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Extension of multipliers by periodicity

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A theorem proved by de Leeuw for $\Gamma = R^n$ and later generalized by Lohoué and Saeki states that if Γ is an LCA group, Γ_0 a closed subgroup thereof, π the canonical mapping from Γ onto Γ/Γ_0 and ϕ a Fourier multiplier of type (p, p) on Γ/Γ_0, then $\phi \circ \pi$ is a Fourier multiplier of type (p, p) on Γ. We show here that if $1 \leq p < q \leq \infty$, Γ_0 is a compact subgroup of Γ and ϕ is a Fourier multiplier of type (p, q) on Γ/Γ_0, then $\phi \circ \pi$ is a Fourier multiplier of type (p, q) on Γ; and if Γ_0 is a non-compact subgroup of Γ and $\phi \circ \pi$ is a Fourier multiplier of type (p, q) on Γ for some p and q satisfying $1 \leq p < q \leq \infty$, then ϕ is zero. We prove also that if ϕ is a Fourier multiplier of type (p, q) on Γ/Γ_0, where $1 \leq q < p \leq \infty$ and Γ is discrete, then $\phi \circ \pi$ is a Fourier multiplier of type (p, q) on Γ.

1. Introduction

Before stating our results formally, we introduce some notation. For a topological space X, $C(X)$ denotes the space of continuous complex-valued functions on X, and $C_c(X)$ is the subspace of $C(X)$ consisting of the functions with compact supports. We shall denote by G

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and Γ dual LCA (locally compact abelian) groups with Haar measures dx and $d\chi$ respectively. Haar measures on dual groups will be assumed to be normalised so that the inversion theorem holds. Throughout this paper, p and q are used to denote extended real numbers satisfying $1 \le p \le \infty$ and $1 \le q \le \infty$; in Sections 2 and 3 we assume further that $p < q$, and in Section 4, we shall take $p > q$. The conjugate indices p' and q' are defined by the equations $p'^{-1} + p^{-1} = q'^{-1} + q^{-1} = 1$. $L^p(G)$ and $L^q(G)$ are the usual Lebesgue spaces on G ; $M_{bd}(G)$ is the space of bounded Radon measures on G . The Fourier transform of a function f is denoted by $\hat{f}$; $A(G)$ is the space of Fourier transforms of elements of $L^1(\Gamma)$.

A Fourier multiplier of type (p, q) , hereinafter called a multiplier of type (p, q) , is defined to be a locally integrable function ϕ on Γ such that, for some constant C ,

(1) $$\left|\int_\Gamma \phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| \le C\|f\|_p\|g\|_{q'}$$

for all $f, g \in C_c(G)$. It is assumed that the function $\phi\hat{f}\hat{g}$ is integrable for all $f, g \in C_c(G)$. The space of multipliers of type (p, q) is called $M_p^q(\Gamma)$; $M_p^q(\Gamma)$ is a normed vector space if the norm of ϕ , written $\|\phi\|_{p,q}$, is defined to be the least admissible value of C in the inequality (1), and we identify functions which are equal locally almost everywhere.

If ϕ is a multiplier of type (p, q) , then, for fixed $f \in C_c(G)$, the linear functional

$$\Phi_f : g \to \int_\Gamma \phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi$$

is defined on the dense subspace $C_c(G)$ of $L^{q'}(G)$ if $q \ne 1$, and on the dense subspace $C_c(G)$ of $C_0(G)$ if $q = 1$ ($C_0(G)$ is the space of continuous functions on G which vanish at infinity), and satisfies

$$|\Phi_f(g)| \le \|\phi\|_{p,q}\|f\|_p\|g\|_{q'} .$$

Therefore there exists an element $t_\phi f$ of $L^q(G)$ if $q \ne 1$, and of

$M_{bd}(G)$ if $q = 1$ so that

(2) $$\int_\Gamma \phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi = t_\phi f * g(0) .$$

One may show quite easily that $t_\phi f$ is actually in $L^1(G)$ if $q = 1$, using the fact that the only bounded measures for which translation is a (norm) continuous operation are those generated by integrable functions. Evidently, the operator t_ϕ is linear and

$$\|t_\phi f\|_q \le \|\phi\|_{p,q}\|f\|_p ,$$

so t_ϕ may be extended to a continuous operator from $L^p(G)$ (or $C_0(G)$ if $p = \infty$) to $L^q(G)$ of norm $\|\phi\|_{p,q}$. If $p = \infty$, t_ϕ is in fact continuous from $C_0(G)$ to $L^q(G)$ with the weak topology on $C_0(G)$ induced by $L^1(G)$, and so may be extended to a continuous operator from $L^\infty(G)$ to $L^q(G)$ in this case, as $C_0(G)$ is dense in $L^\infty(G)$ with this weak topology. Now for $h, k \in L^2(G)$, Plancherel's formula may be written in the form

$$h * k(0) = \int_\Gamma \hat{h}(\chi)\hat{k}(\chi)d\chi .$$

Consequently, the formula (2) leads us to believe that, in some sense

(3) $$(t_\phi f)\hat{} = \phi\hat{f} .$$

This formula can be shown to be correct provided a general notion of Fourier transform, involving distributional methods, is used. We have no use for the formula (3), so shall avoid this complication; the interested reader is referred to Gaudry [2] and Gluck [3] for details of distributional Fourier transforms. Our definition of $M_p^q(\Gamma)$ differs somewhat from the conventional definition, which involves these methods - see for example, Gaudry [2] and Hörmander [5]; however, our space is a vector subspace of the usual space, with the same norm, provided that we identify functions which are equal locally almost everywhere.

Let Γ_0 be a closed subgroup of the LCA group Γ, and let π be the canonical mapping of Γ onto Γ/Γ_0. Associated naturally with π is the induced mapping π^*, which maps the measurable function ϕ on Γ/Γ_0 to the measurable function $\phi \circ \pi$ on Γ, effectively "extending ϕ by periodicity". As mentioned above, de Leeuw [1] showed that, for $\Gamma = R^n$, π^* is a continuous mapping of $M_p^p(\Gamma/\Gamma_0)$ into $M_p^p(\Gamma)$, and Lohoué [6] and Saeki [8] extended this result independently to general LCA groups. We prove the following theorems.

THEOREM 1. *If* Γ_0 *is a compact subgroup of* Γ *and* $1 \le p < q \le \infty$, *then* π^* *maps* $M_p^q(\Gamma/\Gamma_0)$ *continuously into* $M_p^q(\Gamma)$, *and for any* $\psi \in M_p^q(\Gamma)$ *which is constant on cosets of* Γ_0 *in* Γ, *there exists* $\phi \in M_p^q(\Gamma/\Gamma_0)$ *such that* $\pi^*\phi = \psi$.

