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# ON A FRIEDRICHS EXTENSION RELATED TO UNBOUNDED SUBNORMALS-II

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**Abstract.** We study the Friedrichs extensions of unbounded cyclic subnormals. The main result of the present paper is the identification of the Friedrichs extensions of certain cyclic subnormals with their closures. This generalizes as well as complements the main result obtained in [**5**]. Such characterizations lead to abstract Galerkin approximations, generalized wave equations, and bounded $\mathcal{H}^\infty$-functional calculi.

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**1. Preliminaries.** The present paper is a sequel to [**2**], [**5**], and continues the study of unbounded cyclic subnormals in the same spirit. The main result of the paper is the identification of the Friedrichs extensions of certain cyclic subnormals with their closures. As a corollary, we obtain a generalization of the main result of [**5**]. All the results in this paper rely heavily on the ideas developed in [**5**] and [**2**]. Also, in the present investigations, the notion of the minimal normal extension of spectral type ([**9**]) turns out to be an essential ingredient.

The paper is organized as follows. In Section 2, we give a sufficient condition for unbounded subnormals to admit Friedrichs extensions, and characterize the Friedrichs extensions of certain cyclic subnormals. In Section 3, we discuss several applications of Theorem 2.3. These are the Galerkin approximation in the functional model space, existence and uniqueness of the Hilbert space valued solutions of a generalized wave equation, and an $\mathcal{H}^\infty$-functional calculus for certain cyclic subnormals. In the last section, we obtain generalizations of some results obtained in [**5**]. In the present section, we fix the notation, and record a few requisites pertaining to unbounded subnormals and sectorial forms.

**1.1. Unbounded subnormals.** For a subset $A$ of the complex plane $\mathcal{C}$, let $A^*$, $\text{int}(A)$, $\overline{A}$ and $A^c$ respectively denote the conjugate, the interior, the closure and the complement of $A$ in $\mathcal{C}$. We use $\mathcal{R}$ to denote the real line, and $\text{Re}z$ and $\text{Im}z$ respectively denote the real and imaginary parts of a complex number $z$. Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$. If $S$ is a densely defined linear operator in $\mathcal{H}$ with domain $\mathcal{D}(S)$, then we use $\sigma(S)$, $\sigma_p(S)$, $\sigma_{ap}(S)$ to respectively denote the spectrum, the point spectrum and the approximate point spectrum of $S$. It may be recalled that $\sigma_p(S)$ is the set of eigenvalues of $S$, that $\sigma_{ap}(S)$ is the set of those $\lambda$ in $\mathcal{C}$ for which $S-\lambda$ is not bounded below, and that $\sigma(S)$ is the complement of the set of those $\lambda$ in $\mathcal{C}$ for which

$(T-\lambda)^{-1}$ exists as a bounded linear operator on $\mathcal{H}$. For a normal operator $N$ in $\mathcal{H}$, $\sigma(N) = \sigma_{ap}(N)$. For a non-negative measure $\mu$ on the complex plane $\mathcal{C}$, we will use $\mathrm{supp}(\mu)$ to denote the support of $\mu$.

A densely defined linear operator $S$ in $\mathcal{H}$ with domain $\mathcal{D}(S)$ is said to be *cyclic* if there is a vector $f_0 \in \mathcal{D}^\infty(S) \equiv \cap_{n=0}^\infty \mathcal{D}(S^n)$ (referred to as a *cyclic vector of* $S$) such that $\mathcal{D}(S)$ is the linear span $\mathrm{lin}\{S^n f_0 : n \geq 0\}$ of the set $\{S^n f_0 : n \geq 0\}$.

If $S$ is a densely defined linear operator in $\mathcal{H}$ with domain $\mathcal{D}(S)$, then $S$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a densely defined normal operator $N$ in $\mathcal{K}$ with domain $\mathcal{D}(N)$ such that $\mathcal{D}(S) \subset \mathcal{D}(N)$ and $Sf = Nf,\ f \in \mathcal{D}(S)$.

If a linear operator $S$ in $\mathcal{H}$ is subnormal then its normal extension $N$ in $\mathcal{K}$ is said to be *minimal of spectral type* if the only closed subspace of $\mathcal{K}$ reducing $N$ and containing $\mathcal{H}$ is $\mathcal{K}$ itself. Given a normal extension $N$ of $S$, one can always guarantee a minimal normal extension of spectral type (refer to Section 2 of [**9**]). Let $S$ in $\mathcal{H}$ be subnormal with a normal extension $N$ in $\mathcal{K}$. Let $\mathcal{H}_s[N] \equiv$ the closure of lin $\{E(\sigma)f\ : f \in \mathcal{H}, \sigma$ is a Borel subset of $\mathcal{C}\}$, where $E(\cdot)$ denotes the spectral measure of $N$. Since $\mathcal{H}_s[N]$ reduces $N$, one can define a linear operator $N_s$ in $\mathcal{H}_s[N]$ by $N_s f \equiv Nf$ for every $f \in \mathcal{D}(N_s) \equiv \mathcal{D}(N) \cap \mathcal{H}_s[N]$. It then follows from [**9**, Proposition 1] that $N_s$ is a minimal normal extension of $S$ of spectral type. We will refer to $N_s$ as the *minimal normal extension of $S$ of spectral type associated with the normal extension $N$*.

Suppose $S$ is a cyclic operator in $\mathcal{H}$ with $f_0$ being a cyclic vector of $S$. As established in [**8**, Proposition 3], $S$ is subnormal if and only if there exists a non-negative measure $\mu$ on the complex plane $\mathcal{C}$ (referred to as the *representing measure of* $S$) such that

$$\langle S^m f_0, S^n f_0 \rangle_{\mathcal{H}} = \int_{\mathcal{C}} z^m \overline{z}^n d\mu(z) \text{ for all } m, n \geq 0. \tag{0}$$

If $S$ is a cyclic subnormal operator in $\mathcal{H}$ so that (0) is satisfied, then $S$ is unitarily equivalent to $M_{z,\mu}$ in $\mathcal{H}_\mu$, where $\mathcal{H}_\mu$ is the $L^2(\mu)$-closure of the vector space $\mathcal{C}[z]$ of complex polynomials in $z$ and where $M_{z,\mu}$ is the operator of multiplication by $z$ with domain $\mathcal{C}[z]$ (refer to [**9**, Theorem 5]); the triple $(M_{z,\mu}, \mathcal{C}[z], \mathcal{H}_\mu)$ will be referred to as a *functional model* of the cyclic subnormal operator $S$. If $S$ is a cyclic subnormal with the cyclic vector $f_0$ and with a minimal normal extension $N$ of spectral type then it follows from [**9**, Theorem 5] that the triple $(M_{z,\mu_N}, \mathcal{C}[z]|_{\sigma(N)}, \mathcal{H}_{\mu_N})$ is a functional model of $S$, where the positive Borel measure $\mu_N(\cdot)$ supported on $\sigma(N)$ is given by $\langle E(\cdot)f_0, f_0 \rangle$ for the spectral measure $E(\cdot)$ of $N$.

If $S$ is a densely defined closable operator in $\mathcal{H}$, then we use $\overline{S}$ to denote the closure of $S$ in $\mathcal{H}$. It is known that every subnormal operator is closable (refer to [**8**]). Let $S$ be a densely defined cyclic subnormal operator in $\mathcal{H}$ with the cyclic vector $f_0$. It follows from [**9**, Proposition 6] that, for any point $\lambda \in \sigma_p(S^*)^*$, there exists a unique vector $h_\lambda$ in $\mathcal{H}$ such that $p(\lambda) = \langle p(S)f_0, h_\lambda \rangle_{\mathcal{H}}$ for every complex polynomial $p$. One can then define a function $k_S$ on $\mathcal{C}$ by setting $k_S(\lambda)$ equal to $\|h_\lambda\|^2$ if $\lambda \in \sigma_p(S^*)^*$, and equal to $\infty$ otherwise. If $\mathrm{int}(\sigma_p(S^*))$ is non-empty and if one defines

$$\gamma(S) = \{\lambda \in \mathcal{C} : k_S \text{ is finite and continuous in a neighbourhood of } \lambda\}, \tag{1}$$

then it follows from [**9**, Theorem 7] that $\gamma(S)$ is an open subset of $\mathrm{int}(\sigma_p(S^*)^*)$ and that $\mathrm{int}(\sigma_p(S^*)^*) \setminus \gamma(S)$ is a nowhere dense subset of $\mathcal{C}$; further, as pointed out in Footnote 9 following [**9**, Theorem 9], $\sigma(\overline{S}) \setminus \sigma_{ap}(\overline{S}) \subset \gamma(S)$.

