Can. J. Math. Vol. 47 (5), 1995 pp. 1037–1050

RESIDUE INTEGRALS AND THEIR MELLIN TRANSFORMS

MIKAEL PASSARE AND AUGUST TSIKH

ABSTRACT. Given an almost arbitrary holomorphic map we study the structure of the associated residue integral and its Mellin transform, and the relation between these two objects. More precisely, we relate the limit behaviour of the residue integral to the polar structure of the Mellin transform. We consider also ideals connected to non-isolated singularities.

The following notation will be used in this note. We write $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}_+$ for the sets of natural, integer and positive real numbers respectively, and we consider them all as subsets of $\mathbb{C}$, the set of complex numbers. The letter X denotes a connected complex manifold of complex dimension n and f is a holomorphic map $X \longrightarrow \mathbb{C}^p$ with $p \leq n$. Given vectors a and b in $\mathbb{C}^p$ we write (a, b) for the bilinear pairing $\sum a_j b_j$.

For any smooth compactly supported $(n, n-p)$ form φ on X and for any ε in $\mathbb{R}_+^p$ we consider the residue integral

$$(1) \qquad I_f^\varphi(\varepsilon) = \int_{T_f(\varepsilon)} \frac{\varphi}{f_1 \cdots f_p}, \quad T_f(\varepsilon) = \{z \in X\,;\, |f_1(z)| = \varepsilon_1, \ldots, |f_p(z)| = \varepsilon_p\},$$

with the orientation of the tube $T_f(\varepsilon)$ depending in an alternating fashion on the ordering of the functions f_j. For suitable λ in $\mathbb{C}^p$ we shall also be dealing with the function

$$(2) \qquad \Gamma_f^\varphi(\lambda) = \frac{1}{\lambda_1 \cdots \lambda_p} \int_X \frac{\bar{\partial}|f_1|^{\lambda_1} \wedge \cdots \wedge \bar{\partial}|f_p|^{\lambda_p} \wedge \varphi}{f_1 \cdots f_p},$$

which admits the alternative representation

$$\Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^p} \varepsilon_1^{\lambda_1 - 1} \cdots \varepsilon_p^{\lambda_p - 1} I_f^\varphi(\varepsilon)\, d\varepsilon.$$

We shall call Γ_f^φ the *Mellin transform* of I_f^φ.

Of particular interest to us are the possible limits of the residue integral I_f^φ as ε tends to zero, and the connection between these limits and the singularity at the origin of the Mellin transform Γ_f^φ. Given s in $\mathbb{R}_+^p$ we shall use the notation

$$(3) \qquad R_f^\varphi(s) = \lim_{\delta \to 0} I_f^\varphi(\delta^{s_1}, \ldots, \delta^{s_p}), \quad \delta \in \mathbb{R}_+,$$

provided the limit exists. We shall call R_f^φ the *residue function* associated to I_f^φ. The following theorem was proved in [6].

Both authors supported by a joint Swedish/Soviet research project financed by the Royal Swedish Academy of sciences and the Academy of Sciences of the USSR.

Received by the editors January 19, 1994.

AMS subject classification: Primary: 32A27, 32H02; secondary: 13J07, 44A30.

https://doi.org/10.4153/CJM-1995-055-4 Published online by Cambridge University Press

THEOREM 1. *There is a finite number of non-zero vectors b_j in $\mathbb{Z}^p$, depending only on f and the support of φ, such that the residue function R_f^φ given by (1) and (3) is well defined on the complement in $\mathbb{R}_+^p$ of the hyperplanes $(s, b_j) = 0$. It is locally constant, i.e. constant in each connected cone defined by the given family of hyperplanes.*

The proof of Theorem 1 which is given in [6] relies on the resolution of singularities. It turns out that it is then essentially enough to consider maps f of the type

$$f_j(z) = u_j(z)z^{a_j}, \quad 1 \leq j \leq p, \tag{4}$$

where the a_j are vectors in $\mathbb{N}^n$ and the u_j are non-vanishing holomorphic functions. There is a duality between the vectors b_j in Theorem 1 and the column vectors a^k in Theorem 2 below. In order to render this duality more transparent we are going to indicate the proof of the following more special but also more exact version of Theorem 1, which should be compared to Proposition 2.

PROPOSITION 1. *Let X be the unit polydisc D^n in $\mathbb{C}^n$ and suppose that f is given by (4). Let also $u_1 = \cdots = u_r \equiv 1$, where r is the rank of the $(p \times n)$ matrix A whose rows are the vectors a_j. Then the residue function R_f^φ defined by (1) and (3) is zero on the complement in $\mathbb{R}_+^p$ of the closed convex cone spanned by the column vectors $a^1, \dots, a^n$ of the matrix A. If $r = p$ and φ is given by*

$$\varphi = \varphi(z)\, d\bar{z}_{p+1} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n,$$

where the coefficient $\varphi(z)$ is a smooth compactly supported function $D^n \longrightarrow \mathbb{C}$, then R_f^φ vanishes outside the closed convex cone spanned by $a^1, \dots, a^p$ and in the interior of this cone it takes on a constant value which we denote by $R_f^\varphi(A)$.

Notice that, unless $r = p$ with $a^1, \dots, a^p$ linearly independent, the cone occurring in Proposition 1 does not have any interior points.

PROOF. Fix s in $\mathbb{R}_+^p$. If there is a $\delta_0 \in \mathbb{R}_+$ such that the tube $T_f(\delta^s) \subset D^n$ is empty for $\delta < \delta_0$, then certainly the limit $R_f^\varphi(s)$ is equal to zero. Assume now that this is not the case. Then to each $N \in \mathbb{N}$ we can find $\delta_N \leq e^{-N}$ and $z^{(N)} \in D^n$ in the tube $T_f(\delta_N^s)$. Notice that, unless $a^k = 0$, we must have $z_k^{(N)} \neq 0$. We can also choose $c \in \mathbb{R}_+$ such that

$$e^{-c} \leq \max |u_j(z)| \leq e^c, \quad 1 \leq j \leq p,$$

the maximum being taken over the support of φ. Keeping in mind how the tube T_f^φ is defined, we get by a straight-forward computation that

$$\max_{1 \leq j \leq p} \Big| \sum_{k=1}^n a_j^k t_k^{(N)} - s_j \Big| \leq \frac{c}{N}, \quad \text{with} \quad t_k^{(N)} = \log |z_k^{(N)}| / \log \delta_N > 0.$$

In other words, we have found a sequence of vectors $\sum a^k t_k^{(N)}$ in the cone spanned by $a^1, \dots, a^n$, which converges to s. The first part of the proposition is thereby proved.