THEOREM 2. *If* Γ_0 *is a non-compact closed subgroup of* Γ *and, for some locally integrable function* ϕ *on* Γ/Γ_0, $\pi^*\phi \in M_p^q(\Gamma)$ *for some* p *and* q *satisfying* $1 \le p < q \le \infty$, *then* $\phi = 0$ *locally almost everywhere.*

THEOREM 3. *If* Γ_0 *is a subgroup of the discrete* LCA *group* Γ, *and* $1 \le q < p \le \infty$, *then* π^* *maps* $M_p^q(\Gamma/\Gamma_0)$ *continuously into* $M_p^q(\Gamma)$, *and for any* $\psi \in M_p^q(\Gamma)$ *which is constant on cosets of* Γ_0 *in* Γ, *there exists* $\phi \in M_p^q(\Gamma/\Gamma_0)$ *such that* $\pi^*\phi = \psi$.

2. Extension over compact subgroups (Theorem 1)

The subgroup Γ_0 is assumed to be compact, and so G_0, its annihilator in G, is an open subgroup of G. The Haar measure dx_0 of G_0 may therefore be taken to be that of G restricted to G_0. If we assign to G/G_0 the natural measure $d\dot{x}$ ascribing unit mass to each point of G/G_0, then for any $f \in C_c(G)$,

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$$\int_G f(x)dx = \int_{G/G_0}\left[\int_{G_0} f(x+x_0)dx_0\right]d\dot{x} .$$

Our assumption that the inversion theorem holds for dual pairs of groups implies that Γ_0 has total Haar measure one, and for any $\gamma \in C_c(\Gamma)$,

$$\int_\Gamma \gamma(\chi)d\chi = \int_{\Gamma/\Gamma_0}\left[\int_{\Gamma_0} \gamma(\chi+\chi_0)d\chi_0\right]d\dot{\chi} ,$$

(with the obvious notation). It is well-known that if $f \in C_c(G)$ is supported in G_0 , then $\hat{f} = \pi^*\hat{f}_0$, f_0 denoting the function f restricted to G_0 (whose dual is Γ/Γ_0).

Let f and g be in $C_c(G)$. Since G_0 is an open subgroup of G , the supports of f and g have non-void intersection with only a finite number of cosets of G_0 in G . Therefore there exist an integer n , points $x_1, x_2, \ldots, x_n$ in G and functions

$f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n$ in $C_c(G)$ such that $f = \sum_{j=1}^{n} f_j$,

$g = \sum_{k=1}^{n} g_k$, f_j is supported in $x_j + G_0$, g_k is supported in $-x_k + G_0$ and the sets $x_j + G_0$ $(j = 1, 2, \ldots, n)$ are pairwise disjoint. Then, denoting by T_x the translation operator $T_x f(y) = f(y-x)$, we see that

$$\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi$$

$$= \sum_{j,k=1}^{n} \int_\Gamma \pi^*\phi(\chi)\hat{f}_j(\chi)\hat{g}_k(\chi)d\chi$$

$$= \sum_{j,k=1}^{n} \int_\Gamma \pi^*\phi(\chi)\overline{x_j(\chi)}\left(T_{-x_j}f_j\right)^\wedge(\chi)x_k(\chi)\left(T_{x_k}g_k\right)^\wedge(\chi)d\chi$$

$$= \sum_{j,k=1}^{n} \int_\Gamma \pi^*\phi(\chi)\overline{x_j(\chi)}\pi^*\left(T_{-x_j}f_j\right)\hat{}_0(\chi)x_k(\chi)\pi^*\left(T_{x_k}g_k\right)\hat{}_0(\chi)d\chi$$

$$= \sum_{j,k=1}^{n} \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\left(T_{-x_j}f_j\right)\hat{}_0(\dot{\chi})\left(T_{x_k}g_k\right)\hat{}_0(\dot{\chi})\left[\int_{\Gamma_0} \left(\overline{x_j}\cdot x_k\right)(\chi+\chi_0)d\chi_0\right]d\dot{\chi} .$$

It is known that, if K is a compact LCA group, ξ a character of K, and dy its Haar measure, then $\int_K \xi(y)dy = 0$ unless $\xi(y) = 1$ for all $y \in K$. Thus

$$(4) \qquad \int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi = \sum_{j=1}^{n} \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\left(T_{-x_j}f_j\right)\hat{}_0(\dot{\chi})\left(T_{x_j}g_j\right)\hat{}_0(\dot{\chi})d\dot{\chi}$$

and so

$$\begin{aligned}\left|\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| &\le \sum_{j=1}^{n} \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\left(T_{-x_j}f_j\right)\hat{}_0(\dot{\chi})\left(T_{x_j}g_j\right)\hat{}_0(\dot{\chi})d\dot{\chi}\right| \\ &\le \sum_{j=1}^{n} \|\phi\|_{p,q}\left\|\left(T_{-x_j}f_j\right)0\right\|_p\left\|\left(T_{x_j}g_j\right)0\right\|_{q'} \\ &\le \|\phi\|_{p,q}\left[\sum_{j=1}^{n} \left\|\left(T_{-x_j}f_j\right)0\right\|_p^p\right]^{1/p}\left[\sum_{j=1}^{n} \left\|T_{x_j}g_j\right\|_{q'}^{p'}\right]^{1/p'}\end{aligned}$$

by Hölder's inequality. The space $l^{q'}$ is contained continuously in $l^{p'}$ since $q' < p'$, and so

$$(5) \qquad \left|\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| \le \|\phi\|_{p,q}\left[\sum_{j=1}^{n} \left\|T_{-x_j}f_j\right\|_p^p\right]^{1/p}\left[\sum_{j=1}^{n} \left\|T_{x_j}g_j\right\|_{q'}^{q'}\right]^{1/q'} = \|\phi\|_{p,q}\|f\|_p\|g\|_{q'},$$

if $p > 1$. A similar argument applies if $p = 1$, and so, as claimed, $\pi^*\phi \in M_p^q(\Gamma)$, and $\|\pi^*\phi\|_{p,q} \le \|\phi\|_{p,q}$.

To conclude the proof of the theorem, suppose that ϕ is a function on Γ/Γ_0 such that $\psi = \pi^*\phi \in M_p^q(\Gamma)$. Let f_0 and g_0 be in $C_c(G_0)$ and denote by f and g the functions in $C_c(G)$ which are supported in G_0 and agree with G_0 and f_0 on G_0. Then

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$$(6)\quad \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right|$$
$$= \left|\int_{\Gamma/\Gamma_0}\left[\int_{\Gamma_0} \pi^*\phi(\chi+\chi_0)\pi^*\hat{f}_0(\chi+\chi_0)\pi^*\hat{g}_0(\chi+\chi_0)d\chi_0\right]d\dot{\chi}\right|$$
$$= \left|\int_{\Gamma} \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right|$$
$$\leq \|\psi\|_{p,q}\|f\|_p\|g\|_{q'}$$
$$= \|\psi\|_{p,q}\|f_0\|_p\|g_0\|_{q'} ,$$

completing the proof of the theorem.