Given a Hilbert space $\mathcal{H}$ and a cyclic operator $S$ in $\mathcal{H}$ with a cyclic vector $f_0$ of $S$, one can come up with a sequence $r = \{r_n\}_{n\geq 0}$ of complex polynomials such that $e = \{e_n\}_{n\geq 0}$ with $e_n = r_n(S)f_0$ is an orthonormal basis for $\mathcal{H}$ and such that

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$\text{lin}\{e_n : n \geq 0\} = \text{lin}\{S^n f_0 : n \geq 0\}$ (refer to [**9**]). As observed in the proof of [**9**, Proposition 6], the polynomials $r_n$ so obtained form a Hamel basis for $\mathcal{C}[z]$, the vector space of complex polynomials in $z$. If $\omega_r$ is the set $\{z \in \mathcal{C} : \sum_{n=0}^{\infty} |r_n(z)|^2 < \infty\}$, then we define $K_r(\cdot, \cdot)$ on $\omega_r \times \omega_r$ by $K_r(z, w) = \sum_{n=0}^{\infty} \overline{r_n(z)} r_n(w)$ $(z, w \in \omega_r)$. Since $K_r$ is a positive definite kernel on $\omega_r$, we can associate with $K_r$ a reproducing kernel Hilbert space $\mathcal{H}_r$ as described in [**1**]. The following theorem is [**9**, Theorem 6].

THEOREM 1.1. *Suppose $\mathcal{H}$, $S$, $r$, $e$, $\omega_r$ and $\mathcal{H}_r$ are as described in the preceding paragraph. If the point spectrum $\sigma_p(S^*)$ of $S^*$ is non-empty, then the following statements are true:*

(a) *$\mathcal{P}_r$, the set of restrictions of members of $\mathcal{C}[z]$ to $\omega_r$, is dense in $\mathcal{H}_r$,*
(b) *the operator $M_z$ of multiplication by $z$ defined on $\mathcal{P}_r$ is cyclic with the cyclic vector the constant polynomial* 1,
(c) *there is a unique partial isometry $W : \mathcal{H} \to \mathcal{H}_r$ with its initial space being the closure of lin$\{\sum_{n=0}^{\infty} \overline{r_n(\lambda)} e_n : \lambda \in \sigma_p(S^*)^*\}$ and its final space being $\mathcal{H}_r$ and such that $WS = M_z W$, and*
(d) $\omega_r = \sigma_p(S^*)^* = \sigma_p(M_z^*)^*$.

Suppose that, for a cyclic operator $S$ in $\mathcal{H}$ having non-empty point spectrum $\sigma_p(S^*)$, $W$ in (c) of Theorem 1.1 turns out to be a unitary of $\mathcal{H}$ onto $\mathcal{H}_r$; in this case the triple $(M_z, \mathcal{P}_r, \mathcal{H}_r)$ will be referred to as an *analytic model* of the cyclic operator $S$. Define $M_z^{max}$ in $\mathcal{H}_r$ by $(M_z^{max} f)(z) = z f(z)$ $(z \in \omega_r)$ where $f \in \mathcal{D}(M_z^{max}) = \{f \in \mathcal{H}_r : zf \in \mathcal{H}_r\}$. Using the reproducing property of $\mathcal{H}_r$, it can be easily seen that $M_z^{max}$ is a closed linear operator in $\mathcal{H}_r$.

**1.2. Sectorial forms.** Let $\mathcal{H}$ be a complex infinite-dimensional separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$. Let $\Gamma$ be a dense subspace of $\mathcal{H}$ such that $\Gamma$ itself is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\Gamma}$ and the corresponding norm $\|\cdot\|_{\Gamma}$. Let there exist a positive number $M_1$ satisfying

$$\|x\|_{\mathcal{H}} \leq M_1 \|x\|_{\Gamma} \text{ for all } x \in \Gamma. \tag{2}$$

Let also $\mathcal{F} : \Gamma \times \Gamma \to \mathcal{C}$ be a sesquilinear form and assume that there exist positive numbers $M_2$, $M_3$ and a real number $a$ such that

$$|\mathcal{F}(x, y)| \leq M_2 \|x\|_{\Gamma} \|y\|_{\Gamma} \text{ for all } x, y \in \Gamma, \tag{3}$$

$$\text{Re } \mathcal{F}(x, x) \geq M_3 \|x\|_{\Gamma}^2 + a\|x\|_{\mathcal{H}}^2 \text{ for all } x \in \Gamma. \tag{4}$$

Such $\mathcal{F}$ are referred to as *sectorial sesquilinear forms*. Corresponding to any such $\mathcal{F}$ we can define a linear operator $A$ in $\mathcal{H}$ as follows: $x \in \mathcal{D}(A)$ if and only if $x \in \Gamma$ and there exists $z$ in $\mathcal{H}$ such that $\mathcal{F}(x, y) = \langle z, y \rangle_{\mathcal{H}}$ for all $y$ in $\Gamma$; we set $Ax = z$. The linear operator $A$ so defined is referred to as the *operator associated with* $\mathcal{F}$ and it clearly satisfies $\mathcal{F}(x, y) = \langle Ax, y \rangle_{\mathcal{H}}$ for all $x \in \mathcal{D}(A)$ and for all $y$ in $\Gamma$. For the basic properties of the operator $A$ associated with a sectorial sesquilinear form $\mathcal{F}$, the reader is referred to [**7**, Chapter 2]. In particular, it follows from [**7**, Theorem 2.8.2] that $A$ is a closed densely defined operator in $\mathcal{H}$ and that $\sigma(A)$ is contained in the "truncated cone"

$$\Lambda = \left\{\zeta \in \mathcal{C} : \text{Re } \zeta \geq a + M_3/M_1^2 \text{ and } |\text{Im } \zeta| \leq M_2(\text{Re } \zeta - a)/M_3\right\}. \tag{5}$$

The following result is Theorem 2.12.1 of [**7**].

THEOREM 1.2. *Suppose $S$ is a densely defined linear operator in a complex Hilbert space $\mathcal{H}$ such that, for some $r \in \mathcal{R}$ and $M \in (0, \infty)$,*

$$|\mathrm{Im}\langle Sx, x\rangle_{\mathcal{H}}| \leq M\ \mathrm{Re}\langle Sx - rx, x\rangle_{\mathcal{H}} \text{ for all } x \in \mathcal{D}(S). \tag{6}$$

*Then there exist a subspace $\Gamma$ of $\mathcal{H}$, an inner product $\langle \cdot, \cdot\rangle_{\Gamma}$ on $\Gamma$ with the corresponding norm $\|\cdot\|_{\Gamma}$, and a sectorial sesquilinear form $\mathcal{F}$ on $\Gamma$ such that the following assertions hold:*

(a) *$\mathcal{D}(S)$ is a dense subspace of $\Gamma$ (in the $\|\cdot\|_{\Gamma}$ norm).*
(b) *$\langle x, y\rangle_{\Gamma} = (1/2)(\langle Sx, y\rangle_{\mathcal{H}} + \langle x, Sy\rangle_{\mathcal{H}}) + (1 - r)\langle x, y\rangle_{\mathcal{H}}$ for all $x, y \in \mathcal{D}(S)$.*
(c) *$\mathcal{F}(x, y) = \langle Sx, y\rangle_{\mathcal{H}}$ for all $x, y \in \mathcal{D}(S)$.*

The linear operator $A$ associated with the sectorial sesquilinear form $\mathcal{F}$ of Theorem 1.2 is clearly an extension of the operator $S$ and is called the *Friedrichs extension* of $S$. Referring to the proof of [**7**, Theorem 2.12.1] one finds that the constants $M_1, M_2, M_3$ and $a$ corresponding to $\mathcal{F}$ as in (2), (3), (4) are given by $M_1 = 1, M_2 = M + 1 + |r - 1|, M_3 = 1$ and $a = r - 1$; in particular (refer to (5)),

$$\sigma(A) \subset \{\zeta \in \mathcal{C} : \mathrm{Re}\,\zeta \geq r \text{ and } |\mathrm{Im}\,\zeta| \leq (M + 1 + |r - 1|)(\mathrm{Re}\,\zeta - r + 1)\}. \tag{7}$$

REMARK 1.3. Let $S$ be as in the statement of Theorem 1.2. Suppose there exist a densely defined linear operator $T$ in $\mathcal{H}'$ and a unitary $U$ from $\mathcal{H}$ onto $\mathcal{H}'$ such that $US = TU$. Let $y$ be in $\mathcal{D}(T)$. Since $U$ is onto, $y = Ux$ for some $x \in \mathcal{H}$. Hence $\langle Ty, y\rangle_{\mathcal{H}'} = \langle TUx, Ux\rangle_{\mathcal{H}'} = \langle USx, Ux\rangle_{\mathcal{H}'} = \langle Sx, x\rangle_{\mathcal{H}}$. Since $S$ satisfies (6), so does $T$. Thus the Friedrichs extension of $T$ is guaranteed. Moreover, the Friedrichs extension of $T$ satisfies (7).

