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Bifurcation and chain recurrence

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Abstract. We show that there is a residual subset of the set of C^1 diffeomorphisms on any compact manifold at which the map

$$f \to (\text{number of chain components for } f)$$

is continuous. As this number is apt to be infinite, we prove a localized version, which allows one to conclude that if f is in this residual set and X is an isolated chain component for f, then

(i) there is a neighbourhood U of X which isolates it from the rest of the chain recurrent set of f, and

(ii) all g sufficiently C^1 close to f have precisely one chain component in U, and these chain components approach X as g approaches f.

(ii) is interpreted as a generic non-bifurcation result for this type of invariant set.

0. *Introduction*

A classical set of problems in the study of dynamical systems is concerned with understanding the structure of various invariant sets of a given system, and to describe how these sets change as one changes the system. This bifurcation problem is well understood in some instances. For example, the theorems of Kupka & Smale and Hartman & Grobman tell us that for each f in a residual subset of $\mathrm{Diff}^r(M)$ $(r \geq 1)$, and each $n \geq 1$, f^n has a finite number of fixed points, and that if g is C^r close enough to f, (how close depends on n), then g^n has exactly the same number of fixed points as f^n, and the fixed point set of g^n approaches that of f^n as g approaches f. See [**6**] or [**8**] for more details.

The main result of the present paper is an analogue of this well-known result. The setting is a compact Riemannian manifold M, with metric d. We consider the chain recurrent set of a map f in $\mathrm{Diff}^r(M)$, $r \geq 0$. ($\mathrm{Diff}^0(M)$ is the set of homeomorphisms of M to itself with the uniform metric d_0; $\mathrm{Diff}^1(M)$ is the set of C^1 diffeomorphisms on M with the uniform C^1 metric d_1, and so on.)

A point x in M is *α-chain recurrent* for f if for each $\beta > \alpha \geq 0$ there is a β-chain from x back to x, that is, a finite set of points

$$x_0, x_1, \ldots, x_p, \qquad \text{with } x = x_0 = x_p,$$

and

$$d(f(x_i), x_{i+1}) < \beta \qquad \text{for } i = 0, 1, 2, \ldots, p-1.$$

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We will sometimes represent such a chain by the notation

$$x_0 \to x_1 \to \cdots \to x_p$$

and say that this β-chain goes from x_0 to x_p.

The *chain recurrent set* of f, CR(f), is the set of points in M that are α-chain recurrent for all $\alpha \geq 0$, (so chain recurrent and 0-chain recurrent mean the same thing). There is a natural equivalence relation that is defined on CR(f) by calling two points *equivalent* if for any β there is a periodic β-chain containing both points. More precisely, $x \sim y$ if for each $\beta > 0$ there is a β-chain going from x to y and a β-chain going from y to x. Each equivalence class is called a *chain component*. Let $N(f)$ denote the (possibly infinite) number of chain components of f.

The basic new result of this paper is part (b) of the following theorem. Recall that a subset S of a topological space X is *residual* if S can be realized as a countable intersection of open, dense subsets of X.

THEOREM A. *There is a residual subset I of* Diff^1 (M) *such that whenever f is an element of I, then*

(a) (Conley, Takens) *f is a continuity point of* CR(_);

(b) *N(_) is continuous at f.*

One interprets the continuity in part (a) of the theorem by viewing CR(_) as a map from (Diff^1 (M), d_1) to (FM, d_H), where d_1 is the uniform C^1 metric on Diff^1 (M), FM is the set of all closed non-void subsets of M, and d_H is the Hausdorff metric on FM,

$$d_H(F, F') = \inf \{t > 0 \mid F' \subset B(F, t) \text{ and } F \subset B(F', t)\},$$

where

$$B(F, t) = \{x \in M \mid d(x, F) < t\}.$$

The relevant facts are that (Diff^1 (M), d_1) is a Baire space [**2**], so that any residual subset is dense; and that the metric topology makes (FM, d_H) a compact metric space [**4**]. A more detailed description is contained in [**3**]. The proof of (a) is essentially due to C. Conley. His description of CR(f) in terms of the attractors of f (see II.6 and II.7 of [**1**], especially 6.2.A on page 37) shows that the map $f \to$ CR(f) is upper semicontinuous; from this it is a standard argument to establish (a). More details are contained in the discussion surrounding lemma 1, below. See also [**10**, theorem 1], and corollary 3(a), below.