It is perhaps worthy of note that π^* is an isometry from $M_p^p(\Gamma/\Gamma_0)$ to $M_p^p(\Gamma)$ regardless of the normalisations of the Haar measures concerned. However, if $p < q$, the norm of $\pi^*\phi$ is dependent on the choice of Haar measure on Γ_0. This is little more than a restatement of the fact that if T is a continuous linear operator from $L^p(G)$ to $L^q(G)$ and the Haar measure of G is changed from dx to Cdx, then the norm of T changes from $\|T\|$ to $C^{1/q-1/p}\|T\|$.

3. Extension over non-compact subgroups (Theorem 2)

Theorem 2 is proved by contradiction. Estimates akin to those for the Dirichlet kernel are used to invalidate inequality (8) below. We prove the theorem first for continuous functions ϕ, and then generalize our result.

LEMMA. *Suppose that* $\phi \in C(\Gamma/\Gamma_0)$ *and* $\pi^*\phi \in M_p^q(\Gamma)$, *where* Γ_0 *is a non-compact subgroup of* Γ, *and* $p < q$. *Then* $\phi = 0$.

Proof. Suppose that $\phi \neq 0$. Evidently, without any loss of generality, we may suppose that

(a) $\phi(0) \neq 0$ since $T_\chi\pi^*\phi = \pi^*T_{\dot{\chi}}\phi \in M_p^q(\Gamma)$ if $\pi^*\phi \in M_p^q(\Gamma)$;

(b) $\phi(0) = 1$, since $M_p^q(\Gamma)$ is stable under scalar multiplication;

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(c) ϕ is real-valued, since if $\pi^*\phi \in M_p^q(\Gamma)$, $\overline{\pi^*\phi} \in M_p^q(\Gamma)$ and so $\frac{\pi^*\phi+\overline{\pi^*\phi}}{2} \in M_p^q(\Gamma)$.

Because ϕ is continuous, there exists a neighbourhood V of $\dot{0}$ in Γ/Γ_0 such that $\phi(\dot{\chi}) > 0$ for any $\dot{\chi} \in V$. Let $v \in A(\Gamma/\Gamma_0)$ be a non-negative function which is supported in V , does not vanish at $\dot{0}$, and satisfies $\|v\|_{A(\Gamma/\Gamma_0)} = 1$. Therefore [7, 4.1.3], π^*v is the Fourier transform of a bounded measure μ on G , and $\int_G |d\mu| = 1$. If $f \in C_c(G)$, $f * \mu \in L^1 \cap C(G)$, and so, for some real constant C_1 depending only on ϕ ,

$$\left|\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{\mu}(\chi)\hat{f}(\chi)d\chi\right| \le C_1\|f*\mu\|_p\|f\|_q, \\ \le C_1\|f\|_p\|f\|_q,$$

by an obvious extension of the inequality (1).

Let $f \in C_c(G)$ be such that $\hat{f}$ is non-negative and non-vanishing at 0 . Given a Γ_0-valued sequence $(\chi_j)_{j\in Z^+}$ (Z^+ is the set of positive integers), define the $C_c(G)$-valued sequence $(f_n)_{n\in Z^+}$

$$f_n = \left(\sum_{j=1}^{n} \chi_j\right)f \ , \tag{7}$$

the summation being a sum of functions and not the group-theoretic sum. Then $\hat{f}_n = \sum_{j=1}^{n} T_{\chi_j}\hat{f}$, so

$$\left|\int_\Gamma \pi^*\phi(\chi)\hat{f}_n(\chi)\hat{\mu}(\chi)\hat{f}_n(\chi)d\chi\right| = \int_\Gamma \pi^*\phi(\chi)\pi^*v(\chi) \sum_{j=1}^{n} T_{\chi_j}\hat{f}(\chi) \sum_{k=1}^{n} T_{\chi_k}\hat{f}(\chi)d\chi \\ \ge \int_\Gamma \pi^*\phi(\chi)\pi^*v(\chi) \sum_{j=1}^{n} \left[T_{\chi_j}\hat{f}(\chi)\right]^2 d\chi \\ = n\int_\Gamma \pi^*\phi(\chi)\pi^*v(\chi)[\hat{f}(\chi)]^2 d\chi \ ,$$

since $\pi^*\phi$ and $\pi^*\upsilon$ are constant on cosets of Γ_0 in Γ. This last integral is non-zero, so for some constant C_2 independent of n,

(8) $$n \le C_2\|f_n\|_p\|f_n\|_{q'} .$$

We show first that Γ_0 cannot contain a discrete subgroup isomorphic to the integers Z. Suppose that Γ_0 contains a discrete subgroup isomorphic to Z; let ξ_0 be a generator of this subgroup, and set $\chi_j = \xi_0^{j-1}$ for $j \in Z^+$. Then, with the notation (7),

$$\begin{aligned} f_n &= \left(\sum_{j=1}^{n} \chi_j\right)f \\ &= \left(\sum_{j=0}^{n-1} \xi_0^j\right)f \\ &= \frac{\left(1-\xi_0^n\right)}{1-\xi_0} f , \end{aligned}$$

unless $\xi_0(x) = 1$, in which case $f_n(x) = nf(x)$. It transpires that the kernel $\ker\xi_0$ of ξ_0 is of measure zero, so we may neglect this possibility. If the Haar measures dt and $d\dot{x}$ of $\ker\xi_0$ and $G/\ker\xi_0$ are appropriately normalised,

$$\begin{aligned} \|f_n\|_p^p &= \int_G \left|\frac{1-\xi_0^n(x)}{1-\xi_0(x)}\right|^p |f(x)|^p dx \\ &= \int_{G/\ker\xi_0} \left|\frac{1-\xi_0^n(\dot{x})}{1-\xi_0(\dot{x})}\right|^p \left[\int_{\ker\xi_0} |f(x+t)|^p dt\right] d\dot{x} . \end{aligned}$$

Now $|f|^p \in C_c(G)$ and so the function $F : \dot{x} \to \int_{\ker\xi_0} |f(x+t)|^p dt$ is continuous and has compact support [4, 15.21] and is therefore bounded. So for some C_3 depending on f and p but not on n,

(9) $$\|f_n\|_p^p \le C_3 \int_{G/\ker\xi_0} \left|\frac{1-\xi_0^n(\dot{x})}{1-\xi_0(\dot{x})}\right|^p d\dot{x} .$$

The annihilator in G of the subgroup generated by ξ_0 is just $\ker\xi_0$, and so $G/\ker\xi_0$ is isomorphic (topologically and algebraically) to the dual group of Z, namely, the circle group. Therefore

$$\|f_n\|_p^p \le C_3 \int_0^{2\pi} \left|\frac{1-\exp[int]}{1-\exp[it]}\right|^p dt$$
$$= 2C_3 \int_0^{\pi} \left|\frac{1-\exp[int]}{1-\exp[it]}\right|^p dt .$$

The following estimates are readily obtained:

$$\left|\frac{1-\exp[int]}{1-\exp[it]}\right| \le n \quad \text{for} \quad t \in \left(0, \frac{2\pi}{n}\right) ,$$

and

$$\left|\frac{1-\exp[int]}{1-\exp[it]}\right| \le \frac{2}{2\sin(t/2)} \le \frac{\pi}{t} \quad \text{for} \quad t \in \left[\frac{2\pi}{n}, \pi\right] .$$