**2. The Friedrichs extensions of cyclic subnormals.** In this section, we identify the Friedrichs extensions of certain cyclic subnormals with their closures (Theorem 2.3). We begin with the following proposition, which guarantees the Friedrichs extensions of subnormal operators with normal spectra contained in certain cones.

PROPOSITION 2.1. *Let $S$ in $\mathcal{H}$ be subnormal with a normal extension $N$. If $\sigma(N)$ is contained in the cone $\Lambda_{r,M} = \{z \in \mathcal{C} : |\mathrm{Im}z| \leq M(\mathrm{Re}z - r)\}$ where $r \in \mathcal{R}$ and $M \in (0, \infty)$, then there exist a subspace $\Gamma$ of $\mathcal{H}$, an inner product $\langle \cdot, \cdot\rangle_{\Gamma}$ on $\Gamma$ with the corresponding norm $\|\cdot\|_{\Gamma}$, and a sectorial sesquilinear form $\mathcal{F}$ on $\Gamma$, so that assertions* (a), (b), (c) *of Theorem 1.2 hold true. If in addition $S$ admits a functional model $(M_{z,\mu}, \mathcal{C}[z], \mathcal{H}_{\mu})$ then* (a), (b), *and* (c) *can be made explicit in the following way:*

(a) *$\mathcal{D}(M_{z,\mu})$ is a dense subspace of $\Gamma$ (in the $\|\cdot\|_{\Gamma}$ norm).*
(b) *$\langle f, g\rangle_{\Gamma} = \int(\mathrm{Re}z - r + 1)f(z)\overline{g(z)}d\mu(z)$ for all $f, g \in \mathcal{D}(M_{z,\mu})$.*
(c) *$\mathcal{F}(f, g) = \int zf(z)\overline{g(z)}d\mu(z)$ for all $f, g \in \mathcal{D}(M_{z,\mu})$.*

*Proof.* The condition (6) in the statement of Theorem 1.2 can be checked using the Spectral Theorem for unbounded normal operators. To check that let $E(\cdot)$ denote the

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spectral measure of $N$. Since $\sigma(N)$ is contained in the cone

$$\Lambda_{r,M} = \{z \in \mathcal{C} : |\mathrm{Im} z| \leq M(\mathrm{Re} z - r)\},$$

one has

$$\begin{aligned}
|\mathrm{Im}\langle Sx, x\rangle_{\mathcal{H}}| &= |\mathrm{Im}\langle Nx,\ x\rangle_{\mathcal{K}}| \\
&= \left|\mathrm{Im} \int_{\sigma(N)} z d\langle E(z)x, x\rangle_{\mathcal{K}}\right| \\
&\leq \int_{\sigma(N)} |\mathrm{Im} z| d\langle E(z)x, x\rangle_{\mathcal{K}} \\
&\leq \int_{\sigma(N)} M(\mathrm{Re} z - r) d\langle E(z)x, x\rangle_{\mathcal{K}} \\
&= M\ \mathrm{Re}\langle Nx - rx, x\rangle_{\mathcal{K}}.
\end{aligned}$$

Thus $|\mathrm{Im}\langle Sx, x\rangle_{\mathcal{H}}| \leq M\ \ \mathrm{Re}\langle Sx - rx, x\rangle_{\mathcal{H}}$ for all $x \in \mathcal{D}(S)$. Now one may appeal to Theorem 1.2 to derive the first part. The remaining part can be easily deduced using (0). □

Let $S$ and $\mathcal{F}$ be as in the preceding proposition. The linear operator $A$ associated with the sectorial sesquilinear form $\mathcal{F}$ is the Friedrichs extension of $S$.

LEMMA 2.2. *Let $S$ in $\mathcal{H}$ be cyclic with the cyclic vector $f_0$, and subnormal with a normal extension $N$. Let $N_s$ be the minimal normal extension of $S$ of spectral type associated with $N$. Let $H_r$ denote the half-plane $\{\mu \in \mathcal{C}\ :\ Re\mu < r\}$ for some real $r$, and let $\omega$ be the unbounded connected component of $\sigma(N)^c$ in $\mathcal{C}$ that contains $H_r$. Suppose that $S$ admits an analytic model and that $\sigma(N) \subset H_r^c$. If $\{\sum_{n=0}^{k} \frac{(-m)^n S^n f_0}{n!}\}_{k\geq 0}$ converges in $\mathcal{H}$ for every integer $m \geq 1$ then the following statements are true.*

(1) *$\sigma(\overline{S})$ is contained in $\omega^c$.*
(2) *If in addition $\sigma(N)^c$ is connected, then $\sigma(\overline{S}) = \sigma(N_s)$.*

*Proof.* Assume the hypotheses. Assume also that $\{\sum_{n=0}^{k} \frac{(-m)^n S^n f_0}{n!}\}_{k\geq 0}$ converges in $\mathcal{H}$ for every integer $m \geq 1$. The proof here is an adaption of the argument of [**5**, Lemma 1]. Suppose $\sigma(\overline{S})$ is not contained in $\omega^c$, where $\omega$ is the unbounded connected component of $\sigma(N)^c$ that contains $H_r$. From the proof of [**9**, Theorem 2] (refer to [**5**, Remark 1]) one has either $\omega \cap \sigma(\overline{S}) = \emptyset$ or $\omega \subset \sigma(\overline{S})$. Since $\sigma(\overline{S})$ is not contained in $\omega^c$, one has $\omega \subset \sigma(\overline{S})$.

With $\gamma(S)$ as defined in (1), $\sigma(\overline{S}) \setminus \sigma_{ap}(\overline{S}) \subset \gamma(S)$ as recorded in our comments following (1). Since $N$ extends $\overline{S}$, we have $\sigma_{ap}(\overline{S}) \subset \sigma_{ap}(N) = \sigma(N)$ and thus $\sigma(\overline{S}) \setminus \sigma(N) \subset \gamma(S)$. Since $\omega \subset \sigma(\overline{S})$ and $\omega \cap \sigma(N) = \emptyset$, we have $\omega \subset \gamma(S)$. From the observations following (1) in Section 1, we have $\gamma(S) \subset \mathrm{int}(\sigma_p(S^*)^*)$. By hypothesis, $H_r \subset \omega$, so that $H_r \subset \sigma_p(S^*)^*$.

