7

Cayley Flatness

There are two traditional notions of what a "family of varieties" is: the older Cayley–Chow variant (3.5) and the currently ubiquitous Hilbert–Grothendieck variant (3.6), which puts flatness at the center.

For stable varieties, the Hilbert–Grothendieck approach gives the correct moduli theory. That is, a stable morphism $X \to S$ is a flat morphism with additional properties, as in Section 6.2.

A major problem in the moduli theory of stable pairs is that, while the underlying varieties X form flat families, the divisorial parts Δ do not. Neither of the two main traditional methods of parametrizing varieties or schemes gives the right answer for the divisorial part.

- Cayley–Chow theory works only over reduced base schemes.
- Hilbert–Grothendieck theory works only when the coefficients of Δ satisfy various restrictions, as in Sections 6.2 and 6.4.

In this chapter we develop a theory – called K-flatness – that interpolates between these two, managing to keep from both of them the properties that we need. The objects that we parametrize are divisors – so the strong geometric flavor of Cayley–Chow theory is preserved – but one can work over Artinian base schemes. The latter is one of the key advantages of the theory of Hilbert schemes. Quite unexpectedly, the new theory behaves better than either of the classical approaches in several aspects; see especially (7.4–7.5).

One might say that the main new result is Definition 7.1; we discuss its origin and relationship to the classical theory of Chow varieties in (7.2). The rest of this chapter is then devoted to proving that it has all the hoped-for properties. (Actually, we end up with several variants, but we conjecture them to be equivalent; see Section 7.4.)

The definition of K-flatness and its main properties are discussed in Section 7.1, while Section 7.2 reviews divisor theory over Artinian schemes. The key notion of divisorial support is introduced and studied in Section 7.3.

Several versions of K-flatness are investigated in Section 7.4. For our treatment, technically the most important is C-flatness, which is treated in detail in Section 7.5. The main results are proved in Section 7.6.

Sections 7.7–7.9 are devoted to examples. First, we show that a K-flat deformation of a normal variety is flat. Then we describe first order K-flat deformations of plane curves in Section 7.8 and of seminormal curves in Section 7.9. While the computations are somewhat lengthy, the answers are quite nice.

Assumptions In this Chapter we work over an arbitrary field k.

7.1 K-flatness

We eventually introduce several closely related (possibly equivalent) notions in (7.37). The most natural one is C-flatness, which is closest to the ideas of Cayley. Aiming to create a notion that is independent of projective embeddings led to K-flatness. Conveniently, K is also the first syllable of Cayley.

Definition 7.1 (K-flatness) Let $f: X \to S$ be a projective morphism of pure relative dimension n. A relative Mumford divisor $D \subset X$ as in (4.68) is *K-flat* over S iff one of the following–increasingly more general–conditions hold.

(7.1.1) (S local with infinite residue field) For every finite morphism $\pi: X \to \mathbb{P}^n_S$, $\pi_* D \subset \mathbb{P}^n_S$ is a relative Cartier divisor.

(7.1.2) (S local) For some (equivalently every) flat, local morphism $q: S' \to S$, where S' has infinite residue field, the pull-back $q^* D$ is K-flat over S'.

(7.1.3) (S arbitrary) D is K-flat over every localization of S.

Let us start with some comments on the definition.

(7.1.4) The definition of $\pi_* D$ is not always obvious; in essence Section 7.3 is mainly devoted to establishing it. However, $\pi_* D$ equals the scheme-theoretic image of D if $\operatorname{red} D \to \operatorname{red}(\pi(D))$ is birational and π is étale at every generic point of the closed fiber D_s (7.28.2). It is sufficient to check condition (7.1.1) for such morphisms $\pi: X \to \mathbb{P}^n_S$.

(7.1.5) If S is not local, then there may not be any finite morphisms $\pi: X \to \mathbb{P}^n_S$; see (7.7.2) for an example. This is one reason for the three-step definition.

(7.1.6) The residue field extension in (7.1.2) is necessary in some cases; see for example (7.80.9).

(7.1.7) The definition of K-flatness is global in nature, but we show that it is in fact local on X (7.52).