Hence

$$\|f_n\|_p^p \le 2C_3\pi\left[2n^{p-1} + \frac{(n/2)^{p-1}-1}{p-1}\right] \quad \text{if} \quad 1 < p < \infty$$

and

$$\|f_n\|_1 \le 2C_3\pi[2+\log(n/2)] ;$$

that is,

(10) $$\|f_n\|_p = O(\log n) \quad \text{as} \quad n \to \infty \quad \text{if} \quad p = 1 ,$$

and

(11) $$\|f_n\|_p = O\left(n^{1/p'}\right) \quad \text{as} \quad n \to \infty \quad \text{if} \quad p > 1 .$$

Since $p < q$, $p'^{-1} + q^{-1} = 1 - p^{-1} + q^{-1} < 1$, so $\|f_n\|_p\|f_n\|_{q'} = o(n)$ as $n \to \infty$, contradicting the inequality (8). Thus Γ_0 cannot contain a discrete subgroup isomorphic to Z.

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By a well-known structure theorem [4, 9.8], Γ_0 contains an open subgroup of the form $R^n + K$, where K is compact. Since Γ_0 cannot contain a discrete subgroup isomorphic to Z, $n = 0$, that is, Γ_0 contains a compact open subgroup. Denote Γ/K and Γ_0/K by Γ' and Γ'_0 respectively; Γ_0/K is discrete because K is an open subgroup of Γ_0, and because Γ_0 is not compact, Γ_0/K is infinite. Topologically and algebraically, Γ/Γ_0 is isomorphic to Γ'/Γ'_0, so any continuous function ϕ on Γ/Γ_0 naturally defines a continuous function ϕ' on Γ'/Γ'_0. Further, $\pi*\phi$ is in $M_p^q(\Gamma)$ and is constant on cosets in Γ of the compact subgroup K, so by Theorem 1, $\phi' \circ \pi' \in M_p^q(\Gamma')$, where π' is the canonical mapping of Γ' onto Γ'/Γ'_0. Thus, if $0 \neq \pi*\phi \in M_p^q(\Gamma)$, there exists a group Γ' with an infinite discrete subgroup Γ'_0 and $\phi' \in C(\Gamma'/\Gamma'_0)$ such that $\phi' \circ \pi' \in M_p^q(\Gamma')$. Every element of Γ'_0 must be of finite order (since Γ'_0 cannot contain a subgroup isomorphic to Z); thus we may, without loss of generality, assume that $0 \neq \phi \in C(\Gamma/\Gamma_0)$, $\pi*\phi \in M_p^q(\Gamma)$, and Γ_0 is an infinite discrete subgroup, every element of which is of finite order.

Since Γ_0 is infinite, there exists a sequence $(\xi_j)_{j\in Z^+}$ of elements of Γ_0 so that ξ_{k+1} is not a member of the finite group Λ_k of order $m(k)$ generated by the elements $\xi_1, \xi_2, \ldots, \xi_k$. Denote by Λ the group $\bigcup_{k=1}^{\infty} \Lambda_k$, and let $(\chi_j)_{j\in Z^+}$ be an enumeration of the elements of Λ so that

$$\{\chi_j : 1 \leq j \leq m(k)\} = \{\chi : \chi \in \Lambda_k\} .$$

Let H be the annihilator of Λ in G; Λ is discrete, so G/H is compact. Then, as argued for (9)

(12) $$\|f_n\|_p^p \le C_4 \int_{G/H} \left| \sum_{j=1}^{n} \chi_j(\dot{x}) \right|^p d\dot{x} \;;$$

in particular, taking $n = m(k)$,

$$\|f_{m(k)}\|_p^p \le C_4 \int_{G/H} \left| \sum_{j=1}^{m(k)} \chi_j(\dot{x}) \right|^p d\dot{x} \;.$$

We assume that the Haar measure of G/H is normalised so that G/H has measure one. It is easily checked that the Fourier transform of the function F_k on G/H defined to be $m(k)$ times the characteristic function of H_k/H , where H_k is the annihilator of Λ_k in G , is just the characteristic function of Λ_k ; that is, $F_k(\dot{x}) = \sum_{j=1}^{m(k)} \chi_j(\dot{x})$. It follows immediately that

$$\begin{aligned} \|f_{m(k)}\|_p^p &\le C_4 \int_{G/H} |F_k(\dot{x})|^p d\dot{x} \\ &= C_4 m(k)^{p-1} \;, \end{aligned}$$

that is,

(13) $$\|f_{m(k)}\|_p = O\left(m(k)^{1/p'}\right) \;.$$

But from (8),

$$\begin{aligned} m(k) &\le C_2 \|f_{m(k)}\|_p \|f_{m(k)}\|_{q'} \\ &= o\left(m(k)\right) \quad \text{as} \quad m(k) \to \infty \;, \end{aligned}$$

so we have produced the desired contradiction, and the lemma is proved.

Proof of Theorem 2. Suppose that ϕ is a locally integrable function such that $\pi^*\phi \in M_p^q(\Gamma)$. Then if $f, g \in C_c(G)$ and $\|f\|_p = \|g\|_{q'} = 1$,

$$\left| \int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi \right| \le \|\pi^*\phi\|_{p,q} \;.$$

Since $M_p^q(\Gamma)$ is stable under translation, for any $\dot{\xi} \in \Gamma/\Gamma_0$,

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$$\left|\int_\Gamma \pi^* T_{\dot\xi}\phi(\chi)\hat f(\chi)\hat g(\chi)d\chi\right| \le \|\pi^*\phi\|_{p,q} ,$$

so

$$\left|\int_{\Gamma/\Gamma_0} \phi(\dot\chi-\dot\xi)\left[\int_{\Gamma_0} \hat f(\chi+\chi_0)\hat g(\chi+\chi_0)d\chi_0\right]d\dot\chi\right| \le \|\pi^*\phi\|_{p,q} ,$$

whence

$$\left|\int_{\Gamma/\Gamma_0} \phi(\dot\chi-\dot\xi)\gamma(\dot\xi)\left[\int_{\Gamma_0} \hat f(\chi+\chi_0)\hat g(\chi+\chi_0)d\chi_0\right]d\dot\chi\right| \le \|\pi^*\phi\|_{p,q}|\gamma(\dot\xi)|$$

for any $\gamma \in C_c(\Gamma/\Gamma_0)$ and $\dot\xi \in \Gamma/\Gamma_0$. Integrating with respect to $\dot\xi$ over Γ/Γ_0 and applying Fubini's Theorem, we see that

$$\left|\int_{\Gamma/\Gamma_0} \phi * \gamma(\dot\chi)\left[\int_{\Gamma_0} \hat f(\chi+\chi_0)\hat g(\chi+\chi_0)d\chi_0\right]d\dot\chi\right| \le \|\pi^*\phi\|_{p,q}\|\gamma\|_1 ,$$

that is,

$$\left|\int_\Gamma \pi^*(\phi * \gamma)(\chi)\hat f(\chi)\hat g(\chi)d\chi\right| \le \|\pi^*\phi\|_{p,q}\|\gamma\|_1 .$$

Thus $\pi^*(\phi * \gamma) \in M_p^q(\Gamma)$ for any $\phi \in C_c(\Gamma/\Gamma_0)$. Furthermore, $\phi * \gamma$ is a continuous function for any $\gamma \in C_c(\Gamma/\Gamma_0)$, and so $\phi * \gamma$ is zero by the lemma. Hence $\phi = 0$ locally almost everywhere, completing the theorem.