In part (b) of the theorem, the range of the map N(_) is $\{1, 2, \ldots, \infty\}$ viewed as the usual one-point compactification of the positive integers.

Theorem A more or less provides the kind of non-bifurcation result we spoke of in the opening paragraph of this paper. If N(_) is continuous and finite at f, then diffeomorphisms that are C^1 close to f have exactly the same number of chain components as does f. In other words, one cannot break apart any of the chain components of f by using a C^1 perturbation. Unfortunately, the conclusion that one can draw from the continuity of N(_) at f is much less informative if $N(f)$ is infinite. Of course, this would not be a problem if $N(g)$ were finite for all g in

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a residual subset of $\text{Diff}^1(M)$, but it is not known whether or not this is the case (unless the dimension of M is small). Moreover, S. Newhouse has shown, [**5**], that any residual subset of $\text{Diff}^r(M)$ must contain diffeomorphisms f with $N(f)=\infty$ whenever both r and the dimension of M are at least 2. This fact motivates a different approach to improving the conclusions that can be drawn when $N(f)$ is infinite. Instead of viewing the dynamics on all of M, we can consider only the chain-recurrent behaviour that is contained in a specified subset Y of M. Specifically, we say that an α-chain

$$x_0 \to x_1 \to \cdots \to x_p$$

is an *(α, Y)-chain* if each x_i lies in the closure of Y. $\text{CR}(f; Y)$ is then defined to be the set of points x such that for each $\alpha > 0$ there is an (α, Y)-chain from x back to x,

$$x = x_0 \to x_1 \to \cdots \to x_p = x.$$

Just as in the original case, one can say that x and y in $\text{CR}(f; Y)$ are equivalent if for each $\alpha > 0$ there is a periodic (α, Y)-chain

$$x_0 \to \cdots \to x_p = x_0 \qquad \text{with } x = x_0 \quad \text{and} \quad y = x_i \qquad \text{for some } i.$$

An equivalence class under this relation will be called a *Y-chain component* of f. Let $N(f; Y)$ denote the (again, possibly infinite) number of Y-chain components of f.

THEOREM B. *There is a countable basis $\mathcal{U}$ for the topology on M, and a residual subset J of* $\text{Diff}^1(M)$ *with the property that whenever f is an element of J and U is an element of $\mathcal{U}$, then*

(a) *f is a continuity point of* $\text{CR}(\_; U)$;
(b) *$N(\_; U)$ is continuous at f.*

Note that by taking $U = M$ in theorem B one obtains theorem A.

COROLLARY. *Suppose f is in J, X is a chain component of f, and B is an open neighbourhood of X with*

$$\text{clos}(B) \cap \text{CR}(f) = X.$$

Then there is a neighbourhood G of f in $\text{Diff}^1(M)$ *such that each g in G has exactly one chain component X_g contained in B, and no other chain component of g meets* $\text{clos}(B)$.

The proof of theorem B relies on the affirmative answer to the following stabilization question for invariant sets:

If U is open in M and contains an f-invariant set, can f be C^r approximated by g such that any diffeomorphism sufficiently C^r close to g has an invariant set contained in U?

We use the closing lemma [**7**] to obtain an affirmative answer to this question. This is the reason we restrict ourselves to C^1 diffeomorphisms in theorems A and B.

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As an application of the above results, we obtain the following theorem. To say that A in M is a *chain transitive attractor* of f means that

(1) A is a chain component of f;

(2) A is an attractor of f; that is, there is an open neighbourhood U of A with $f(\mathrm{clos}\,(U))$ contained in U and

$$\bigcap_{n\geq 0} f^n(U)=A.$$

THEOREM C. *There is a residual subset* $\mathcal{A}$ *of* $\mathrm{Diff}^1(M)$ *such that if* f *is in* $\mathcal{A}$, f_n *converges to* f *in the* C^1 *topology, and* A *is a chain transitive attractor of* f, *then there are chain transitive attractors* A_n *of* f_n *with* A_n *converging to* A *in the Hausdorff topology. For large* n *there is only one* A_n *that is near* A.

The C^r version of theorem C was stated in [**3**] (theorem A(c)). However, the proof there contains an error. The C^r version of theorem C for r greater than 1 is still an unsettled question.