Let $h \in H_r$. Choose $\epsilon > 0$ such that $h - \epsilon \in H_r$. Consider the sequence $\{g_k \equiv \sum_{n=0}^{k} \exp\{n(h-\epsilon)\} E_n(S, f_0)\}_{k\geq 0}$ in $\mathcal{H}$, where $E_n(S, f_0)$ is the limit of the sequence $\{\sum_{m=0}^{k} \frac{(-n)^m S^m f_0}{m!}\}_{k\geq 0}$ in $\mathcal{H}$. We claim that $\{g_k\}_{k\geq 0}$ is convergent in $\mathcal{H}$. Let $N_s$ be the minimal normal extension of $S$ of spectral type associated with $N$. Then $S$ admits the functional model $(M_{z,\mu_{N_s}}, \mathbb{C}[z]|_{\sigma(N_s)}, \mathcal{H}_{\mu_{N_s}})$ (refer to the discussion following (0)). We check that $\left\|f - \sum_{k=0}^{m} \exp\{(-z + h - \epsilon)k\}\right\|_{\mathcal{L}}$ converges to 0 as $m$ tends to $\infty$

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for some $f$ in $\mathcal{L} = L^2(\sigma(N_s), \mu_{N_s})$. Since $N_s$ is a minimal normal extension of spectral type of $S$, by [9, Theorem 5], one has $\mathrm{supp}(\mu_{N_s}) = \sigma(N_s)$. Also, since $\mathcal{D}(N_s) \subset \mathcal{D}(N)$ and since $Nx = N_s x$ $(x \in \mathcal{D}(N_s))$, it follows that $\sigma(N_s) = \sigma_{ap}(N_s) \subset \sigma_{ap}(N) = \sigma(N)$. Hence $\mathrm{supp}(\mu_{N_s}) \subset \sigma(N)$. Note that for any integer $m \geq 0$ and for any $z \in \sigma(N) \subset H_r^c$, one has

$$\begin{aligned} \left| \sum_{k \geq m+1} \exp\{(-z+h-\epsilon)k\} \right| &\leq \sum_{k \geq m+1} |\exp\{(-z+h-\epsilon)k\}| \\ &= \sum_{k \geq m+1} \exp\{(-\mathrm{Re}z + \mathrm{Re}h - \epsilon)k\} \\ &\leq \sum_{k \geq m+1} \exp(-\epsilon k), \end{aligned}$$

and the last expression tends to 0 as $m$ tends to $\infty$. In view of $\mathrm{supp}(\mu_{N_s}) \subset \sigma(N)$, one has

$$\begin{aligned} \left\| \sum_{k \geq m+1} \exp\{(-z+h-\epsilon)k\} \right\|_{\mathcal{L}}^2 &= \int \left| \sum_{k \geq m+1} \exp\{(-z+h-\epsilon)k\} \right|^2 d\mu_{N_s}(z) \\ &= \int_{\sigma(N)} \left| \sum_{k \geq m+1} \exp\{(-z+h-\epsilon)k\} \right|^2 d\mu_{N_s}(z) \\ &\leq \left( \sum_{k \geq m+1} \exp(-\epsilon k) \right)^2 \mu_{N_s}(\sigma(N)). \end{aligned}$$

The preceding arguments show that $f \equiv \sum_{k=0}^{\infty} \exp\{(-z+h-\epsilon)k\} \in \mathcal{L}$ and that $\|f - \sum_{k=0}^{m} \exp\{(-z+h-\epsilon)k\}\|_{\mathcal{L}}$ converges to 0 as $m$ tends to $\infty$. Since $g_k = \sum_{n=0}^{k} \exp\{(h-\epsilon)n\} E_n(S, f_0) \in \mathcal{H}$ $(k \geq 1)$, it follows that for every integer $k \geq 1$, $\sum_{n=0}^{k} \exp\{(-z+h-\epsilon)n\} \in \mathcal{H}_{\mu_{N_s}}$. It is now clear that $f \in \mathcal{H}_{\mu_{N_s}}$. Since $\mu_{N_s}$ is a representing measure of $S$, there exists a unique unitary $V : \mathcal{H}_{\mu_{N_s}} \to \mathcal{H}$ such that $Vp = p(S)f_0$ for every complex polynomial $p$ in $\mathcal{H}_{\mu_{N_s}}$. Thus $g \equiv Vf$ is such that $\|g - \sum_{k=0}^{m} \exp\{(h-\epsilon)k\} E_k(S, f_0)\|_{\mathcal{H}}$ converges to 0 as $m$ tends to $\infty$. Thus the claim stands verified.

Since $H_r \subset \sigma_p(S^*)^*$ and since $h - \epsilon \in H_r$, one has $h - \epsilon \in \sigma_p(S^*)^*$. By [9, Lemma 2], there exists a constant $c_h > 0$ such that

$$|p(h-\epsilon)| \leq c_h \|p(S)f_0\|_{\mathcal{H}} \text{ for every polynomial p.} \tag{8}$$

By hypothesis, $S$ admits the analytic model $(M_z, \mathcal{P}_r, \mathcal{H}_r)$. Since $E_n(S, f_0)$ belongs to $\mathcal{H}$, $\exp\{(-z+h-\epsilon)n\} \in \mathcal{H}_r$. Hence by part (a) of Theorem 1.1, there exists a sequence of complex polynomials $\{p_{n,k}\}$ such that $\|p_{n,k} - \exp\{(-z+h-\epsilon)k\}\|_{\mathcal{H}_r} \to 0$ as $n$ tends to $\infty$ for every integer $k \geq 1$. Since $h - \epsilon \in \sigma_p(S^*)^*$ and since $\mathcal{H}_r$ is a reproducing kernel Hilbert space, it follows that $|p_{n,k}(h-\epsilon) - 1|$ converges to 0 as $n$ tends to $\infty$. Hence $|\sum_{k=0}^{m} p_{n,k}(h-\epsilon) - (m+1)|$ converges to zero for all integers $m \geq 1$. Now if we let

$p \equiv \sum_{n=0}^{m} p_{n,k}$ in (8), we get

$$\left|\sum_{k=0}^{m} p_{n,k}(h-\epsilon)\right| \leq c_h \left\|\sum_{k=0}^{m} p_{n,k}(S)f_0\right\|_{\mathcal{H}} = c_h \left\|\sum_{k=0}^{m} p_{n,k}\right\|_{\mathcal{H}_r}.$$

The passage $n$ tends to $\infty$ in the preceding inequality yields

$$m+1 \leq \left\|\sum_{k=0}^{m} \exp\{(-z+h-\epsilon)k\}\right\|_{\mathcal{H}_r} = \left\|\sum_{k=0}^{m} \exp\{(h-\epsilon)k\}E_k(S,f_0)\right\|_{\mathcal{H}}.$$

This contradicts the fact that $\left\|\sum_{k=0}^{m} \exp\{(h-\epsilon)k\}E_k(S,f_0)\right\|_{\mathcal{H}}$ converges to $\|g\|_{\mathcal{H}}$ as $m$ tends to $\infty$. Thus we must have $\sigma(\overline{S}) \subset \omega^c$.

Now assume in addition that $\sigma(N)^c$ is connected so that $\omega^c = \sigma(N)$. From what we saw above, for any normal extension $N$ of $S$, one has $\sigma(\overline{S}) \subset \sigma(N)$. In particular, one has $\sigma(\overline{S}) \subset \sigma(N_s)$, where $N_s$ is the minimal normal extension of $S$ of spectral type associated with $N$. Since $N_s$ is a normal extension of spectral type of $\overline{S}$, by the Spectral Inclusion Theorem ([**9**, Theorem 1]) $\sigma(N_s) \subset \sigma(\overline{S})$. Hence $\sigma(\overline{S}) = \sigma(N_s)$ as desired. □

THEOREM 2.3. *Let $S$ in $\mathcal{H}$ be cyclic with the cyclic vector $f_0$, and subnormal with a normal extension $N$. Let $\Lambda_{r,M}$ denote the cone $\{z \in \mathcal{C} : |\mathrm{Im}z| \leq M(\mathrm{Re}z - r)\}$ for $r \in \mathcal{R}$ and $M \in (0,\infty)$. Suppose that $S$ admits an analytic model and that $\sigma(N) \subset \Lambda_{r,M}$. If $\{\sum_{n=0}^{k} \frac{(-m)^n S^n f_0}{n!}\}_{k\geq 0}$ converges in $\mathcal{H}$ for every integer $m \geq 1$ then the Friedrichs extension of $S$ is equal to $\overline{S}$.*

*Proof.* Let $S$, $N$ be as in the statement of the theorem and let $A$ denote the Friedrichs extension of $S$ as guaranteed by Proposition 2.1. By Lemma 2.2, we have $\sigma(\overline{S}) \subset \omega^c$, where $\omega$ is the unbounded connected component of $\sigma(N)^c$ in $\mathcal{C}$ that contains $H_r$. Since $A$ is the Friedrichs extension of $S$, one has $\sigma(A) \subset \Lambda$ as noted earlier (refer to (7)). Since $A$ is closed, $A$ extends $\overline{S}$; further, $\sigma(\overline{S}) \subset \omega^c$ and $\sigma(A) \subset \Lambda$ imply that $\sigma(\overline{S})^c \cap \sigma(A)^c$ is non-empty. It then follows from [**7**, Lemma 1.6.14] that $A = \overline{S}$. □

REMARK. Assume the hypotheses of Lemma 2.2, and for a positive real $t$ let $\exp(-t\overline{S})f_0 \equiv \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-t)^n S^n f_0}{n!}$. An examination of the proof of Theorem 2.3 reveals that $\overline{S} - r$ is an m-accretive operator. Hence by a result due to R. S. Phillips ([**7**, Corollary 4.3.11]), $\overline{S} - r$ is the generator of a contraction semigroup. It now follows from [**7**, Theorem 4.3.1] that $\exp(-m\overline{S})f_0 \in \mathcal{H}$ for every positive integer $m$ if and only if $\{\exp(-t\overline{S})\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathcal{H}$ (refer to [**7**]).