4. Extension in discrete groups (Theorem 3)

The proof of Theorem 3 is similar to that of Theorem 1.

Let Γ_0 be a subgroup of the discrete group Γ, and G_0 its annihilator in G. We may assume that the Haar measures $d\chi$, $d\chi_0$ and $d\dot\chi$ of Γ, Γ_0 and Γ/Γ_0 respectively assign unit mass to each point of these groups; then, for any $\gamma \in C_c(\Gamma)$,

$$\int_\Gamma \gamma(\chi)d\chi = \int_{\Gamma/\Gamma_0}\left[\int_{\Gamma_0} \gamma(\chi+\chi_0)d\chi_0\right]d\dot\chi .$$

Our assumption that the inversion theorem holds for dual pairs of groups implies that the Haar measures of G, G/G_0 and G_0 are normalised so

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that each group has unit measure. Further, for any $f \in C(G)$,

$$\int_G f(x)dx = \int_{G/G_0} \left[\int_{G_0} f(x+x_0)\,dx_0\right] d\dot{x} .$$

Because G is compact, the set of trigonometric polynomials on G , denoted by $T(G)$, is dense in $C(G)$. Thus $\psi \in M_p^q(\Gamma)$ if and only if there exists a constant C such that

(14) $$\left|\int_\Gamma \psi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| \le C\|f\|_p\|g\|_q$$

for all $f, g \in T(G)$, and the least possible value of C in the inequality is $\|\psi\|_{p,q}$. An analogous criterion, to decide whether a function ϕ on Γ/Γ_0 belongs to $M_p^q(\Gamma/\Gamma_0)$, will also be used.

Let $h \in A(G)$ and $x_0 \in G_0$. By the inversion theorem,

$$\begin{aligned} h(x_0) &= \int_{\Gamma/\Gamma_0} \left[\int_{\Gamma_0} (\chi+\chi_0)(x_0)\hat{h}(\chi+\chi_0)d\chi_0\right] d\dot{\chi} \\ &= \int_{\Gamma/\Gamma_0} \dot{\chi}(x_0)\left[\int_{\Gamma_0} \hat{h}(\chi+\chi_0)d\chi_0\right] d\dot{\chi} , \end{aligned}$$

since Γ_0 annihilates G_0 . The function h_0 , defined to be h restricted to G_0 , therefore satisfies

$$h_0\hat{}(\dot{\chi}) = \int_{\Gamma_0} \hat{h}(\chi+\chi_0)d\chi_0 .$$

In particular, if $f, g \in C(G)$, then $f * g \in A(G)$, so

(15) $$(f * g)\hat{}_0(\dot{\chi}) = \int_{\Gamma_0} \hat{f}(\chi+\chi_0)\hat{g}(\chi+\chi_0)d\chi_0 .$$

Therefore, if $f, g \in T(G)$, the orthogonality relations show that

$$\begin{aligned} (f * g)\hat{}_0(\dot{\chi}) &= \int_{\Gamma_0} \left[\int_{\Gamma_0} \int_G \hat{f}(\chi+\chi_0)\hat{g}(\chi+\xi_0)\overline{(\chi+\chi_0)(y)}(\chi+\xi_0)(y)dy\,d\xi_0\right] d\chi_0 \\ &= \int_G \left[\int_{\Gamma_0} \hat{f}(\chi+\chi_0)\overline{(\chi+\chi_0)(y)}d\chi_0\right]\left[\int_{\Gamma_0} \hat{g}(\chi+\xi_0)(\chi+\xi_0)(y)d\xi_0\right] dy \\ &= \int_G (T_yf)\hat{}_0(\dot{\chi})(T_{-y}g)\hat{}_0(\dot{\chi})dy . \end{aligned}$$

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So, using Fubini's Theorem,

$$\left|\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| = \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\left[\int_{\Gamma_0} \hat{f}(\chi+\chi_0)\hat{g}(\chi+\chi_0)d\chi_0\right]d\dot{\chi}\right|$$

$$= \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\left[\int_G (T_yf)_0^\wedge(\dot{\chi})(T_{-y}g)_0^\wedge(\dot{\chi})dy\right]d\dot{\chi}\right|$$

$$= \left|\int_G \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})(T_yf)_0^\wedge(\dot{\chi})(T_{-y}g)_0^\wedge(\dot{\chi})d\dot{\chi}dy\right|$$

$$\le \int_G \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})(T_yf)_0^\wedge(\dot{\chi})(T_{-y}g)_0^\wedge(\dot{\chi})d\dot{\chi}\right|dy$$

$$\le \int_G \|\phi\|_{p,q}\left\|(T_yf)_0\right\|_p\left\|(T_{-y}g)_0\right\|_{q'}dy$$

$$\le \|\phi\|_{p,q}\left[\int_G \left\|(T_yf)_0\right\|_p^p dy\right]^{1/p}\left[\int_G \left\|(T_{-y}g)_0\right\|_{q'}^{p'} dy\right]^{1/p'} ,$$

by Hölder's inequality. Since G is compact and $p' < q'$, $L^{p'}(G)$ is contained continuously in $L^{q'}(G)$, so, if $p < \infty$ and $q' < \infty$,

(16) $$\left|\int_\Gamma \pi^*\phi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right|$$

$$\le \|\phi\|_{p,q}\left[\int_G \left\|(T_yf)_0\right\|_p^p dy\right]^{1/p}\left[\int_G \left\|(T_{-y}g)_0\right\|_{q'}^{q'} dy\right]^{1/q'}$$

$$= \|\phi\|_{p,q}\left[\int_G \int_{G_0} |f(x_0-y)|^p dx_0dy\right]^{1/p}\left[\int_G \int_{G_0} |g(x_0+y)|^{q'} dx_0dy\right]^{1/p'}$$

$$= \|\phi\|_{p,q}\|f\|_p\|g\|_{q'} ,$$

by Fubini's Theorem, the translation and reflection invariance of Haar measures, and the normalisation of the Haar measure of G_0. If $p = \infty$ or $q' = \infty$, a similar argument will give the appropriate estimate. The first half of the theorem is now proved.

We show now that if $\pi^*\phi \in M_p^q(\Gamma)$, then $\phi \in M_p^q(\Gamma/\Gamma_0)$, and $\|\phi\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. Since we have just shown that $\|\pi^*\phi\|_{p,q} \le \|\phi\|_{p,q}$, it will follow that π^* is an isometry (with the normalisation we have assumed). An inductive argument is employed.