§ 1 contains the proof of theorem B as well as a study of the stabilization question that is the key to that result. § 2 contains the application to attractors.

1. *Proof of theorem B*

We begin with three simple lemmas.

LEMMA 1. *Suppose* X_1 *and* X_2 *are metric spaces, with* X_2 *compact. Let* $h: X_1\to FX_2$ *be either upper or lower semicontinuous. Then the set of continuity points of* h *is a residual subset of* X_1.

This is [**9**, lemma 2.3]. The map h is lower semicontinuous at x in X_1 if whenever x_n approaches x and y is in $h(x)$, then there are points y_n in $h(x_n)$ with y_n converging to y. h is upper semicontinuous at x if for any sequence $x_n\to x$, if y_n is in $h(x_n)$ and $y_n\to y$, then y is in $h(x)$. If g is a map from some topological space X_1 into the extended half-line $S=[0,\infty]$ (viewed as the one-point compactification of $[0,\infty)$), then g is lower (upper) semicontinuous at x in X_1 if and only if the induced map

$$g^*: X_1\to \mathrm{FS} \qquad \text{given by} \quad g^*(z)=[0,g(z)]$$

is lower (upper) semicontinuous at x. This agrees with the usual definition of lower (upper) semicontinuity for real-valued functions:

$$\liminf_{z\to x} g(z)\geq g(x) \qquad (\limsup_{z\to x} g(z)\leq g(x)).$$

LEMMA 2. *Fix* f *in* $\mathrm{Diff}^r(M)$ *and a closed subset* Y *of* M. *Define a map* $G: Y\to[0,\infty)$ *by*

$$G(x)=\inf\{\alpha>0 \mid x \textit{ lies on a periodic } \alpha\textit{-chain for } f \textit{ which is contained in } Y\}.$$

Then G *is continuous.*

Proof. Let $\gamma>0$ be given, and choose $\delta<\gamma$ small enough that whenever x, y are in M and $d(x,y)<\delta$ then

$$d(f(x),f(y))<\gamma.$$

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Now suppose that x, y are in Y and within δ of each other, and that

$$x \to x_1 \to \cdots \to x_p \to x$$

is an α-chain for f in Y. Then

$$d(f(y), x_1) \leq d(f(y), f(x)) + d(f(x), x_1) < \gamma + \alpha,$$

and

$$d(f(x_p), y) \leq d(f(x_p), x) + d(x, y) < \alpha + \gamma.$$

Thus $d(x, y) < \delta$ implies that

$$G(y) \leq G(x) + \gamma.$$

By the symmetry in the assumptions on x and y we conclude that

$$|G(x) - G(y)| < \gamma$$

whenever $d(x, y) < \delta$, so G is continuous. □

Note that for Y closed in M and G defined as in the lemma,

$$\mathrm{CR}(f; Y) = G^{-1}(0),$$

so the lemma shows that $\mathrm{CR}(f; Y)$ is closed in M. In addition, it shows that distinct Y-chain components of f are bounded apart in the sense that if X_1 and X_2 are disjoint Y-chain components of f then for some $\alpha > 0$ there is no periodic α-chain for f that is both contained in Y and meets each of the sets X_1 and X_2.

LEMMA 3. *Suppose f_n converges to f in the C^0 topology, $y_n \to y$ and $z_n \to z$ in M, and that for each n and each positive α there is an (α, Y)-chain for f_n going from y_n to z_n. Then for each positive α there is an (α, Y)-chain for f going from y to z.*

Proof. Since (α, Y)-chains and $(\alpha, \mathrm{clos}(Y))$-chains are the same thing, we may as well assume that Y is closed. Let $\alpha > 0$ be given. Choose n large enough that

(i) $d_0(f, f_n) < \alpha/3$;
(ii) $d(f(y), f(y_n)) < \alpha/3$;
(iii) $d(y, y_n) < \alpha/3$;
(iv) $d(z, z_n) < \alpha/3$.