LEMMA 2.4. *Let $S$ be as in the statement of Theorem 2.3. Suppose there exist a densely defined linear operator $T$ in $\mathcal{H}'$ and a unitary $U$ from $\mathcal{H}$ onto $\mathcal{H}'$ such that $US = TU$. Then the Friedrichs extension of $T$ is equal to $\overline{T}$ (refer to Remark 1.3 of Section 1).*

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*Proof.* By Remark 1.3, the Friedrichs extension, say, B of $T$ is guaranteed. Since $S$ is closable, so is $T$. Moreover, $U\overline{S} = \overline{T}U$. By Theorem 2.3, the Friedrichs extension $A$ of $S$ is $\overline{S}$. Thus $UA = \overline{T}U$; in particular $\sigma(A) = \sigma(\overline{T})$. Thus $\sigma(\overline{T})^c \cap \sigma(B)^c \neq \emptyset$ in view of (7) and Remark 1.3. Since $\overline{T} \subset B$, one may appeal to [**7**, Lemma 1.6.14] to conclude that $B = \overline{T}$. □

COROLLARY 2.5. *Let $(M_z, \mathcal{P}_r, \mathcal{H}_r)$ be an analytic model of a cyclic subnormal $S$ with a normal extension $N$. Let $\Lambda_{r,M}$ denote the cone $\{z \in \mathcal{C} : |\mathrm{Im} z| \leq M(\mathrm{Re} z - r)\}$ for $r \in \mathcal{R}$ and $M \in (0, \infty)$. Suppose that $\sigma(N) \subset \Lambda_{r,M}$ and that $\{\sum_{n=0}^{k} \frac{(-m)^n z^n}{n!}\}_{k \geq 0}$ converges in $\mathcal{H}_r$ for every integer $m \geq 1$. If int$(\sigma_p(M_z^*))$ is non-empty then the Friedrichs extension of $M_z$ is equal to $M_z^{max}$.*

*Proof.* Since $S$ admits the analytic model $(M_z, \mathcal{P}_r, \mathcal{H}_r)$, $W : \mathcal{H} \to \mathcal{H}_r$ is unitary and $WS = M_z W$. By Remark 1.3, the Friedrichs extension, say, B of $M_z$ is guaranteed. By the preceding lemma, $B = \overline{M_z}$. Since $\mathrm{int}(\sigma_p(M_z^*)) \neq \emptyset$, it follows from [**9**, Proposition 11] that $\overline{M_z} = M_z^{max}$. Hence the Friedrichs extension of $M_z$ is equal to $M_z^{max}$. □

**3. Applications: Galerkin approximations, generalized wave equations, and $H^\infty$-functional calculi.** We discuss here several applications of Theorem 2.3. These are Proposition 3.1, Proposition 3.3, Proposition 3.4, and Corollary 3.5. We mention that Proposition 3.1 and Proposition 3.3 below generalize Proposition 3 and Proposition 4 of [**5**] respectively. Our first application of Theorem 2.3 is a Galerkin approximation result in the functional model space $\mathcal{H}_\mu$ (refer to [**7**, sections 2.8 and 2.12]).

PROPOSITION 3.1. *Let $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ denote a functional model of cyclic subnormal $S$. Let $N$ be a normal extension of $S$ and let $\Lambda_{r,M}$ denote the cone $\{z \in \mathcal{C} : |\mathrm{Im} z| \leq M(\mathrm{Re} z - r)\}$ for $r \in \mathcal{R}$ and $M \in (0, \infty)$. Suppose that $\sigma(N) \subset \Lambda_{r,M}$ and that $\{\sum_{n=0}^{k} \frac{(-m)^n z^n}{n!}\}_{k \geq 0}$ converges in $\mathcal{H}_\mu$ for every integer $m \geq 1$. Suppose also that $S$ admits an analytic model. For $n \geq 1$, let $\nu_n = \mathrm{lin}\{z_1, \ldots, z_n\}$ where $z_1$ is equal to the constant function 1 in $\mathcal{H}_\mu$ and $z_k = M_{z,\mu}^{k-1} 1$ $(k \geq 2)$. Let $\lambda \in \mathcal{C}$ be such that $\mathrm{Re}\lambda < r$. If $f \in \mathcal{H}_\mu$ then, for each $n \geq 1$, there exists a unique $f_n$ in $\nu_n$ such that*

$$\int (M_{z,\mu} - \lambda) f_n(z) \overline{z_{k-1}(z)} d\mu = \int f(z) \overline{z_{k-1}(z)} d\mu, \ 1 \leq k \leq n.$$

*Moreover,*

$$\lim_{n \to \infty} \int \left| f_n(z) - \frac{f(z)}{z - \lambda} \right|^2 d\mu \leq \lim_{n \to \infty} \int (\mathrm{Re} z - r + 1) \left| f_n(z) - \frac{f(z)}{z - \lambda} \right|^2 d\mu = 0.$$

*Proof.* Let $\mathcal{H}' = \mathcal{H}_\mu$ and let $T = M_{z,\mu}$ in $\mathcal{H}'$ with $\mathcal{D}(T) = \cup_{n \geq 1} \nu_n = \mathcal{C}[z]$. Since $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ is a functional model for $S$ there exists a unitary $U$ from $\mathcal{H}$ onto $\mathcal{H}'$ such that $US = TU$. The relation $US = TU$, guarantees, as in Remark 1.3, that $|\mathrm{Im}\langle Tf, f\rangle_{\mathcal{H}'}| \leq M \mathrm{Re}\langle Tf - rf, f\rangle_{\mathcal{H}'}$ for all $f$ in $\mathcal{D}(T)$. An appeal to [**7**, Theorem 2.12.6] now yields the first part. Since the Friedrichs extension of $T$ is $\overline{M_{z,\mu}}$ (Lemma 2.4), one more appeal to [**7**, Theorem 2.12.6] yields the remaining part in view of part (b) of Proposition 2.1. □

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Our second application of Theorem 2.3 is obtaining a 'smooth' $\mathcal{H}_\mu$-valued solution of a generalized wave equation (refer to Section 4.4 of [**7**]). For that purpose we establish a lemma (cf. [**5**]).

LEMMA 3.2. *Let $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ denote a functional model of cyclic subnormal $S$ and let $N$ be a normal extension of $S$. Suppose that $\sigma(N)$ is contained in $\Lambda_{r,M} \cap P$ where $\Lambda_{r,M} = \{z \in \mathcal{C} : |\mathrm{Im}z| \leq M(\mathrm{Re}z - r)\}$ and $P = \{z \in \mathcal{C} : (\mathrm{Im}z)^2 \leq M(\mathrm{Re}z - r + 1)\}$ with $r \in \mathcal{R}$ and $M \in (0, \infty)$. Let $\mathcal{F}$ be the sesquilinear sectorial form defined on $\Gamma$ and let $A$ be the associated operator (that is, the Friedrichs extension of $M_{z,\mu}$ in $\mathcal{H}_\mu$) as guaranteed by* Remark 1.3 *of* Section 1. *Then the following statements are true:*

(i) $|\mathcal{F}(f, g) - \overline{\mathcal{F}(g, f)}| \leq 2M||f||_\Gamma||g||_{\mathcal{H}_r}$ *for all* $f, g \in \Gamma$.