Let us consider the case $r = p$ with all the u_j identically equal to 1. Writing $|a^k| = a_1^k + \cdots + a_p^k$ and using multi-index notation, we then have

$$I_f^\varphi(\delta^s) = \int_{D^n \cap \{|z|^a = \delta^s\}} z^{-|a|} \varphi \, d\bar{z}'' \wedge dz, \tag{5}$$

where $d\bar{z}'' = d\bar{z}_{p+1} \wedge \cdots \wedge d\bar{z}_n$.

We are going to use the fact that, given any smooth function $\varphi \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ and any $\alpha \in \mathbb{N}^n$, one can always find a decomposition of the type

$$\varphi(z) = \sum_{j=1}^{n} \sum_{k+\ell<\alpha_j} \varphi_{k\ell}^j(z) z_j^k \bar{z}_j^\ell + \sum_{K+L=\alpha} \varphi_{KL}(z) z^K \bar{z}^L, \tag{6}$$

where all the coefficient functions are smooth and the $\varphi_{k\ell}^j$ are independent of the variable z_j. To prove (6) one can use induction with respect to the dimension n. For $n = 1$ it suffices to take the Taylor development up to order α_1 of φ with respect to z_1. Then, if (6) has been obtained for $n = m - 1$ it also holds for $n = m$ with $\alpha_m = 0$, and to complete the induction it is enough to consider a Taylor development of the last sum in (6) with respect to z_m, see [2, p. 65].

Let us take $\alpha = (|a^1| - 1, \ldots, |a^n| - 1)$ and substitute the decomposition (6) of the function φ into the equation (5). Notice that we may assume that all the $|a^k|$ are positive. Indeed, if some $|a^k|$ is zero then so is the exponent of z_k in (5). But then we can safely perform the integration with respect to this variable and obtain a similar integral over D^{n-1} instead. Introducing polar coordinates $z_j = r_j e^{i\theta_j}$ we first observe that the terms with coefficients $\varphi_{k\ell}^j$ will give rise to inner integrals of the form

$$\int_0^{2\pi} e^{i(-|a^j|+1+k-\ell)\theta_j} \, d\theta_j \quad \text{and} \quad \int_0^{2\pi} e^{i(-|a^j|+k-\ell)\theta_j} \, d\theta_j, \tag{7}$$

which are all equal to zero, since $k - \ell \le k + \ell < |a^j| - 1 < |a^j|$. We are therefore left with

$$I_f^\varphi(\delta^s) = \int_{[0,1]^n \cap \{r^a = \delta^s\}} \Phi(r) \, dr'',$$

with $dr'' = dr_{p+1} \wedge \cdots \wedge dr_n$ and

(8)

$$\Phi(r) = 2^{n-p} i^n \sum_{K+L=|a|-1} \int_{[0,2\pi]^n} e^{i(-|a|+1'+K-L,\theta)} \varphi_{KL}(re^{i\theta}) \, d\theta, \quad 1' = (\overbrace{1, \ldots, 1}^{p}, \overbrace{0, \ldots, 0}^{n-p}),$$

which is a smooth function on $[0, 1]^n$. Since the rank of A is maximal, the set $\{r^a = \delta^s\}$ is a smooth graph with respect to p of the variables. However, if the first $p \times p$ minor A_p is not invertible then one can find a linear combination of the vectors a_j of the form $(0, b)$, with $b \in \mathbb{N}^{n-p}$ non zero. So then the differential form dr'' vanishes on $\{r^a = \delta^s\}$. Taking into account the alternating dependence with respect to the ordering of the vectors a_j and reasoning as in [6, p. 50], we finally get

$$R_f^\varphi(s) = \begin{cases} R_f^\varphi(A) & \text{if } s \text{ is in the image of } \mathbb{R}_+^p \text{ under the linear map } A_p, \\ 0 & \text{if } s \text{ is not in the closure of this set,} \end{cases}$$

where

$$R_f^{\varphi}(A) = \operatorname{sgn}\det A_p \int_{[0,1]^{n-p}} \Phi(0,r'')\,dr''. \tag{9}$$

This completes the proof of the proposition.

THEOREM 2. *The Mellin transform Γ_f^{φ} defined by (2) is holomorphic for* $\operatorname{Re}\lambda$ *in* $\mathbb{R}_+^p$ *and it has a meromorphic continuation to all of $\mathbb{C}^p$. There is a finite number of non-zero vectors a^k in $\mathbb{N}^p$, depending only on f and the support of φ, such that the poles of Γ_f^{φ}, which are all simple, are contained in the hyperplanes $(a^k,\lambda) = -m$, $m \in \mathbb{N}$. Moreover, near the origin one has*

$$\Gamma_f^{\varphi}(\lambda) = \sum_{|K|=p} \frac{c_K}{(a^{k_1},\lambda)\cdots(a^{k_p},\lambda)} + Q(\lambda),$$

where the c_K are constants and Q is a finite sum of functions with simple poles along fewer than p hyperplanes.

PROOF. An application of a partition of unity to the test form φ shows that the theorem is local, so there is no loss of generality in assuming that there is a resolution $\pi\colon \tilde{X} \longrightarrow X$ of the singularities of the product $f_1\cdots f_p$, see [4]. Here $\tilde{X}$ is another complex manifold of the same dimension and π is a proper holomorphic surjective map, which is biholomorphic outside the zero set of $f_1\cdots f_p$, and such that the preimage of this zero set is given, in suitable local coordinates on $\tilde{X}$, by $z_1\cdots z_k = 0$, for some $k \le n$. It follows that the pull-backs to $\tilde{X}$ of the functions f_j, which we again denote simply by f_j, locally are of the form (4).