Suppose that Γ_0 is a finite group of order n; then G/G_0 is a

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finite group of order n. Let $\{x_1, x_2, \ldots, x_n\}$ be any subset of G containing exactly one element of each coset of G_0 in G. If f_0 is any member of $C(G_0)$, we shall write f for the function on G which is supported in G_0 and coincides there with f_0; also, if h is any member of $C(G)$, we shall denote by h_0 its restriction to G_0. We note that dx_0, the Haar measure on G_0, is just n times the restriction to G_0 of the Haar measure on G. Consequently, for $f_0, g_0 \in C(G_0)$,

(17) $$\|f_0\|_p = n^{1/p}\|f\|_p$$

and

(18) $$n(f * g)_0 = f_0 * g_0 .$$

Then

$$\int_\Gamma \pi*\phi(\chi)\left[\sum_{j=1}^n \left(T_{x_j}f\right)^\wedge(\chi)\right]\left[\sum_{k=1}^n \left(T_{-x_k}g\right)^\wedge(\chi)\right]d\chi$$
$$= \sum_{j=1}^n \sum_{k=1}^n \int_{\Gamma/\Gamma_0} \phi(\dot\chi)\left[\int_{\Gamma_0} \left(T_{x_j}f\right)^\wedge(\chi+\chi_0)\left(T_{-x_k}g\right)^\wedge(\chi+\chi_0)d\chi_0\right]d\dot\chi$$
$$= \sum_{j=1}^n \sum_{k=1}^n \int_{\Gamma/\Gamma_0} \phi(\dot\chi)\left(T_{x_j}f * T_{-x_j}g\right)_0^\wedge(\dot\chi)d\dot\chi ,$$

by (15). $\left(T_{x_j}f * T_{-x_k}g\right)_0 = 0$ unless $j = k$, and $T_{x_j}f * T_{-x_j}g = f * g$, so

$$\int_\Gamma \pi*\phi(\chi)\left[\sum_{j=1}^n \left(T_{x_j}f\right)^\wedge(\chi)\right]\left[\sum_{k=1}^n \left(T_{-x_k}g\right)^\wedge(\chi)\right]d\chi = \sum_{j=1}^n \int_{\Gamma/\Gamma_0} \phi(\dot\chi)(f * g)_0^\wedge(\dot\chi)d\dot\chi$$
$$= \int_{\Gamma/\Gamma_0} \phi(\dot\chi)n(f * g)_0^\wedge(\dot\chi)d\dot\chi$$
$$= \int_{\Gamma/\Gamma_0} \phi(\dot\chi)\hat f_0(\dot\chi)\hat g_0(\dot\chi)d\dot\chi ,$$

by (18). Therefore

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$$(20) \quad \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})f_0^\wedge(\dot{\chi})g_0^\wedge(\dot{\chi})d\dot{\chi}\right|$$

$$= \left|\int_\Gamma \pi^*\phi(\chi)\left[\sum_{j=1}^n \left(T_{x_j}f\right)^\wedge(\chi)\right]\left[\sum_{k=1}^n \left(T_{-x_k}g\right)^\wedge(\chi)\right]d\chi\right|$$

$$\leq \|\pi^*\phi\|_{p,q}\left\|\sum_{j=1}^n T_{x_j}f\right\|_p\left\|\sum_{k=1}^n T_{-x_k}g\right\|_{q'},$$

$$= \|\pi^*\phi\|_{p,q}n^{1/p}\|f\|_p n^{1/q'}\|g\|_{q'},$$

$$= \|\pi^*\phi\|_{p,q}\|f_0\|_p\|g_0\|_{q'} ,$$

the penultimate step because the functions $T_{x_j}f$ $(j = 1, 2, \ldots, n)$ have pairwise disjoint supports, and the last step by (17). This establishes the second half of the theorem if Γ_0 is a finite group.

Assume now that $\Gamma_0 = Z$ (the integers), and hence that $G/G_0 = T$, the circle group, which we view as the unit circle in the complex plane. For any positive integer n, let $S_n = \{x_1, x_2, \ldots, x_n\}$ be a subset of G such that

$$\dot{x}_k = \exp\left[\frac{2\pi ik}{n}\right] , \quad k = 1, 2, \ldots, n .$$

Define the subintervals I_n and J_n of the real line:

$$I_n = \left[\frac{\pi}{n^2} - \frac{\pi}{n}, \frac{\pi}{n} - \frac{\pi}{n^2}\right] ,$$

$$J_n = \left(-\frac{\pi}{n}, \frac{\pi}{n}\right) ,$$

and set $K_n = \exp\left[iI_n\right]$ and $U_n = \exp\left[iJ_n\right]$. Then K_n is a compact subset of T containing the identity and contained in the open set U_n. Let ψ_n be a continuous function supported in U_n satisfying $0 \leq \psi_n \leq 1$ and $\psi_n\left(K_n\right) = \{1\}$. Denote by ψ_n' the periodic extension of ψ_n to G; this is constant on cosets of G_0. If $h \in C(G)$, then

$$n\int_G \psi_n'(x)h(x)dx = n\int_{G/G_0} \psi_n(\dot{x})\left[\int_{G_0} h\left(x+x_0\right)dx_0\right]d\dot{x} .$$

The integral inside the brackets is a continuous function of $\dot{x}$, and

$$1 - 1/n \le n \int_{G/G_0} \psi_n(\dot{x})d\dot{x} \le 1 \ ,$$

so, as $n \to \infty$, the limit $\lim_{n\to\infty} n \int_G \psi_n'(x)h(x)dx$ exists, and

$$\lim_{n\to\infty} n \int_G \psi_n'(x)h(x)dx = \int_{G_0} h(x_0)dx_0 \ . \tag{21}$$

Let $f_0, g_0 \in T(G_0)$, and let f and g be trigonometric polynomials on G which agree with f_0 and g_0 respectively on G_0 . Put $f_n = \psi_n'f$, $g_n = \psi_n'g$, $F_n = \sum_{j=1}^{n} T_{x_j}f_n$ and $G_n = \sum_{k=1}^{n} T_{-x_k}g_n$. Let $(h_l)_{l\in L}$ be an approximate identity in $L^1(G_0)$ so that $\|h_l\|_1 \le 1$ and $\hat{h}_l \in C_c(\Gamma/\Gamma_0)$. Then

$$\begin{aligned}\int_\Gamma \pi^*\phi(\chi)\pi^*\hat{h}_l(\chi)\hat{F}_n(\chi)\hat{G}_n(\chi)d\chi &= \sum_{j=1}^{n}\sum_{k=1}^{n} \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{h}_l(\dot{\chi})\left[\int_{\Gamma_0} \left(T_{x_j}f_n\right)^\wedge(\chi+\chi_0)\left(T_{-x_k}g_n\right)^\wedge(\chi+\chi_0)d\chi_0\right]d\dot{\chi} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{h}_l(\dot{\chi})\left(T_{x_j}f_n * T_{-x_k}g_n\right)_0^\wedge(\dot{\chi})d\dot{\chi} \ ,\end{aligned}$$

by (15). $\left(T_{x_j}f_n * T_{-x_k}g_n\right)_0 = 0$ unless $j = k$, and so

$$\int_\Gamma \pi^*\phi(\chi)\pi^*\hat{h}_l(\chi)\hat{F}_n(\chi)\hat{G}_n(\chi)d\chi = n \int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{h}_l(\dot{\chi})(f_n * g_n)_0^\wedge(\dot{\chi})d\dot{\chi} \ . \tag{22}$$