By assumption there is an $(\alpha/3, Y)$-chain for f_n that goes from y_n to z_n; denote it by

$$y_n = x_0 \to x_1 \to \cdots \to x_p = z_n$$

(of course the x_j's depend on n, but we suppress this dependence from the notation). Now (i)–(iv) combined with the triangle inequality show that

$$y \to x_1 \to x_2 \to \cdots \to x_{p-1} \to z$$

is a α-chain in Y for f. □

COROLLARY 3(a). *The map $f \to \mathrm{CR}(f; Y)$ is upper semicontinuous at all f in* $\mathrm{Diff}^r(M)$, *for any $r \geq 0$ and any subset Y of M.*

COROLLARY 3(b). *If f, f_n, y, y_n, z, z_n are as in lemma 3 and for each n, y_n and z_n lie in a single Y-chain component of f_n, then y and z lie in a single Y-chain component of f.*

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Proof. 3(a), in the case $Y = M$, is essentially (although not explicitly) contained in [**1**]; see also [**10**] for a very closely related result. To prove 3(a) just use the lemma with $y_n = z_n$ for each n.

As for 3(b), the assumptions ensure that for each n and each positive α there are $(\alpha; Y)$-chains for f_n from y_n to z_n and from z_n to y_n. Applying the lemma twice gives $(\alpha; Y)$-chains for f from y to z and from z to y, which establishes 3(b). □

PROPOSITION 4. *The set*

$$\mathscr{C}_r(Y) = \{f \in \mathrm{Diff}^r(M) \mid \mathrm{CR}(\_; Y) \textit{ is continuous at } f\}$$

is residual in $\mathrm{Diff}^r(M)$ *for any subset* Y *of* M, *and* r *in* $\{0, 1, \ldots, \infty\}$.

Proof. Combine lemma 1 and corollary 3(a). □

Recall that $N(f; Y)$ is the number of Y-chain components of f.

PROPOSITION 5. *Suppose f is in $\mathscr{C}_r(Y)$. Then $N(\_; Y)$ is lower semicontinuous at f.*

Proof. Let f_n approach f in $\mathrm{Diff}^r(M)$. Suppose first that $N(f; Y)$ in finite, so that we can list the Y-chain components, $X_1, \ldots, X_k$. Since these sets are closed and disjoint, we can find open sets $G_1, \ldots, G_k$ (open in Y) such that X_j is contained in G_j and the closures of the sets G_j are pairwise disjoint. Since $\mathrm{CR}(f_n; Y)$ is assumed to approach $\mathrm{CR}(f; Y)$ in the Hausdorff topology, we can conclude that, for large n, at least one Y-chain component of f_n meets each of the sets G_j. We can then apply corollary 3(b) to see that for large enough n no Y-chain component of f_n can meet more than one of the sets G_j. Thus

$$\liminf N(f_n; Y) \geq \text{number of } G_j\text{'s} = k = N(f; Y).$$

Now suppose that $N(f; Y)$ is infinite. By arguing as above for arbitrary finite collections of Y-chain components of f, we can show that $\liminf N(f_n; Y)$ is also infinite. □

In what follows it will become necessary to restrict the possible subsets Y that we will consider. Accordingly, we fix a certain countable basis $\mathscr{U}$ for the topology on M. We require that $\mathscr{U}$ contains enough open sets so that any two disjoint, closed subsets of M can be separated by elements U, U' of $\mathscr{U}$ with

$$\mathrm{clos}(U) \cap \mathrm{clos}(U') = \varnothing.$$

We also require that each element of $\mathscr{U}$ be an open subset of M whose topological boundary is a smooth codimension-one submanifold of M. Obtaining such a basis is no problem, since a compact manifold is always second countable, and obtaining the smoothness condition involves only some elementary arguments in differential topology (see, e.g. [**2**, exercise 1, p. 55]). This smoothness condition will facilitate certain technical arguments, and it does not interfere with the applications we have in mind (specifically, the corollary to theorem B, and theorem C).

Define $\mathscr{C}_r$ to be the intersection of all the sets $\mathscr{C}_r(U)$ for U in $\mathscr{U}$. By proposition 4 and the fact that $\mathscr{U}$ is countable, $\mathscr{C}_r$ is a residual subset of $\mathrm{Diff}^r(M)$.