Let A be the $(p \times n)$ matrix which has the vectors a_j for its rows, and assume that the rank of A is equal to r. We then claim that the local coordinates on $\tilde{X}$ may in fact be chosen in such a way that r of the functions u_j are identically equal to 1. Indeed, after a preliminary relabelling of the coordinates we can assume that the first columns $a^1,\dots,a^r$ of A are linearly independent. We can then choose a permutation σ of $\{1,\dots,p\}$ such that the first $(r\times r)$ minor of the corresponding matrix A_σ becomes invertible. Let (a_{jk}^{-1}) denote the inverse of this minor, pick a branch of each logarithm $\log u_j$ and put

$$\tilde{z}(z) = \bigl(v_1(z)z_1,\dots,v_r(z)z_r,z_{r+1},\dots,z_n\bigr), \quad \text{with} \quad \log v_j = \sum_{k=1}^{r} a_{jk}^{-1}\log u_{\sigma(k)}.$$

The Jacobian of the map $\tilde{z}$ equals $v_1(0)\cdots v_r(0) \ne 0$ at the origin so we can take the components $\tilde{z}_j$ as new local coordinates. It is now an easy matter to check that the composed functions $f_j\bigl(z(\tilde{z})\bigr)$ are again of the form (4) with respect to $\tilde{z}$, and that the new $u_{\sigma(j)}(\tilde{z})$ are identically equal to 1 for $1 \le j \le r$.

A second partition of unity, applied this time to the pull-back to $\tilde{X}$ of the form φ, then shows that it suffices to prove the theorem in the case where X is a neighborhood of the origin and f is given by (4) with r of the u_j being constant. Since a reordering of the f_j merely corresponds to a multiplication by ± 1, we may in fact assume that $u_1 = \cdots = u_r \equiv 1$. Considering each term of the form φ separately and once again relabelling the coordinates, which also affects only the sign of Γ_f^{φ}, we can now deduce the theorem from the following proposition.

PROPOSITION 2. *Let X be the unit polydisc D^n in $\mathbb{C}^n$ and suppose that f is given by (4). Let also $u_1 = \cdots = u_r \equiv 1$, where r is the rank of the $(p \times n)$ matrix A whose rows are the vectors a_j. Finally let φ be given by*

$$\varphi = \varphi(z)\, d\bar{z}_{p+1} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n,$$

where the coefficient $\varphi(z)$ is a smooth compactly supported function $D^n \longrightarrow \mathbb{C}$. Then the Mellin transform Γ_f^φ defined by (2) is holomorphic for $\operatorname{Re}\lambda$ in $\mathbb{R}_+^p$ and it has a meromorphic continuation to all of $\mathbb{C}^p$. Its poles, which are all simple, are contained in the hyperplanes $(a^k, \lambda) = -m$, $1 \le k \le n$, $m \in \mathbb{N}$, where the a^k denote the column vectors of A. Moreover, near the origin one has

$$\Gamma_f^\varphi(\lambda) = \sum_{|K|=r} \frac{g_K(\lambda)}{(a^{k_1}, \lambda) \cdots (a^{k_r}, \lambda)} + h(\lambda), \quad K \subset \{1, \ldots, p\},$$

the functions g_K and h being holomorphic. If $r = p$ then g_K is of the form

$$|\det A_p| R_f^\varphi(A) + \sum_{k=1}^{p} (a^k, \lambda) G_k(\lambda),$$

where the G_k are holomorphic functions, A_p is the first $(p \times p)$ minor of A and $R_f^\varphi(A)$ is the same constant that occurs in Proposition 1.

PROOF. Taking into account the relations $\bar\partial |f_j|^{\lambda_j} / f_j = \lambda_j |f_j|^{\lambda_j - 2} d\bar{f}_j / 2$ and the particular form of the functions f_j, we obtain the following expression for the Mellin transform:

$$\begin{aligned} \Gamma_f^\varphi(\lambda) &= 2^{-p} \int_{D^n} |f_1|^{\lambda_1 - 2} \cdots |f_p|^{\lambda_p - 2} d\bar{f}_1 \wedge \cdots \wedge d\bar{f}_p \wedge \varphi \\ &= 2^{-p} \int_{D^n} |z_1|^{\Sigma a_j^1 (\lambda_j - 2)} \cdots |z_n|^{\Sigma a_j^n (\lambda_j - 2)} \Big(\bigwedge_{j=1}^{p} |u_j(z)|^{\lambda_j - 2} d\big(\bar{u}_j(z) \bar{z}^{a_j}\big) \Big) \wedge \varphi. \end{aligned} \tag{10}$$

If we expand the exterior product and use multi-index notation we can rewrite (10) as

$$\Gamma_f^\varphi(\lambda) = 2^{-p} \int_{D^n} |z|^{(a,\lambda) - 2|a|} \bar{z}^{|a| - 1'} \tilde{\varphi}(z, \lambda)\, d\bar{z} \wedge dz, \tag{11}$$

where $\tilde\varphi$ is now a smooth function $D^n \times \mathbb{C}^p \longrightarrow \mathbb{C}$, compactly supported in z and holomorphic in λ. In case $r = p$, so that all the u_j are $\equiv 1$, we have

$$\tilde\varphi(z, \lambda) = \det A_p \varphi(z). \tag{12}$$

In fact, if the rank of A_p is equal to $q < p$ then (11) may be written as a finite sum of similar integrals, but with $1'$ replaced by vectors with 1 only in q places and 0 elsewhere. Notice further that, just as in the proof of Proposition 1, we may assume that all the $|a^k|$ are positive. If not, we integrate with respect to z_k in (10) which gives us a similar integral over D^{n-1} instead.

Now, given any smooth function $\tilde\varphi : \mathbb{C}^n \times \mathbb{C}^p \longrightarrow \mathbb{C}$ and any $\alpha \in \mathbb{N}^n$ one can find a decomposition of the type

$$\tilde\varphi(z, \lambda) = \sum_{j=1}^{n} \sum_{k+\ell < \alpha_j} \varphi_{k\ell}^j(z, \lambda) z_j^k \bar{z}_j^\ell + \sum_{K+L=\alpha} \varphi_{KL}(z, \lambda) z^K \bar{z}^L, \tag{13}$$

where, just as in (6), the $\varphi^j_{k\ell}$ are independent of the variable z_j, all the coefficient functions being smooth. They may also be taken holomorphic in λ or independent of λ if the given function $\tilde{\varphi}$ has that property. The proof of (13) is identical to that of (6), one just has to verify that all Taylor coefficients and remainder terms are holomorphic in λ.