By an obvious analogue of (21), if $x_0 \in G_0$,

$$\begin{aligned}\lim_{n\to\infty} n(f_n * g_n)(x_0) &= \int_{G_0} f(x_0-y_0)g(y_0)dy_0 \\ &= f_0 * g_0(x_0) \ ,\end{aligned}$$

and

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$$\begin{aligned}|n(f_n * g_n)(x_0)| &\le n\|f_n * g_n\|_\infty \\ &\le n\|f_n\|_1\|g_n\|_\infty \\ &\le \|f\|_\infty\|g\|_\infty ,\end{aligned}$$

so by Lebesgue's Dominated Convergence Theorem, $n(f_n * g_n)_0$ converges in $L^1(G_0)$ to $f_0 * g_0$. Consequently $n(f_n * g_n)\hat{}_0$ converges pointwise to $\hat{f}_0\hat{g}_0$. The group G is compact and so $\pi^*\phi \in M_p^q(\Gamma)$ implies that $\pi^*\phi$ (and hence ϕ) is bounded. Further, $\hat{h}_l$ has finite support, whence

$$\begin{aligned}\lim_{n\to\infty}\int_\Gamma \pi^*\phi(\chi)\pi^*\hat{h}_l(\chi)\hat{F}_n(\chi)\hat{G}_n(\chi)d\chi &= \lim_{n\to\infty}\int_{\Gamma/\Gamma_0}\phi(\dot{\chi})\hat{h}_l(\dot{\chi})n(f_n * g_n)\hat{}_0(\dot{\chi})d\dot{\chi} \\ &= \int_{\Gamma/\Gamma_0}\phi(\dot{\chi})\hat{h}_l(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi} .\end{aligned}$$

Therefore

$$\begin{aligned}\left|\int_{\Gamma/\Gamma_0}\phi(\dot{\chi})\hat{h}_l(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right| &\le \liminf_{n\to\infty}\left|\int_\Gamma \pi^*\phi(\chi)\pi^*\hat{h}_l(\chi)\hat{F}_n(\chi)\hat{G}_n(\chi)d\chi\right| \\ &\le \liminf_{n\to\infty}\|\pi^*\phi\|_{p,q}\|\mu_l * F_n\|_p\|G_n\|_{q'} ,\end{aligned}$$

where μ_l denotes the measure whose Fourier transform is $\pi^*\hat{h}_l$ and whose norm is $\|h_l\|_1$ [7, 4.1.3]. Then

$$\begin{aligned}\left|\int_{\Gamma/\Gamma_0}\phi(\dot{\chi})\hat{h}_l(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right| &\le \liminf_{n\to\infty}\|\pi^*\phi\|_{p,q}\|h_l\|_1\|F_n\|_p\|G_n\|_{q'} \\ &\le \liminf_{n\to\infty}\|\pi^*\phi\|_{p,q}n^{1/p}\|f_n\|_p n^{1/q'}\|g_n\|_{q'} \\ &= \|\pi^*\phi\|_{p,q}\|f_0\|_p\|g_0\|_{q'} ,\end{aligned}$$

since $\lim_{n\to\infty} n^{1/p}\|f_n\|_p = \|f_0\|_p$ by (21). The net $(h_l)_{l\in L}$ is an approximate identity, so $\hat{h}_l$ converges pointwise to 1. $\hat{f}_0$ has finite support, and so

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$$(23)\quad \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right| = \lim_L \left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{h}_l(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right|$$
$$\le \|\pi^*\phi\|_{p,q}\|f_0\|_p\|g_0\|_{q'} ,$$

completing the proof in the case $\Gamma_0 = Z$.

Suppose now that whenever Γ_0 is finitely generated by at most m elements $(m \ge 1)$ and $\pi^*\phi \in M_p^q(\Gamma)$, then $\phi \in M_p^q(\Gamma/\Gamma_0)$ and $\|\phi\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. Let Γ_0 be a group generated by $(m+1)$ elements $\chi_1, \chi_2, \ldots, \chi_{m+1}$. The inductive hypothesis shows that, if Γ_1 is the group generated by $\chi_1, \chi_2, \ldots, \chi_m$, the function ψ on Γ/Γ_1 , obtained in the natural way from $\pi^*\phi$, belongs to $M_p^q(\Gamma/\Gamma_1)$ and that $\|\psi\|_{p,q} \le \|\pi^*\phi\|_{p,q}$, provided that, as usual, the discrete groups Γ/Γ_1 and Γ are taken with counting measure. The Second Isomorphism Theorem [4, 2.2] states that Γ/Γ_0 is isomorphic to $(\Gamma/\Gamma_1)/(\Gamma_0/\Gamma_1)$. Since Γ_0/Γ_1 is generated by one element and ψ is constant on the cosets of Γ_0/Γ_1 in Γ/Γ_1 , it follows that $\phi \in M_p^q(\Gamma/\Gamma_0)$ and that $\|\phi\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. The theorem is now established whenever Γ_0 is finitely generated.

To demonstrate the general result enunciated, we need another lemma. Let Γ_1 be any subgroup of Γ , G_1 its annihilator in G , and π_1 the canonical mapping $G \to G/G_1$. We shall assume that the Haar measures dx_1 and $d\dot{x}_1$ of G_1 and G/G_1 are such that each group has unit mass; the Haar measure on Γ_1 must therefore be that on Γ restricted to Γ_1 .

LEMMA. *If* $\psi \in M_p^q(\Gamma)$ *, then* $\psi|_{\Gamma_1} \in M_p^q(\Gamma_1)$ *, and*

$$\|\psi|_{\Gamma_1}\|_{p,q} \le \|\psi\|_{p,q} .$$

Proof. Let $f \in C(G/G_1)$. It is well known that

$$(24)\quad (f \circ \pi_1)^\wedge(\chi) = \hat{f}(\chi) \quad \text{if} \quad \chi \in \Gamma_1$$

and

$$(f \circ \pi_1)^\wedge(\chi) = 0 \quad \text{otherwise.}$$

Further,

$$(25) \qquad \|f \circ \pi_1\|_p = \left[\int_G |f \circ \pi_1(x)|^p dx\right]^{1/p} = \left[\int_{G/G_1} |f(\dot{x}_1)|^p \int_{G_1} dx_1 d\dot{x}_1\right]^{1/p} = \|f\|_p ,$$

and so

$$(26) \qquad \left|\int_{\Gamma_1} \psi(\chi)\hat{f}(\chi)\hat{g}(\chi)d\chi\right| = \left|\int_{\Gamma} \psi(\chi)(f \circ \pi_1)^\wedge(\chi)(g \circ \pi_1)^\wedge(\chi)d\chi\right| \le \|\psi\|_{p,q}\|f \circ \pi_1\|_p\|g \circ \pi_1\|_{q'} = \|\psi\|_{p,q}\|f\|_p\|g\|_{q'}$$

for any $f, g \in C(G/G_1)$, proving the lemma.