(ii) $\Gamma = \mathcal{D}((A - a)^{1/2})$, *where* $a = r - 1$.

*Proof.* (i) Since $\mathrm{supp}(\mu) \subset \sigma(N)$ (the proof of Lemma 2.2), and $\sigma(N) \subset \Lambda_{r,M} \cap P$ (hypothesis), one has $\mathrm{supp}(\mu) \subset \Lambda_{r,M} \cap P$. In view of this, one has, for any $f, g \in \Gamma$,

$$\begin{aligned}
\left|\mathcal{F}(f, g) - \overline{\mathcal{F}(g, f)}\right| &= \left|\int zf(z)\overline{g(z)}d\mu - \overline{\int zg(z)\overline{f(z)}d\mu}\right| \\
&= \left|\int (z - \overline{z})f(z)\overline{g(z)}d\mu\right| \\
&\leq 2\int |\mathrm{Im}z||f(z)||g(z)|d\mu \\
&\leq 2\left(\int |\mathrm{Im}z|^2|f(z)|^2d\mu(z)\right)^{1/2}\left(\int |g(z)|^2d\mu(z)\right)^{1/2} \\
&\leq 2M^{1/2}||f||_\Gamma||g||_{\mathcal{H}_\mu}.
\end{aligned}$$

(ii) The desired result follows from part (i) above, [**7**, Theorem 2.8.12 and Corollary 6.1.14], and the choice $a = r - 1$ as recorded at the end of Section 1. $\square$

For a Hilbert space $\mathcal{H}$ and for a positive integer $k$, let $C^k([0, \infty), \mathcal{H})$ denote the vector space of $\mathcal{H}$-valued $k$ times continuously differentiable functions on $[0, \infty)$. For $u \in C^k([0, \infty), \mathcal{H})$ and for a positive integer $m \leq k$, let $u^{(m)}(t)$ denote the $m$th derivative of $u$ at $t$.

PROPOSITION 3.3. *Let $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ denote a functional model of cyclic subnormal $S$ and let $N$ be a normal extension of $S$. Suppose that $\sigma(N)$ is contained in $\Lambda_{r,M} \cap P$ where $\Lambda_{r,M} = \{z \in \mathcal{C} : |\mathrm{Im}z| \leq M(\mathrm{Re}z - r)\}$ and $P = \{z \in \mathcal{C} : (\mathrm{Im}z)^2 \leq M(\mathrm{Re}z - r + 1)\}$ with $r \in \mathcal{R}$ and $M \in (0, \infty)$. Suppose that $S$ admits an analytic model and that $\{\sum_{n=0}^k \frac{(-m)^n z^n}{n!}\}_{k\geq 0}$ converges in $\mathcal{H}_\mu$ for every integer $m \geq 1$. If $\Gamma \equiv \mathcal{D}((\overline{M_{z,\mu}} - r + 1)^{1/2})$, then for each $f \in \mathcal{D}(\overline{M_{z,\mu}})$, $g \in \Gamma$ there exists a unique $u \in C^2([0, \infty), \mathcal{H}_\mu) \cap C^1([0, \infty), \Gamma)$ such that $u(0) = f$, $u^{(1)}(0) = g$, $u(t) \in \mathcal{D}(\overline{M_{z,\mu}})$ for all $t \in [0, \infty)$, and $u^{(2)}(t) = -zu(t)$ for all $t \in [0, \infty)$.*

*Proof.* As shown by Lemma 2.4, the action of the Friedrichs extension $A$ of $M_{z,\mu}$ is multiplication by $z$. The desired conclusion now follows from the previous lemma and [**7**, Theorem 4.4.2]. $\square$

Set $\Lambda_\theta = \{z \in \mathcal{C} : |\mathrm{Im}z| \leq \tan\theta \cdot \mathrm{Re}z\}$ for some $\theta \in (0, \pi)$. Let $H^\infty(\Lambda_\theta)$ denote the Banach algebra of all bounded holomorphic functions on $\Lambda_\theta$ endowed with the supremum norm $\|\cdot\|_{\infty,\Lambda_\theta}$.

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Our next application of Theorem 2.3 is a bounded $\mathcal{H}^\infty$-functional calculus for certain cyclic subnormals.

PROPOSITION 3.4. *Let S denote a cyclic subnormal operator with a normal extension N. Suppose that $\sigma_p(S^*)^*$ has non-empty interior and that $\sigma(N)$ is contained in $\Lambda_\omega$ where $\Lambda_\omega = \{z \in \mathcal{C} : |\mathrm{Im}z| \leq \tan w \cdot \mathrm{Re}z\}$ for some $\omega \in (0, \frac{\pi}{2})$. Suppose further that S admits an analytic model and that $\{\sum_{n=0}^{k} \frac{(-m)^n S^n f_0}{n!}\}_{k\geq 0}$ converges in $\mathcal{H}$ for every integer $m \geq 1$. If $\theta \in (\omega, \frac{\pi}{2})$ then S admits an $H^\infty(\Lambda_\theta)$-functional calculus such that*

$$\|f(\overline{S})\| \leq \left(2 + \frac{2}{\sqrt{3}}\right) \|f\|_{\infty,\Lambda_\theta} \ (f \in H^\infty(\Lambda_\theta)).$$

*Proof.* By Theorem 2.3 the Friedrichs extension of $S$ is $\overline{S}$. Further, an examination of the proof of Proposition 2.1 reveals that

$$\{\langle Sh, h\rangle : h \in \mathcal{D}(S) \text{ and } \|h\| = 1\} \subset \{z \in \mathcal{C} : |\mathrm{Im}z| \leq \tan\omega \cdot \mathrm{Re}z\}.$$

A routine limit argument now shows that

$$\{\langle \overline{S}h, h\rangle : h \in \mathcal{D}(\overline{S}) \text{ and } \|h\| = 1\} \subset \{z \in \mathcal{C} : |\mathrm{Im}z| \leq \tan\omega \cdot \mathrm{Re}z\}.$$

Furthermore, in view of

$$\sigma(\overline{S}) \subset \{\zeta \in \mathcal{C} : \mathrm{Re}\,\zeta \geq 0 \ \text{ and } |\mathrm{Im}\,\zeta| \leq (\tan\omega + 2)(\mathrm{Re}\,\zeta + 1)\}$$

(see (7)), $\overline{S} + I$ is invertible. In view of these observations and [**6**, Corollary 7.1.17], it suffices to check that $\overline{S}$ is injective. Since there exists a unitary $W$ from $\mathcal{H}$ onto $\mathcal{H}_r$ such that $W\overline{S} = \overline{M_z}W$ (see the discussion following Theorem 1.1), it is sufficient to verify that $\overline{M_z}$ is injective. To see that suppose $\overline{M_z}f = 0$ for some $f \in \mathcal{D}(\overline{M_z})$. Since the action of $\overline{M_z}$ is multiplication by $z$, using the reproducing property of $\mathcal{H}_r$ one has $zf(z) = 0$ for every $z \in \sigma_p(M_z^*)^*$. Thus $f(z) = 0$ for every $z \in \sigma_p(M_z^*)^* \setminus \{0\}$. By [**9**, Theorem 7], the set $\gamma(M_z)$ as defined in (1) is the maximal open subset of $\sigma_p(M_z^*)^*$ on which all functions in $\mathcal{H}_r$ are holomorphic. It follows that $f$ is identically zero. □

Proposition 3.4 yields an important polynomial approximation result in the functional model space.