Let us insert the decomposition (13) of the function $\tilde{\varphi}$, with $\alpha = (|a^1|-1, \ldots, |a^n|-1)$, into the integral (11). Introducing polar coordinates $z_j = r_j e^{i\theta_j}$ we observe that the terms with coefficients $\varphi^j_{k\ell}$ produce inner integrals of the form (7), which all vanish for the same reason as before. So all that remains of (11) is

$$\Gamma^{\varphi}_f(\lambda) = \int_{[0,1]^n} r^{(a,\lambda)-1'} \Phi(r,\lambda)\, dr \tag{14}$$

where Φ is given by (8) with $\varphi_{KL}(re^{i\theta})$ replaced by $\varphi_{KL}(re^{i\theta},\lambda)$. This function is smooth in r and holomorphic in λ.

From the expression (14) it is now evident that Γ^{φ}_f is holomorphic for $\operatorname{Re}\lambda$ in $\mathbb{R}^p_+$. To obtain a meromorphic continuation to all of $\mathbb{C}^p$ one can use elementary integration by parts. One then sees that the poles are indeed simple and contained in the union of the hyperplanes $(a^k,\lambda) = -m$ for $1 \le k \le n$ and any natural number m. It follows from what we have already observed that, if the rank of A_p is less than p, then (14) can be decomposed into a finite sum of integrals of the same form, but with fewer than p of the exponents being subtracted by 1. Hence, for each such term, fewer than p integrations by parts will suffice to achieve meromorphic extension past the origin.

The only remaining case to consider is when A_p is invertible. Since the vectors $a^{p+1}, \ldots, a^n$ are then linear combinations of the vectors $a^1, \ldots, a^p$ it follows that Γ^{φ}_f is in fact a function of the variables $(a^1,\lambda), \ldots, (a^p,\lambda)$. All that we need to check is therefore that

$$(a^1,\lambda)\cdots(a^p,\lambda)\Gamma^{\varphi}_f(\lambda)|_{\lambda=0} = |\det A_p| R^{\varphi}_f(A), \tag{15}$$

with $R^{\varphi}_f(A)$ given by (9). In view of (12) the decompositions (13) and (6) are then identical, except for the non-zero factor $\det A_p$. Hence the function Φ in (14) is equal to the function Φ in (8) multiplied by $\det A_p$. Integration by parts in (14) gives the explicit formula

(16)

$$(a^1,\lambda)\cdots(a^p,\lambda)\Gamma^{\varphi}_f(\lambda) = \sum_{\ell=0}^{p}(-1)^\ell \sum_{|I|=\ell} \int_{[0,1]^{\ell+n-p}} r_I^{(a,\lambda)} \frac{\partial^\ell}{\partial r_I}\Phi(1,\ldots,r_I,\ldots,1,r'')\, dr_I\, dr'',$$

where

$$r_I^{(a,\lambda)} = r_{i_1}^{(a^{i_1},\lambda)} \cdots r_{i_\ell}^{(a^{i_\ell},\lambda)} r_{p+1}^{(a^{p+1},\lambda)} \cdots r_n^{(a^n,\lambda)},$$

and $1,\ldots,r_I,\ldots,1$ means r_{i_j} in the place i_j for $1 \le j \le \ell$ with 1 in the other places. Elementary integration now shows that for $\lambda = 0$ the right hand side in (16) is equal to

$$\int_{[0,1]^{n-p}} \Phi(0,r'')\, dr'',$$

and (15) follows.

Our next theorem establishes a relation between the singularity at the origin of the Mellin transform of the residue integral (1) and the different limits (3).

THEOREM 3. *For any σ in $\mathbb{R}_+^p$ the restriction of the function $\lambda_1 \cdots \lambda_p \Gamma_f^\varphi(\lambda)$ to the complex line $\{t\sigma \; ; \; t \in \mathbb{C}\}$ is holomorphic in a half plane containing the origin. Its value at the origin is equal to the mean value of the residue function R_f^φ over the simplex $\{s \in \mathbb{R}_+^p \; ; \; (s,\sigma) = 1\}$.*

PROOF. From Theorem 2 it follows that the function

$$t \longmapsto \sigma_1 \cdots \sigma_p t^p \Gamma_f^\varphi(t\sigma) \tag{17}$$

is meromorphic in all of $\mathbb{C}$ with poles in a discrete set of negative real numbers. From the proof of Theorem 2 it follows that it is enough to consider maps f of the form (4). By Proposition 2 the value of (17) at the origin is equal to $\sigma_1 \cdots \sigma_p |\det A_p| R_f^\varphi(A)/(a^1,\sigma)\cdots(a^p,\sigma)$ if the matrix A_p is invertible, and zero otherwise.

On the other hand, it follows from Proposition 1 that the residue function R_f^φ is also zero almost everywhere on the simplex $\{s \in \mathbb{R}_+^p \; ; \; (s,\sigma) = 1\}$ unless the matrix A_p is invertible. What remains to be shown is that the volume of the simplex spanned by $a^1/(a^1,\sigma), \ldots, a^p/(a^p,\sigma)$ divided by the volume of the whole simplex $(s,\sigma) = 1$ is equal to $\sigma_1 \cdots \sigma_p |\det A_p|/(a^1,\sigma)\cdots(a^p,\sigma)$. To see this it suffices to consider the linear map $L\colon \mathbb{R}^p \longrightarrow \mathbb{R}^p$ determined by the conditions

$$e_j/\sigma_j \longmapsto a^j/(a^j,\sigma),$$

where $e_1, \ldots, e_p$ are the usual basis vectors in $\mathbb{R}^p$. The absolute value of the Jacobian of L is indeed equal to $\sigma_1 \cdots \sigma_p |\det A_p|/(a^1,\sigma)\cdots(a^p,\sigma)$, and the proof is complete.

It follows from Theorem 3 that the function $\sigma \longmapsto t^p \Gamma_f^\varphi(t\sigma)|_{t=0}$ is essentially equal to the Radon transform of the residue function $R_f^\varphi(s)$. Since the Radon transform is injective on the space of homogeneous functions in $\mathbb{R}_+^p$ this means in particular that it is possible to recover $R_f^\varphi(s)$ from the Mellin transform Γ_f^φ. For $p = 2$ we have a particularly simple inversion formula.