Suppose that $f_0, g_0 \in T(G_0)$. Let $\{\chi_1, \chi_2, \ldots, \chi_n\}$ be a finite subset of Γ containing elements of each coset of Γ_0 in Γ on which $\hat{f}_0$ or $\hat{g}_0$ is non-zero, and define Γ_1 to be the group generated by $\chi_1, \chi_2, \ldots, \chi_n$. A subgroup of a finitely generated abelian group is also finitely generated [9, II.3.k], so $\Gamma_1 \cap \Gamma_0$ is finitely generated. By the lemma, $\pi^*\phi|_{\Gamma_1} \in M_p^q(\Gamma_1)$, and $\|\pi^*\phi|_{\Gamma_1}\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. Since $\pi^*\phi|_{\Gamma_1}$ is constant on cosets of the finitely generated group $\Gamma_1 \cap \Gamma_0$, ϕ', defined to be the function on $\Gamma_1/(\Gamma_1 \cap \Gamma_0)$ whose periodic extension to Γ_1 is $\pi^*\phi|_{\Gamma_1}$, satisfies $\phi' \in M_p^q\left(\Gamma_1/(\Gamma_1 \cap \Gamma_0)\right)$ and $\|\phi'\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. The First Isomorphism Theorem [4, 2.1] states that the group $(\Gamma_1+\Gamma_0)/\Gamma_0$ is isomorphic to the group $\Gamma_1/(\Gamma_1 \cap \Gamma_0)$; the natural isomorphism θ maps the coset $\chi + \Gamma_0$ of $(\Gamma_1+\Gamma_0)/\Gamma_0$ $(\chi \in \Gamma_1)$ to the coset $\chi + \Gamma_1 \cap \Gamma_0$ of $\Gamma_1/(\Gamma_1 \cap \Gamma_0)$. We have (implicitly) normalised the Haar measure of

$\Gamma_1/(\Gamma_1\cap\Gamma_0)$ so that each point has unit mass; if we also normalise $(\Gamma_1+\Gamma_0)/\Gamma_0$ so that each point has unit mass, the mapping $\theta^* : \psi \to \psi \circ \theta$ must be an isometric isomorphism of $M_p^q\left(\Gamma_1/(\Gamma_1\cap\Gamma_0)\right)$ onto $M_p^q\left((\Gamma_1+\Gamma_0)/\Gamma_0\right)$. In particular, $\phi|_{(\Gamma_1+\Gamma_0)/\Gamma_0} = \phi' \circ \theta \in M_p^q\left((\Gamma_1+\Gamma_0)/\Gamma_0\right)$ and $\|\phi|_{(\Gamma_1+\Gamma_0)/\Gamma_0}\|_{p,q} \le \|\pi^*\phi\|_{p,q}$. Let Γ_2 be the group $(\Gamma_1+\Gamma_0)$, G_2 the annihilator in G_0 of Γ_2/Γ_0, and π_2 the canonical projection $G_0 \to G_0/G_2$. The dual group of Γ_2/Γ_0 is G_0/G_2, so, for any $h, k \in C(G_0/G_2)$,

$$\left|\int_{\Gamma_2/\Gamma_0} \phi(\dot{\chi})\hat{h}(\dot{\chi})\hat{k}(\dot{\chi})d\dot{\chi}\right| \le \|\pi^*\phi\|_{p,q}\|h\|_p\|k\|_{q'} .$$

In particular, considering $h, k \in T(G_0/G_2)$ such that

$$\hat{h} = \hat{f}_0\big|_{\Gamma_2/\Gamma_0} \quad\text{and}\quad \hat{k} = \hat{g}_0\big|_{\Gamma_2/\Gamma_0} ,$$

$$\left|\int_{\Gamma_2/\Gamma_0} \phi(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right| \le \|\pi^*\phi\|_{p,q}\|h\|_p\|k\|_{q'} .$$

However, Γ_1 was defined so that $\hat{f}_0$ and $\hat{g}_0$ were both supported in $(\Gamma_1+\Gamma_0)/\Gamma_0$. Consequently, as in (24), $f_0 = h \circ \pi_2$ and $g_0 = k \circ \pi_2$. The normalisations are such that $\|h \circ \pi_2\|_p = \|h\|_p$ and $\|k \circ \pi_2\|_{q'} = \|k\|_{q'}$, as in (25). Thus

$$\left|\int_{\Gamma/\Gamma_0} \phi(\dot{\chi})\hat{f}_0(\dot{\chi})\hat{g}_0(\dot{\chi})d\dot{\chi}\right| \le \|\pi^*\phi\|_{p,q}\|f_0\|_p\|g_0\|_{q'} ,$$

which, since f_0 and g_0 were arbitrary trigonometric polynomials on G_0, proves the theorem.

Finally, we should note that, if G is not compact (that is, if Γ is not discrete) then $M_p^q(\Gamma) = \{0\}$ if $1 \le q < p \le \infty$. Hörmander [5] demonstrates this if $G = R^n$, and the generalisation of his proof to arbitrary non-compact groups is obvious. So when $p > q$, the only case of

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interest in connection with periodic extensions of multipliers is that where Γ is discrete.

References

[1] Karel de Leeuw, "On L_p multipliers", *Ann. of Math.* (2) 81 (1965), 364-372.

[2] G.I. Gaudry, "Quasimeasures and operators commuting with convolution", *Pacific J. Math.* 18 (1966), 461-476.

[3] Lawrence Gluck, Draft of Ph.D. thesis, Illinois Institute of Technology, Chicago, 1970.

[4] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis, Vol. I* (Die Grundlehren der mathematischen Wissenschaften, Band 115. Academic Press, New York; Springer-Verlag, Berlin, Göttingen, Heidelberg, 1963).

[5] Lars Hörmander, "Estimates for translation invariant operators in L^p spaces", *Acta Math.* 104 (1960), 93-140.

[6] Noël Lohoué, "La synthèse des convoluteurs sur un groupe abélien localement compact", *C.R. Acad. Sci. Paris Sér. A* 272 (1971), 27-29.

[7] Walter Rudin, *Fourier analysis on groups* (Interscience, New York, London, 1962).

[8] Sadahiro Saeki, "Translation invariant operators on groups", *Tôhoku Math. J.* 22 (1970), 409-419.

[9] Eugene Schenkman, *Group theory* (Van Nostrand, Princeton, New Jersey; Toronto; New York; London; 1965).

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