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THEOREM 6. *There is a residual subset J of* $\mathrm{Diff}^1(M)$ *with the property that whenever f is in J and U is in* $\mathcal{U}$, *then*

(a) *f is a continuity point of* CR(_; *U*);

(b) $N(\_;U)$ *is continuous at f.*

Proposition 5 and lemma 1 combine to give a residual subset J of $\mathscr{C}_1$ at which the restriction of $N(\_;U)$ to $\mathscr{C}_1$ is continuous for each U in $\mathcal{U}$. Since a residual subset of a residual set is residual, J is residual in $\mathrm{Diff}^1(M)$. We need to show that the unrestricted map $N(\_;U)$ is continuous at any f in J. If $N(f;U)$ is infinite, this is equivalent to the lower semicontinuity of proposition 5, so we may assume that f_n approaches f in $\mathrm{Diff}^1(M)$ with

$$\infty \geq N(f_n;U) > N(f;U) \qquad \text{for all } n,$$

and look for a contradiction. By assumption on f, there is a $\delta>0$ such that $d_1(f,g)<\delta$ and g in $\mathscr{C}_1$ imply that $N(g;U)$ is equal to $N(f;U)$, so to get a contradiction it will suffice to show that we can perturb any f_n to a map g_n in $\mathscr{C}_1$ with

$$N(g_n;U) \geq N(f_n;U).$$

This is a type of stabilization problem; it may be phrased in a stronger way as follows.

Stabilization question. If X is a U-chain component for f, is there an open set G in $\mathrm{Diff}^r(M)$, containing f in its closure, such that every g in G has at least one U-chain component?

By using the closing lemma [**7**], we can give an affirmative answer in the case $r=1$.

CLOSING LEMMA (Pugh). *Suppose that* $\{f^n(x) \mid -\infty<n<\infty\}$ *is a recurrent orbit for a* C^1 *diffeomorphism f, that V is an open set containing the closure of this orbit, and that* $\delta>0$. *Then there is a diffeomorphism g,* C^1-δ-*close to f, with a hyperbolic periodic orbit in V.*

Proof. The proof is contained in [**7**], although the statement that the periodic orbit lies in V is not explicitly made there. The argument in [**7**] proceeds by choosing a finite segment of the recurrent orbit, and then making perturbations in a small neighbourhood, W, of this orbit segment. The closed orbit that is produced consists of two segments, the first contained in W, and the second being a segment of the original recurrent orbit. Hence one only has to ensure that W is contained in V. For further details and for the definitions of 'hyperbolic' and 'recurrent', see [**7**] or [**8**]. □

LEMMA 7. *Let U be an element of* $\mathcal{U}$, *f be in* $\mathrm{Diff}^1(M)$, *and suppose* $N(f;U)\geq 1$. *Then there is an open set W in* $\mathrm{Diff}^1(M)$ *with f in the closure of W and* $N(g;U)\geq 1$ *for all g in W.*

Proof. $N(f;U)\geq 1$ implies that CR $(f;U)$ is a non-void invariant set in the closure of U, so that clos (U) contains a recurrent orbit $\{f^n(x)\}$. The smoothness conditions on U in $\mathcal{U}$ ensure that there is a smooth diffeomorphism h, C^1-close to the identity, with $h(\text{clos}(U))$ contained in U. If we let $y=h(x)$ then y is a recurrent point of

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the diffeomorphism

$$g_0 = h \circ f \circ h^{-1},$$

and the closure of the g_0-orbit of y lies in U. By the closing lemma, we can find g C^1-close to g_0 such that g has a hyperbolic periodic orbit contained in U. By the local stability of hyperbolic orbits (see [**6**] for example), there is an open neighbourhood $W(g)$ of g such that each map in $W(g)$ also has a periodic orbit in U. Let W be the union of $W(g_n)$ for a sequence g_n converging to f. □

Because the perturbations involved in the proof of lemma 7 are local (that is, they can be required to be the identity away from U, we obtain as an immediate corollary:

PROPOSITION 8. *Let $\delta > 0$ be given, and suppose that $X_1, \ldots, X_k$ are distinct U-chain components of h (U in $\mathscr{U}$), with each X_j contained in an open set U_j in $\mathscr{U}$, $U_j \subset U$, with the closures of the various U_j's pairwise disjoint. Then there is a g in $\mathscr{C}_1$ with $d_1(h, g) < \delta$, $\mathrm{CR}\,(g; U_j)$ non-void for each j, and $N(g; U) \geq k$.*