THEOREM 4. *Let $p = 2$ and take $s \in \mathbb{R}_+^p$ such that $(s, b_j) \neq 0$ where the b_j are as in Theorem 1. Then for small enough $\delta \in \mathbb{R}_+$ one has*

$$R_f^\varphi(s) = \frac{1}{(2\pi i)^2} \int_{|s_1\lambda_1| = |s_2\lambda_2| = \delta} \Gamma_f^\varphi(\lambda_1,\lambda_2)\, d\lambda_1 \wedge d\lambda_2.$$

PROOF. As before it suffices to consider maps f of the form (4). There is no contributions unless A_2 is invertible, so we can assume this to be the case. The Mellin transform is then equal to $|\det A_2| R_f^\varphi(A)/(a_1^1\lambda_1 + a_2^1\lambda_2)(a_1^2\lambda_1 + a_2^2\lambda_2)$ and if we introduce the new coordinates $\lambda = \lambda_1/\lambda_2$, $\mu = \lambda_2$, we get

$$\begin{aligned}
&\frac{1}{(2\pi i)^2} \int_{\substack{|s_1\lambda_1| = \delta \\ |s_2\lambda_2| = \delta}} \Gamma_f^\varphi(\lambda_1,\lambda_2)\, d\lambda_1 \wedge d\lambda_2 \\
&\qquad = \frac{1}{(2\pi i)^2} \int_{\substack{|s_1\lambda/s_2| = 1 \\ |s_2\mu| = \delta}} \frac{|\det A_2| R_f^\varphi(A)}{(a_1^1\lambda + a_2^1)(a_1^2\lambda + a_2^2)}\, d\lambda \wedge \frac{d\mu}{\mu} \\
&\qquad = \frac{\operatorname{sgn}\det A_2}{2\pi i} \int_{|\lambda| = s_2/s_1} \Big(\frac{1}{\lambda+\alpha} - \frac{1}{\lambda+\beta}\Big) R_f^\varphi(A)\, d\lambda,
\end{aligned} \tag{18}$$

where $\alpha = a_2^1/a_1^1$ and $\beta = a_2^2/a_1^2$. It is now obvious that (18) is equal to $\operatorname{sgn}\det A_2 R_f^\varphi(A)$ if s_2/s_1 is between α and β, and zero otherwise. But this is precisely the value of $R_f^\varphi(s)$, so the theorem follows.

The case when f is a complete intersection map, that is $\dim f^{-1}(0) = n - p$, is quite special. The residue function is then constant and the following theorem, first obtained by Berenstein and Yger, shows that the Mellin transform has also a relatively simple structure.

THEOREM 5. *Let $p = 2$ and assume that f is a complete intersection map. The function*

$$(\lambda_1, \lambda_2) \longmapsto \lambda_1\lambda_2\Gamma_f^\varphi(\lambda_1, \lambda_2)$$

is then holomorphic at the origin.

PROOF. Here we consider only the case $n = 2$, a proof of the general case may be found in [1, proof of Theorem 3.18], and we can therefore assume that the origin is the only common zero of f_1 and f_2. First we note that, in view of the representation

$$\Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^2} \varepsilon_1^{\lambda_1-1}\varepsilon_2^{\lambda_2-1} I_f^\varphi(\varepsilon)\, d\varepsilon,$$

the theorem follows at once if φ happens to be holomorphic near the origin, for I_f^φ is constant for small ε then. Just as in the proof of Theorem 2 we have a desingularization map π giving us finitely many local models of the form (4), with one or both of the functions u_j being identically equal to 1 depending on whether the rank of A is one or two. We are thus dealing with a finite sum of integrals of the form

$$\lambda_1\lambda_2 \int_{[0,1]^2} r_1^{(a^1,\lambda)-1} r_2^{(a^2,\lambda)-1}\Phi(r,\lambda)\, dr, \tag{19}$$

with Φ obtained as in the proof of Proposition 2.

Now we claim that if the original test function φ vanishes to sufficiently high order at the origin, then (19) is holomorphic for λ near the origin. Indeed, we have only to bother with those vectors a^k which have both components different from zero, for only those can give rise to poles that are not parallel to the axes. But if both components of a^k are non-zero then the pullback of φ will vanish to high order along $z_k = 0$, and hence Φ will contain r_k as a factor, and this compensates the exponent -1 in (19).

It now remains to consider φ which are non-holomorphic monomials. Again it suffices to consider vectors a^k with both components different from zero. Since π then maps the axis $z_k = 0$ to the origin, it follows that the pullback of a non-holomorphic monomial necessarily contains $\bar{z}_k$ as a factor. Recalling the proof of Proposition 2, we find that the exponent of $\bar{z}_k$ in (11) can be taken equal to $|a^k|$ instead of $|a^k| - 1$, and consequently, that the exponent of r_k in (19) may in fact be reduced to (a^k, λ). This finishes the proof.

Our next theorem generalizes results from [8]. Let $p = n$ and consider the local ring O at the origin in $\mathbb{C}^n$. Let us also assume that $f_1, \dots, f_n \in O$ are such that $f' = (f_1, \dots, f_{n-1})$ is a complete intersection map. Given $g \in O$ we then write

$$R_f(g) = I_f^\varphi(\varepsilon), \tag{20}$$

https://doi.org/10.4153/CJM-1995-055-4 Published online by Cambridge University Press

with the right hand side given by (1) with $\varphi = g(z)\,dz$ and $\varepsilon_1 = \cdots = \varepsilon_{n-1} \ll \varepsilon_n$. Let finally

$$f'^{-1}(0) = K \cup L, \quad \text{with} \quad f_n|_L \equiv 0,\, f_n|_K \not\equiv 0 \text{ a.e.},$$

and consider the corresponding decomposition $k \cap \ell$ of the ideal $\langle f' \rangle$. That is, we let k denote the intersection of all primary components of the ideal $\langle f' \rangle$ which vanish on K, and ℓ denotes the intersection of those primary components that vanish on L.

