\varepsilon,V_n)+\limsup_{n\to\infty}\frac{1}{n}\log v^0(n,\varepsilon)$$

We evaluate $v(n)$ By Lemma 3 the topological degree of the map $g^n\iota$ on $\partial D_{2\kappa}$ is d^n By Lemma 4 we have

$$\text{area}\,(g^n\iota(D_{2\kappa})\cap V)\ge\text{constant}\quad\deg(g^n\iota)=\text{constant}\quad d^n$$

Thus the left hand side is greater than or equal to $\log d$ and we have

$$\log d\le\limsup_{n\to\infty}\frac{1}{n}\log M(n,\varepsilon,V_n)+\limsup_{n\to\infty}\frac{1}{n}\log v^0(n,\varepsilon)$$

Taking limits as ε goes to zero gives

$$\log d\le\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log M(n,\varepsilon,V_n)+\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log v^0(n,\varepsilon)$$

Yomdin shows ([**Y**] Theorem 1 8) that

$$\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}n^{-1}\log v^0(n,\varepsilon)$$

is zero for C^∞ maps This result is stated for compact manifolds but it holds in our situation. The following modification is required in the proof A bound of the form B^k on the norm of the first derivative of the kth iterate of the map is needed In our case if B is a bound for the norm of the derivative of $g|V$ then B^k is a bound for the norm of the derivative of $g^k|V_k$

It remains for us to relate the quantity

$$\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}n^{-1}\log M(n,\varepsilon,V_n)$$

to the entropy of $g|K$ Let $\bar V$ denote the quotient space $(V\cup V^-)/V^-$ Let m be the point corresponding to V^- We define a metric $\bar d(x,y)$ on $\bar V$ by the formula

$$\bar d(x,y)=\min\{d(x,v),d(x,V^-)+d(y,V^-)\}$$
$$\bar d(x,m)=d(x,V^-)$$

Since the set V^- is g invariant, g extends to a continuous map $\bar g$ from $\bar V$ to itself We have

$$\begin{aligned}h(\bar g)&=\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log M(n,\varepsilon,\bar V)\\&\ge\lim_{\varepsilon\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log M(n,\varepsilon,V_n)\\&\ge\log d\end{aligned}$$

The first equality is the definition of entropy The second inequality follows because $V_n \subset V$ and if V_n is sufficiently far from $V \cap V^-$ (relative to the size of ε) then the metrics d and $\bar{d}$ are the same when restricted to V_n Thus an (n, ε) cover of $\bar{V}$ with respect to the $\bar{d}$ metric yields an (n, ε) cover of V_n with respect to d

By a result of Bowen [**B**] the entropy of a map is equal to the entropy of the restriction of the map to the nonwandering set In this case we have $h(\bar{g}) = h(\bar{g} \mid K^+ \cup \{m\})$ because the nonwandering set is contained in $K^+ \cup \{m\}$ Now

$$h(\bar{g} \mid K^+ \cup \{m\}) = h(\bar{g} \mid K^+) + h(\bar{g} \mid \{m\}) = h(\bar{g} \mid K^+)$$

On the set K^+ the maps g and $\bar{g}$ are identical Thus $h(\bar{g}) = h(g \mid K^+)$ The non-wandering set of $g \mid K^+$ is contained in $g \mid K$ so applying Bowen's result again we have $h(g \mid K^+) = h(g \mid K)$ Combining these results gives

$$h(g \mid K) = h(g \mid K^+) = h(\bar{g}) \geq \log d$$

This completes the proof of the theorem □

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