COROLLARY 3.5. *Assume the hypotheses of Proposition 3.4. Let $(M_{z,\mu}, \mathbb{C}[z], \mathcal{H}_\mu)$ denote a functional model of S. Then $H^\infty(\Lambda_\theta)$ is contained in the functional model space $\mathcal{H}_\mu$ for every $\theta \in (\omega, \frac{\pi}{2})$. In particular, for every $\theta \in (\omega, \frac{\pi}{2})$ and for every bounded holomorphic f on $\Lambda_\theta$ there exists a sequence $\{p_n\}_{n\geq 0}$ of complex polynomials such that $\int_{\Lambda_\omega} |p_n(z) - f(z)|^2 d\mu(z)$ converges to 0 as $n \to \infty$.*

*Proof.* Since there exists a unitary $U$ from $\mathcal{H}$ onto $\mathcal{H}_\mu$ such that $US = TU$, we may assume that $S = M_{z,\mu}$ (refer to Remark 1.3 and Lemma 2.4). In view of the preceding proposition, it suffices to verify that the action of $f(\overline{M_{z,\mu}})$ is multiplication by $f$. Since $\sigma(\overline{M_{z,\mu}}) \subset \overline{\Lambda_\omega}$ (Lemma 2.2), it follows that the action of $(z-\mu)^{-1}(\overline{M_{z,\mu}}) \equiv (\overline{M_{z,\mu}} - \mu)^{-1}$ is multiplication by $(z-\mu)^{-1}$ for all $\mu \notin \overline{\Lambda_\theta}$. Thus for every rational function $f$ with poles lying outside $\overline{\Lambda_\theta}$ the action of $f(\overline{M_{z,\mu}})$ is multiplication by $f$ (see [**6**, Proposition F.3.]).

Now let $f \in H^\infty(\Lambda_\theta)$. By [**6**, Proposition F.4] there exists a sequence of rational functions with poles lying outside $\overline{\Lambda_\theta}$ such that $\|r_n\|_{\Lambda_\theta,\infty} \leq \|f\|_{\Lambda_\theta,\infty}$ for all $n$, and $r_n \to f$ pointwise on $\Lambda_\theta$. In particular, $\sup_{n\geq 0} \|r_n\|_{\infty,\Lambda_\theta} < \infty$. Since $\overline{S}$ is injective (see the proof

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of Proposition 3.4), so is $\overline{M_{z,\mu}}$. Hence by [**6**, Proposition 2.1.1(h)] the range-space of $\overline{M_{z,\mu}}$ is dense in $\mathcal{H}$. Also, by the preceding proposition one has

$$\sup_{n\geq 0}\|r_n(\overline{M_{z,\mu}})\|_{\mathcal{H}_\mu} \leq \left(2+\frac{2}{\sqrt{3}}\right)\sup_{n\geq 0}\|r_n\|_{\infty,\Lambda_\theta} < \infty.$$

Now one may appeal to the Convergence Lemma ([**6**, Proposition 5.1.4(b)]) to conclude that $\|r_n(\overline{M_{z,\mu}})x - f(\overline{M_{z,\mu}})x\|_{\mathcal{H}_\mu} \to 0$ for every $x \in \mathcal{H}_\mu$. Since the action of each $r_n(\overline{M_{z,\mu}})$ is multiplication by $z$, by the standard measure theory, so is the action of $f(\overline{M_{z,\mu}})$. This completes the proof of the corollary. □

**4. Examples: multiplication operators with analytic symbols.** In this section, we present a generalization of the main result of [**5**]. To see that, we need some technical jargon. Let $G$ be an open subset of the complex plane $\mathcal{C}$ and let $w : G \to (0, \infty)$ be a positive continuous function. We use $\mu_w$ to denote the weighted area measure on $G$ defined by $d\mu_w(z) = w(z)dz$ $(z \in G)$. Let $L^2(G, \mu_w)$ stand for the Hilbert space of $\mu_w$-square integrable Lebesgue measurable functions on $G$ (with two functions being identified if they are $\mu_w$-almost everywhere equal to each other). Let

$$L^2_a(G, \mu_w) = \{f \in L^2(G, \mu_w) : f \text{ is analytic on } G\}.$$

As in the case of the Bergman space of a bounded domain (see [**3**, Chapter II, Proposition 8.4 and Theorem 8.5]), it can be verified that $L^2_a(G, \mu_w)$ is a Hilbert space, and the point evaluation is bounded on $L^2_a(G, \mu_w)$, so that for any $\lambda \in G$, there exists $k_\lambda \in \mathcal{H} = L^2_a(G, \mu_w)$ such that

$$\langle f, k_\lambda\rangle_{\mathcal{H}} = f(\lambda) \text{ for all } f \text{ in } L^2_a(G, \mu_w). \tag{9}$$

DEFINITION 4.1. A triple $(G, \phi, w)$ is said to be a *weighted analytic domain* if $G$ is a non-empty open subset of the complex plane $\mathcal{C}$, $\phi : G \to \mathcal{C}$ is a non-constant analytic function, $w : G \to (0, \infty)$ is a continuous function, and $\phi^k \in L^2(G, \mu_w)$ for every non-negative integer $k$. (We interpret $\phi^0$ to be the constant function $1_G$ with $1_G(z) = 1$ for all $z$ in $G$.)

Let $(G, \phi, w)$ be a weighted analytic domain with $P^2_\phi(G, \mu_w)$ being the closure of $\text{lin}\{\phi^k : k = 0, 1, 2, \ldots\}$ in $L^2(G, \mu_w)$. Then the operator $M_\phi$ of multiplication by $\phi$ in $P^2_\phi(G, \mu_w)$ with domain $\mathcal{D}(M_\phi) \equiv \text{lin}\{\phi^k : k = 0, 1, \ldots\}$ is subnormal; indeed, a normal extension of $M_\phi$ is $N_\phi$ in $L^2(G, \mu_w)$ defined by $N_\phi f = \phi f$ for $f \in \mathcal{D}(N_\phi) \equiv \{f \in L^2(G, \mu_w) : \phi f \in L^2(G, \mu_w)\}$. It can be easily seen that $\sigma(N_\phi) = \overline{\phi(G)}$. We define $M^{\max}_\phi$ to be the operator of multiplication by $\phi$ on $\mathcal{D}(M^{\max}_\phi) = \{f \in P^2_\phi(G, \mu_w) : \phi f \in P^2_\phi(G, \mu_w)\}$. It is easy to check that $M^{\max}_\phi$ is a closed linear operator in $P^2_\phi(G, \mu_w)$; $M^{\max}_\phi$ obviously extends $\overline{M_\phi}$.

The following lemma borrowed from ([**4**, Lemma 3 of Chapter III]) in particular resolves Question 1 of [**5**] in the negative. For the sake of completeness, we are including the proof of the same.

LEMMA 4.2. *Let $(G, \phi, w)$ be a weighted analytic domain and let $M_\phi$ be as described in the discussion following Definition* 4.1. *Then the following are true.*

1. *$M_\phi$ admits an analytic model, and*
2. *The closure $\overline{M_\phi}$ of $M_\phi$ is equal to $M^{\max}_\phi$.*

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*Proof.* (1) As noted in [**5**, Proposition 1], $\phi(G) \subset \sigma_p(M_\phi^*)^*$; in particular, $\sigma_p(M_\phi^*) \neq \emptyset$. So, by Theorem 1.1, there exists a surjective partial isometry $W : P_\phi^2(G, \mu_w) \to \mathcal{H}_r$, where $\mathcal{H}_r$ is as described in the paragraph preceding Theorem 1.1. It suffices then to check that the null-space of $W$ is trivial. We adapt the argument of [**9**, Proposition 9] to the present situation. Let $f \in P_\phi^2(G, \mu_w)$ be such that $Wf = 0$. Examining the proof of [**9**, Theorem 6], we find that $W(p \circ \phi)(\lambda) = p(\lambda)$ for any polynomial $p$ and any $\lambda$ in $\sigma_p(M_\phi^*)^*$. Since $f$ belongs to $\mathcal{H} = P_\phi^2(G, \mu_w)$, there exists a sequence $p_n$ of polynomials such that $\|p_n \circ \phi - f\|_{\mathcal{H}}$ converges to 0. Then there exists a subsequence $\{q_n\}$ of $\{p_n\}$ such that, $z$-a.e. $[\mu_w]$, $W(q_n \circ \phi)(\phi(z)) = q_n(\phi(z)) = (q_n \circ \phi)(z)$ converges to $f(z)$. Also, since $W$ is continuous, $\| W(q_n \circ \phi) - Wf\|_{\mathcal{H}_r}$ converges to 0. Since $\mathcal{H}_r$ is a reproducing kernel Hilbert space, it follows that $W(q_n \circ \phi)(\phi(z))$ converges to $(Wf)(\phi(z))$; thus, $z$-a.e. $[\mu_w]$, $(Wf)(\phi(z)) = f(z)$. Hence $f$ vanishes almost everywhere $[\mu_w]$; but $f$ is analytic so that $f = 0$.