THEOREM 6. *Assume that k and ℓ given above are regular ideals. The residue R_f given by (20) is then non-degenerate in the following sense:*

a) given $h \in \ell$ one has that $R_f(hg) = 0$ for all $g \in O$ if and only if $h \in \langle f', \ell f_n \rangle$,

b) given $h \in O$ one has that $R_f(hg) = 0$ for all $g \in \ell$ if and only if $h \in \langle k, f_n \rangle$,

c) if $\ell = \mathrm{rad}\, \ell$ then $\ell \supset \langle f \rangle$ and given $h \in \ell$ one has that $R_f(hg) = 0$ for all $g \in \ell$ if and only if $h \in \langle f \rangle$.

PROOF. Let us first choose the coordinates so that the projection of the zero set $\{f' = 0\}$ onto the z_n axis is a branched analytic covering and so that $f_n(0, z_n)$ is not identically equal to zero.

Now we claim that the cycle $T_f(\varepsilon)$ which occurs in (1) and (20) is homotopic, in the complement of $\{f_1 \cdots f_n = 0\}$, to the fibered cycle

$$T_K = \bigcup_{w \in \gamma_K} T_w, \quad \text{where} \quad \gamma_K = \{z \in K\,;\, |z_n| = \delta\},$$

and T_w is a local tube around the zero $z' = w'$ of the map $f'(z', w_n)$.

Indeed, since f_n vanishes on L, the real curve $\gamma = \{z \in K\,;\, |f_n(z)| = \varepsilon_n\}$ may be written as

$$\gamma = \{f_1 = \cdots = f_{n-1} = 0, |f_n| = \varepsilon_n\}.$$

Representing the function f_n as $c_m z_n^m + \chi(z)$, where

$$\chi(z) = \sum_{\alpha \geq m+1} c_\alpha z_n^\alpha + \chi_1(z), \quad \chi_1(0, z_n) \equiv 0,$$

we find that an explicit homotopy, between the curves γ_K and γ in $K \setminus \{0\}$, is provided by

$$\gamma_t = \{z \in K\,;\, |c_m z_n^m + t\chi(z)| = \varepsilon_n\}, \quad t \in [0, 1],$$

if we just put $\delta = |\varepsilon_n / c_m|^{1/m}$. In view of the continuity of the functions f_j it now follows that the family of tubes

$$\{z \in U\,;\, |f_1| = \cdots = |f_{n-1}| = \varepsilon', |c_m z_n^m + t\chi(z)| = \varepsilon_n \gg \varepsilon'\}, \quad t \in [0, 1],$$

where U is an open set containing all the curves γ_t, $0 \leq t \leq 1$, but not intersecting the zero set of f_n, constitutes a homotopy between T_K and $T_f(\varepsilon)$ outside the set $\{f_1 \cdots f_n = 0\}$. This proves our claim.

Next, we define a similar cycle T_L this time fibered over the curve $\gamma_L = \{z \in L\ ;\ |z_n| = \delta\}$ instead. It is clear that $T_K + T_L = T'$, where

$$T' = \{|f_1| = \cdots = |f_{n-1}| = \varepsilon', |z_n| = \delta\},$$

and since none of the cycles T_K, T_L, T' or $T_f(\varepsilon)$ intersect the set $\{f_1 \cdots f_{n-1} = 0\}$, and furthermore $T_f(\varepsilon) \sim T_K$, we find that $T_f(\varepsilon) \sim T' - T_L$ in the complement of $\{f_1 \cdots f_{n-1} = 0\}$. In order to prove the sufficiency of the conditions in a), b) and c) we may certainly assume that the product hg belongs to the ideal $k \cup \langle \ell f_n \rangle$. If it actually happens to be in k then in the integral defining $R_f(hg)$ we can use the cycle T_K, and using its fibered structure together with the local duality for the regular sequence $f'(z', w_n)$, see [3, p. 659], we conclude that the integral vanishes. If instead the product hg is contained in $\langle \ell f_n \rangle$, then we can use the cycle $T' - T_L$ to define $R_f(hg)$, and by a similar argument we again find that the residue is zero. Thus the sufficiency is proved in a), b) and c).

Let us now prove the necessity. Given $h \in O$ we denote by φ_K the quotient h/f_n. We propose to find a function $\varphi_L \in O$ satisfying the equation

$$R_f(hg) + \int_{T_L} \frac{\varphi_L g\, dz}{f_1 \cdots f_{n-1}} = 0, \tag{21}$$

for every germ $g \in O$. To achieve this, we pick generators $\ell_1, \ldots, \ell_{n-1}$ of the ideal ℓ and decompose our original functions as

$$f_i = \sum_{j=1}^{n-1} a_{ij} \ell_j, \quad i = 1, \ldots, n-1.$$

Let Δ be the determinant of the matrix (a_{ij}), and notice that it is non-zero on $L \setminus \{0\}$. For all w in $K \setminus \{0\}$ this determinant belongs to the ideal generated by $f_1, \ldots, f_{n-1}$ in the corresponding local ring O_w, see [3]. The map $g \longmapsto R_f(hg)$ is a linear functional on the local algebra $O/\langle \ell, \Delta \rangle$, and hence by the duality theorem it can be realized by a residue $\mathrm{res}_{(\ell',\Delta)}(\varphi_L \cdot)$. That is, there is a function $\varphi_L \in O$ with the property that

$$R_f(hg) + \int_{T(\ell',\Delta)} \frac{\varphi_L g\, dz}{\ell_1 \cdots \ell_{n-1} \Delta} = 0,$$

where $T(\ell', \Delta)$ is a tube corresponding to the map $(\ell_1, \ldots, \ell_{n-1}, \Delta)$. Considering again a homologous fibered cycle we can re-write this as

$$R_f(hg) + \int_{|z_n|=\delta'} \sum_{w \in L \cap \pi^{-1}(z_n)} \mathrm{res}_{\ell'(z',z_n)}\left(\frac{\varphi_L g}{\Delta}\right) = 0, \tag{22}$$

where π denotes the projection onto the last coordinate. According to the transformation law, see [3, p. 657], we have in fact

$$\mathrm{res}_{\ell'(z',z_n)}\left(\frac{\varphi_L g}{\Delta}\right) = \mathrm{res}_{f'(z',z_n)}(\varphi_L g),$$

and therefore, from (22) we get (21). Consider now the function

$$\phi(z) = \sum_{\pi^{-1}(z_n)\in K} \int_{T_{\pi^{-1}(z_n)}(f')} \varphi_K(\zeta', z_n) W(\zeta', z) + \sum_{\pi^{-1}(z_n)\in L} \int_{T_{\pi^{-1}(z_n)}(f')} \varphi_L(\zeta', z_n) W(\zeta', z),$$

where

$$W(\zeta', z) = (2\pi i)^{-n} H(\zeta', z)\, d\zeta' \Big/ \prod_{j=1}^{n-1} f_j(\zeta', z_n),$$

with $H(\zeta', z)$ denoting the Hefer determinant corresponding to the decomposition

$$f_i(\zeta', z_n) - f_i(z', z_n) = \sum_{j=1}^{n-1} (\zeta_j - z_j) P_{ij}(\zeta', z', z_n), \quad i = 1, \ldots, n-1.$$