Proof of theorem 6. Part (a) of the theorem follows from proposition 4 and the fact that $\mathscr{U}$ is countable, so we turn our attention to the proof of (b). f in J implies that for all g in a neighbourhood of f in $\mathscr{C}_1$

$$N(g; U) = N(f; U) \tag{*}$$

(recall that we can assume that $N(f; U)$ in finite). Since J is contained in $\mathscr{C}_1$, for all h in a C^1 neighbourhood of f

$$N(h; U) \geq N(f; U). \tag{**}$$

If one could find h arbitrarily close to f with strict inequality in (**), proposition 8 would allow one to find g in $\mathscr{C}_1$, arbitrarily close to f, with

$$N(g; U) \geq N(h; U) > N(f; U).$$

This would contradict (*), and so the theorem is established. □

Using theorem 6 and the upper semicontinuity of CR(_), it is not hard to see that if X is an isolated chain component of f in J and g is C^1 close to f, then the single local chain component counted by $N(g; U)$ (here U in $\mathscr{U}$ is an open set that separates X from the rest of $\mathrm{CR}\,(f)$) is in fact a full chain component of g. In other words, if U is in $\mathscr{U}$ and

$$\mathrm{CR}\,(f) \cap U = \mathrm{CR}\,(f) \cap \mathrm{clos}\,(U)$$

is a single chain component of f, then

$$\mathrm{CR}\,(g; U) = \mathrm{CR}\,(g) \cap U$$

is a single chain component of g for all g C^1 close to f.

2. *Proof of theorem C*

Recall that the definition of 'chain transitive attractor' was given in the introduction.

THEOREM 9. *There is a residual subset $\mathscr{A}$ of $\mathrm{Diff}^1(M)$ such that for any f in $\mathscr{A}$ and any sequence f_n which converges to f in the C^1 topology, and any chain transitive attractor A of f, there are attractors A_n of f_n with A_n converging to A in the Hausdorff*

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topology. Moreover, for all large n there is a unique choice of A_n, and it will be chain transitive for f_n.

The uniqueness in the statement of the theorem is to be interpreted as follows. There are neighbourhoods V of A and W of f such that whenever f_n is in W, then A_n can be chosen so that A_n is in V. There is only one such choice, and it is a chain transitive attractor. Since A_n converges to A, this choice must be made for all large n.

LEMMA 10. *Suppose g is in* Diff^1 (M) *and that* A_1 *and* A_2 *are attractors of g with* $A_1 \neq A_2$. *Then* $A_1 \cup A_2$ *contains at least two chain components of g.*

Proof. Recall that the α-limit set of x under g

$$\alpha(x) = \bigcap_{m<0} \text{clos}\,\{g^i(x) \mid i \leq m\},$$

is closed, non-empty, and g-invariant. It is easy to see that $\alpha(x)$ is also chain transitive ([**1**, II.4 and II.6.2]). It follows that any compact, non-void, g-invariant set contains at least one chain component. Consequently we need only show that $A_1 \neq A_2$ forces the existence of two disjoint, compact, non-empty, g-invariant sets in the union of A_1 and A_2. If A_1 and A_2 are disjoint this is immediate, so assume that

$$A_3 = A_1 \cap A_2 \neq \emptyset.$$

By [**1**, II.5.3.A], A_3 is also an attractor of g. Let x be in $A_1 - A_3$ and consider $\alpha(x)$. Since A_1 is compact and g-invariant, $\alpha(x)$ is closed, non-empty, g-invariant, and in A_1. Since A_3 is an attractor, it is not hard to show that $\alpha(x) \cap A_3$ is non-empty if and only if x is in A_3 (see [**1**, II.5.1.A]). Since we are assuming that x is not in A_3, we must conclude that $\alpha(x)$ is contained in $A_1 - A_3$. Hence $\alpha(x)$ and A_3 are the disjoint, non-empty, compact, g-invariant sets we require. □

Proof of theorem 9. By [**3**, 7.15], there is a residual subset T of Diff^1 (M) satisfying all but the uniqueness part of theorem 9. Let $\mathcal{A}$ be the intersection of T with the residual set J of theorem 6. Since A is a chain transitive attractor of f, we can find U in $\mathcal{U}$ with $N(f; U) = 1$, so $N(g; U) = 1$ for all g C^1 close to f. Thus, if g is near f, then g has exactly one attractor in U, for if there were more than one, then the lemma shows that g would also have more than one chain component in U. Let A_g denote this uniquely defined attractor of g. Since A_g is an attractor none of whose proper subsets is an attractor, A_g is chain transitive (this follows from Conley's characterization of CR (g) in terms of the attractors of g; see [**1**, 6.2.A, p. 37]). □

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