(2) The proof of the second part here is an adaptation of the proof of [**9**, Proposition 11]. As recorded in Section 1, $\gamma(M_\phi)$ as defined in (1) is an open subset of $\mathrm{int}(\sigma_p(M_\phi^*)^*)$ and $\Omega = \mathrm{int}\left(\sigma_p(M_\phi^*)^*\right) \setminus \gamma(M_\phi)$ is a nowhere dense subset of $\mathcal{C}$. Further, [**5**, Proposition 1] yields that $\phi(G) \subset \sigma_p(M_\phi^*)^*$. Therefore $\phi(G) \subset \mathrm{int}(\sigma_p(M_\phi^*)^*)$. We claim that $\gamma(M_\phi) \cap \phi(G) \neq \emptyset$. Suppose that $\gamma(M_\phi) \cap \phi(G)$ is empty so that $\phi(G) \subset \gamma(M_\phi)^c$. Hence $\phi(G) \subset \mathrm{int}(\sigma_p(M_\phi^*)^*) \cap \gamma(M_\phi)^c = \mathrm{int}(\sigma_p(M_\phi^*)^*) \setminus \gamma(M_\phi)$. Since $\phi(G)$ is a non-empty open subset of $\mathcal{C}$, this contradicts the fact that $\Omega = \mathrm{int}\left(\sigma_p(M_\phi^*)^*\right) \setminus \gamma(M_\phi)$ is a nowhere dense subset of $\mathcal{C}$. Hence we must have $\gamma(M_\phi) \cap \phi(G) \neq \emptyset$. Next we observe that $\gamma(M_\phi) \cap (\phi(G) \setminus \sigma_{ap}(\overline{M_\phi}))$ is non-empty. By [**9**, Theorem 9], one has $\gamma(M_\phi) = \sigma(\overline{M_\phi}) \setminus \sigma_{ap}(\overline{M_\phi}) = \sigma(\overline{M_\phi}) \cap \sigma_{ap}(\overline{M_\phi})^c$. Hence

$$\begin{aligned}
\gamma(M_\phi) \cap \phi(G) &= (\sigma(\overline{M_\phi}) \cap \sigma_{ap}(\overline{M_\phi})^c) \cap \phi(G) \\
&= (\sigma(\overline{M_\phi}) \cap \sigma_{ap}(\overline{M_\phi})^c) \cap (\sigma_{ap}(\overline{M_\phi})^c \cap \phi(G)) \\
&= (\sigma(\overline{M_\phi}) \setminus \sigma_{ap}(\overline{M_\phi})) \cap (\phi(G) \setminus \sigma_{ap}(\overline{M_\phi})) \\
&= \gamma(M_\phi) \cap (\phi(G) \setminus \sigma_{ap}(\overline{M_\phi})).
\end{aligned}$$

Since $\gamma(M_\phi) \cap \phi(G)$ is non-empty, so is $\gamma(M_\phi) \cap (\phi(G) \setminus \sigma_{ap}(\overline{M_\phi}))$. Thus there exists some $\lambda_0$ in $\gamma(M_\phi) \cap (\phi(G) \setminus \sigma_{ap}(\overline{M_\phi}))$. Let $f$ be in $\mathcal{D}(M_\phi^{\max})$; then $(M_\phi^{\max} - \lambda_0)f \in \mathcal{H} = P_\phi^2(G, \mu_w)$. Thus there exists a sequence $\{q_n \circ \phi\}$ such that $\|q_n \circ \phi - (M_\phi^{\max} - \lambda_0)f\|_{\mathcal{H}}$ converges to 0. Let $z_0 \in G$ be such that $\phi(z_0) = \lambda_0$. Since $P_\phi^2(G, \mu_w) \subset L_a^2(G, \mu_w)$, by (9), $(q_n \circ \phi)(z_0)$ converges to 0; consequently, $t_n = q_n \circ \phi - (q_n \circ \phi)(z_0) \in \mathrm{lin}\{\phi^k : k = 0, 1, 2, \ldots\}$, $t_n$ converges to $(M_\phi^{\max} - \lambda_0)f$, and $t_n$ vanishes at $z_0$ for all $n$. Let $p_n \in \mathrm{lin}\{\phi^k : k = 0, 1, 2, \ldots\}$ be such that $t_n = (\phi - \phi(z_0))p_n$. Since $\lambda_0$ is not in $\sigma_{ap}(\overline{M_\phi})$, there exists a positive number $N$ such that $\|(\overline{M_\phi} - \lambda_0)h\|_{\mathcal{H}} \geq N\|h\|_{\mathcal{H}}$ for every $h \in \mathcal{D}(\overline{M_\phi})$. Putting $h = p_n$ one has $\|t_n\|_{\mathcal{H}} \geq N\|p_n\|_{\mathcal{H}}$ for all $n$. This shows that the sequence $\{p_n\}$ converges to some $g$ in $\mathcal{H}$. Since $(M_\phi - \lambda_0)p_n = t_n$ converges to $(M_\phi^{\max} - \lambda_0)f$, the closedness of $\overline{M_\phi}$ shows that $g \in \mathcal{D}(\overline{M_\phi})$ and $(M_\phi^{\max} - \lambda_0)f = (\overline{M_\phi} - \lambda_0)g$. This in turn implies that $f = g$ except possibly at $z_0$; but both $f$ and $g$ are analytic so that $f = g$. Thus $f$ belongs to $\mathcal{D}(\overline{M_\phi})$ showing that $\mathcal{D}(M_\phi^{\max}) \subset \mathcal{D}(\overline{M_\phi})$. □

EXAMPLE 4.3. Let $\omega \in (0, \frac{\pi}{2})$ and let $\Lambda_\omega = \{z \in \mathcal{C} : |\mathrm{Im} z| \leq \tan\omega \cdot \mathrm{Re} z\}$. Let $\theta \in (\omega, \frac{\pi}{2})$ and let $f$ be a bounded analytic function on $\Lambda_\theta$. It is easy to see that $(G \equiv \Lambda_\omega, \phi \equiv z|_{\Lambda_\omega}, w \equiv \exp(-|z|^2)|_{\Lambda_\omega})$ is a weighted analytic domain, and that $\exp(-k\phi) \in P_\phi^2(G, \mu_w)$ for every integer $k \geq 0$. Since $M_\phi$ is subnormal (see the discussion following

Definition 4.1) and since $M_\phi$ admits an analytic model (Lemma 4.2), by Corollary 3.5 there exists a sequence $\{p_n\}_{n\geq 1}$ of complex polynomials such that

$$\lim_{n\to\infty}\int_{\Lambda_\omega}|p_n(z)-f(z)|^2\exp(-|z|^2)dz = 0.$$

The following result generalizes [**5**, Theorem 3].

THEOREM 4.4. *Let* $(G,\phi,w)$ *be a weighted analytic domain. Suppose that* $\phi(G)$ *is contained in the cone* $\Lambda_{r,M} = \{z\in\mathcal{C} : |\mathrm{Im}z| \leq M(\mathrm{Re}z - r)\}$ *where* $r\in\mathcal{R}$ *and* $M\in(0,\infty)$*, and* $\exp(-k\phi)\in P^2_\phi(G,\mu_w)$ *for every integer* $k\geq 0$*. Then the Friedrichs extension of* $M_\phi$ *is* $M_\phi^{\max}$*.*

*Proof.* Since $\phi(G)\subset\sigma_p(M_\phi^*)^*$ ([**5**, Proposition 1]) and since $\sigma(N_\phi)=\overline{\phi(G)}$, one has $\phi(G)\subset\sigma_p(M_\phi^*)^*\cap\sigma(N_\phi)$. Now arguing as in the proof of Lemma 2.2, it can be seen that the first part of Lemma 2.2 still holds true under the assumption that $\exp(-k\phi)\in P^2_\phi(G,\mu_w)$ for every integer $k\geq 0$. Now one may appeal to Theorem 2.3 with $S=M_\phi$, $f_0=1_G$, and $N=N_\phi$. Since $M_\phi$ admits an analytic model and since $\overline{M_\phi}=M_\phi^{\max}$ (Lemma 4.2), the desired conclusion follows. □

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