Recalling how the cycles T_K and T_L were defined, we deduce the formula

$$\int_{|z_n|=\delta} \phi(z) z_n^k \, dz_n = \int_{T_K} \frac{h(\zeta) H(\zeta', z', \zeta_n)\zeta_n^k \, d\zeta}{f_1(\zeta)\cdots f_n(\zeta)} + \int_{T_L} \frac{\varphi_L(\zeta) H(\zeta', z', \zeta_n)\zeta_n^k \, d\zeta}{f_1(\zeta)\cdots f_{n-1}(\zeta)}, \quad k \in \mathbb{N}.$$

Now the first integral on the right hand side is equal to the residue $R_f(hHz_n^k)$ and hence, by formula (21), we see that the function $\phi(z', z_n)$ is orthogonal to all monomials z_n^k on the circle $|z_n| = \delta$. But an integral representation of ϕ shows, see [9, Proposition on p. 56], that it is also holomorphic outside the hyperplane $z_n = 0$. It follows that ϕ is holomorphic near the origin in $\mathbb{C}^n$.

Now we apply the Weil integral formula, according to which any function Ψ which is holomorphic on a compact polyhedron $\bar{\Pi} = \{|f_1| \le r_1, \ldots, |f_n| \le r_n\}$ may be represented as

$$\Psi(z) = \frac{1}{(2\pi i)^n} \int_{|f_j|=r_j} \frac{\Psi(\zeta) H(\zeta, z)\, d\zeta}{\big(f_1(\zeta) - f_1(z)\big) \cdots \big(f_n(\zeta) - f_n(z)\big)},$$

for all z in the open polyhedron Π. Decomposing the fraction $\big(f_1(\zeta) - f_1(z)\big)^{-1} \cdots \big(f_n(\zeta) - f_n(z)\big)^{-1}$ into a geometric progression

$$\frac{1}{f_1(\zeta)\cdots f_n(\zeta)} \sum_{|\alpha|\ge 0} \left(\frac{f(z)}{f(\zeta)}\right)^{\alpha},$$

we find that

$$\Psi(z) = \frac{1}{(2\pi i)^n} \int \frac{\Psi(\zeta) H(\zeta, z)\, d\zeta}{f_1(\zeta)\cdots f_n(\zeta)} + \text{elements in the ideal } \langle f_1, \ldots, f_n\rangle.$$

Since the function ϕ was defined as a sum of integrals over distinguished boundaries of polyhedra containing K and L we now obtain the relations

$$\phi \equiv \varphi_K \operatorname{mod}\langle f'\rangle \text{ in } O_w, \quad \text{for } w \in K \setminus \{0\}, \tag{23}$$

and

$$\phi \equiv \varphi_L \operatorname{mod}\langle f' \rangle \text{ in } O_w, \quad \text{for } w \in L \setminus \{0\}. \tag{24}$$

Under the condition a), when $R_f(hg) = 0$ for all $g \in O$, we can take $\varphi_L \equiv 0$ in (21). From (24) we then deduce that $\phi \in \ell$. Moreover, since $\varphi_K = h/f_n$, it follows from (23) that $h \equiv \phi f_n \operatorname{mod}\langle f' \rangle$ in O_w for $w \in K \setminus \{0\}$. But since both h and ϕ belong to ℓ this last equivalence holds also for $w \in L \setminus \{0\}$, and hence $h \in \langle f', \ell f_n \rangle$ as desired. A similar argument proves the necessity in b). Finally, if ℓ is a radical ideal, then $f_n \in \ell$. The same reasoning, using (23), again gives that $h - \phi f_n \in \langle f' \rangle$ and we are done.

We observe now that the residue (20) remains unchanged if in the definition of the tube $T_f(\varepsilon)$ we replace the condition

$$\varepsilon_1 = \cdots = \varepsilon_{n-1} \ll \varepsilon_n$$

on the radii by the requirement that

$$\varepsilon_1 \ll \cdots \ll \varepsilon_{n-1} \ll \varepsilon_n. \tag{25}$$

From the results in [1] it follows that under these latter conditions the limit, as $\varepsilon \longrightarrow 0$, of the integrals (20) exists for a general smooth function g, and that such a limit defines a current supported at the origin. Following the paper by Solomin [7] we assign a multiplicity $\mu_0(f)$ to f at the origin by means of the logarithmic residue

$$\mu_0(f) = (2\pi i)^{-n} \int_{T_f(\varepsilon)} \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}, \tag{26}$$

where the ε_j are sufficiently small and satisfy the condition (25). In our situation, when f' is a complete intersection, it follows from the results in [7] that the multiplicity $\mu_0(f)$ is equal to the multiplicity $\mu_0(f_n|_K)$ at the origin of the function f_n restricted to the holomorphic chain K (recall that $f'^{-1}(0) = K \cup L$, with $f_n|_L \equiv 0$ and $f_n|_K \not\equiv 0$ a.e.). We thus find that for any holomorphic function g with the property that $g|_L \equiv 0$ and $g|_K \not\equiv 0$ a.e. we have the relation

$$\mu_0(f) = \mu_0(f_n + g|_{K \cup L}) - \mu_0(g|_L).$$

But in that case the map $(f', f_n + g)$ is a complete intersection and according to [9, Section 19.2] we may write

$$\mu_0(f) = \mu_0(f', f_n + g) - \mu_0(g|_L). \tag{27}$$

We are going to end this paper by exhibiting an algebraic interpretation of the multiplicity $\mu_0(f)$ making use of the formula (27). For comparison we would like to mention the well known result of Palamodov [5] to the effect that for a complete intersection f the multiplicity $\mu_0(f)$ coincides with the dimension of the local algebra of the corresponding germ at the origin:

$$\mu_0(f) = \dim O/\langle f \rangle.$$

In our case, where $f^{-1}(0) = L$ is a curve and f' is a complete intersection, the following statement is valid.

THEOREM 7. *If the ideal ℓ, consisting of the intersection of those primary components of the ideal $\langle f' \rangle$ that vanish on the curve $f^{-1}(0) = L$, is a regular ideal, then*

$$\mu_0(f) = \dim\bigl\{\ell / \bigl\langle \langle f' \rangle, \ell \cdot f_n \bigr\rangle\bigr\}.$$

For the proof of Theorem 7 we shall need the following auxiliary result.

LEMMA. *If $g \in O$ is such that the ideal $\langle f', g \rangle$ is regular, then*

$$O/\langle f', g \rangle \simeq O/\langle \ell, g \rangle \oplus \ell / \bigl\langle \langle f' \rangle, \ell \cdot g \bigr\rangle. \tag{28}$$

PROOF OF LEMMA. An arbitrary germ $h \in O$ may be written as

$$h = c_1 e_1 + \cdots + c_s e_s + h_1 + q_1 g, \quad h_1 \in \ell,\ q_1 \in O,\ c_j \in \mathbb{C},$$

where $e_1, \ldots, e_s$ is a collection of germs that constitute a basis for $O/\langle \ell, g \rangle$. Furthermore, the element h_1 can be represented, up to elements in the ideal $\bigl\langle \langle f' \rangle, \ell \cdot g \bigr\rangle$, as a linear combination of elements $e_{s+1}, \ldots, e_\mu$ which form a basis for the quotient $\ell / \bigl\langle \langle f' \rangle, \ell \cdot g \bigr\rangle$. In this way we obtain the expression

$$h = c_1 e_1 + \cdots + c_\mu e_\mu + h_2 + (q_1 + q_2) g, \quad h_2 \in \langle f' \rangle,\ q_2 \in O.$$

This makes it evident that $\{e_1, \ldots, e_\mu\}$ is a set of generators for $O/\langle f', g \rangle$. Let us show that they are linearly independent. Assume that there is a relation

$$0 = c_1 e_1 + \cdots + c_\mu e_\mu + h_2 + q g, \quad h_2 \in \langle f' \rangle,\ q \in O.$$

Since $e_{s+1}, \ldots, e_\mu \in \ell$ and $h_2 \in \langle f' \rangle \subset \ell$ we must have $c_1 = \cdots = c_s = 0$. Hence $0 = c_{s+1} e_{s+1} + \cdots + c_\mu e_\mu + h_2 + q g$, so if we now recall that the ideals ℓ and $\langle f', g \rangle$ were assumed to be regular, we can conclude that q belongs to ℓ, because the ideal $\langle \ell, g \rangle$ is also regular. Consequently $c_{s+1} = \cdots = c_\mu = 0$ and the lemma is proved.

PROOF OF THEOREM 7. In accordance with formula (27) we have the equation

$$\mu_0(f) = \mu_0(f', f_n + \varphi) - \mu_0(\varphi|_L) \tag{29}$$

for any holomorphic function φ such that $\varphi|_K \equiv 0$ and $\varphi|_L \not\equiv 0$ almost everywhere. In view of the Nullstellensatz a sufficiently high power φ^m of φ will belong to the ideal k. Let us denote the function $f_n + \varphi^m$ by g and apply the above lemma to the ideal $\langle f', g \rangle$. We first observe however, that since $f_n|_L \equiv 0$ we have $\mu_0(\varphi|_L) = \mu_0(g|_L)$, and ℓ being a regular ideal, this latter multiplicity is equal to $\dim O/\langle \ell, g \rangle$. From (29) and (28) we now get

$$\mu_0(f) = \mu_0(f', g) - \mu_0(g|_L) = \dim O/\langle f', g \rangle - \dim O/\langle \ell, g \rangle = \dim \ell / \bigl\langle \langle f' \rangle, \ell \cdot (f_n + \varphi^m) \bigr\rangle.$$

But since $\langle f' \rangle = k \cap \ell$ and $\varphi^m \in k$, we also have $\ell \cdot \varphi^m \in \langle f' \rangle$ and therefore

$$\bigl\langle \langle f' \rangle, \ell \cdot (f_n + \varphi^m) \bigr\rangle = \bigl\langle \langle f' \rangle, \ell \cdot f_n \bigr\rangle,$$

which completes the proof of the theorem.

REFERENCES

1. C. A. Berenstein, R. Gay, A. Vidras and A. Yger, *Residue currents and Bézout identities*, Progress in Math. **114**, Birkhäuser Verlag, Basel, 1993.

2. N. Coleff and M. Herrera, *Les courants résiduels associés à une forme méromorphe*, Lecture Notes in Math. **633**, Berlin, 1978.

3. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Math., John Wiley and Sons, New York, 1978.

4. H. Hironaka, *The resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. **79**(1964), 109–326.

5. V. Palamodov, *On the multiplicity of a holomorphic map (Russian)*, Funktsional. Anal. i Prilozhen. **1:3** (1967), 54–65.

6. M. Passare, *A calculus for meromorphic currents*, J. Reine Angew. Math. **392**(1988), 37–56.

7. J. Solomin, *Le résidu logarithmique dans les intersections non complètes*, C. R. Acad. Sci. Paris **284**(1977), 1061–1064.

8. A. Tsikh, *On the multiplicities of a holomorphic mapping at non-isolated zeros, and the Milnor numbers of a one-dimensional singularity (Russian)*. In: Komleksnyj analiz i matematičeskaja fizika, Physics Institute, Krasnoyarsk, 1988, 113–124.

9. ———, *Multidimensional residues and their applications*, Transl. Math. Monographs **103**, Amer. Math. Soc., Providence, 1992.

Matematiska institutionen
Stockholms universitet
10691 Stockholm
Sweden

Krasnoyarskiĭ gosudarstvennyĭ universitet
Prospekt Svobodnyĭ 79
660 062 Krasnoyarsk
Russia