(7.1.8) We eventually define K-flatness also for families of coherent sheaves in (7.37). This turns out to be quite convenient technically. However, while

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the images $\pi_* D$ carry a lot of information about a Mumford divisor D, much of the sheaf information is lost. Thus it is unlikely that K-flatness can be useful for studying the moduli of sheaves.

7.2 (Why this definition?) The idea in the papers Cayley (1860, 1862) is to associate to a subvariety $Y^{n-1} \subset \mathbb{P}^N_k$ a hypersurface

$$\mathrm{Ch}(Y) := \{L \in \mathrm{Gr}(N-n, \mathbb{P}^N_k) \colon Y \cap L \neq \emptyset\} \subset \mathrm{Gr}(N-n, \mathbb{P}^N_k),$$

we call it the Cayley–Chow hypersurface. In modern terminology, the end result is that, over weakly normal bases, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{well-defined families} \\ \text{of subvarieties} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{flat families of} \\ \text{Cayley–Chow hypersurfaces} \end{array} \right\}; \qquad (7.2.1)$$

see Section 4.8 or Kollár (1996, sec.I.3) for details.

The correspondence (7.2.1) works well for geometrically reduced, pure dimensional subschemes, but for an arbitrary subscheme $Z \subset \mathbb{P}^N$, its Cayley–Chow hypersurface $\mathrm{Ch}(Z)$ detects only $\operatorname{red} Z$ and the multiplicities of Z at the maximal dimensional generic points. This is where the role of X and the Mumford condition become crucial: a Mumford divisor $D \subset X$ is uniquely determined by $\operatorname{red} D$ and the multiplicities.

We know how to define flatness in general, so we try to make the equivalence into a definition over an arbitrary base scheme. So let $f \colon X \to S$ be a flat, projective morphism, say with reduced fibers of pure dimension n. Fix an embedding $X \hookrightarrow \mathbb{P}^N_S$ and let $D \subset X$ be a Mumford divisor. We say that D is C-flat over S iff $\mathrm{Ch}(D/S)$ is flat over S. (This needs a suitable extension of the definition of $\mathrm{Ch}(D/S)$ to allow for multiple fibers; see (7.37) for details.)

There are two immediate disadvantages of C-flatness. Cayley–Chow hypersurfaces are unwieldy objects and the resulting notion is very much tied to the choice of an embedding $X^n \hookrightarrow \mathbb{P}^N_k$.

One can think of a Cayley–Chow hypersurface $\mathrm{Ch}(D/S)$ as encoding the images $\pi(D)$ for all linear projections $\pi \colon \mathbb{P}^N_S \dashrightarrow \mathbb{P}^n_S$. (This also goes back to Cayley; it is worked out in Catanese (1992), Dalbec and Sturmfels (1995), and Kollár (1999).) One can show that the Cayley–Chow hypersurface $\mathrm{Ch}(D/S)$ is flat over S iff $\pi(D) \subset \mathbb{P}^n_S$ is flat over S, for all *linear* projections $\pi \colon \mathbb{P}^N_S \dashrightarrow \mathbb{P}^n_S$ that are finite on $\operatorname{Supp} D$; see (7.47). (In fact, by (7.47), it is enough to check this for a dense set of projections. We need S to be local with infinite residue field to ensure that there are enough projections.)

This suggests three different generalizations of C-flatness. We can work with

- projective morphisms $f \colon X \to S$ and all finite $\pi \colon X \to \mathbb{P}^n_S$,
- affine morphisms $f \colon U \to S$ and all finite $\pi \colon U \to \mathbb{A}^n_S$, or

• morphisms of complete, local schemes $f\colon \widehat{X} \to \widehat{S}$ and all finite $\pi\colon \widehat{X} \to \widehat{\mathbb{A}}^n_S$. The affine version has the problem that, even if S is local, there might not be any finite morphisms $\pi\colon U \to \mathbb{A}^n_S$; see (7.38.4) for more on this. Working with complete, local schemes would be the best theoretically, but several of the technical problems remain unresolved. This leaves us with projective morphisms, which is our definition of K-flatness.

The key technical result (7.40) shows that K-flatness is equivalent to C-flatness for every Veronese embedding $X \hookrightarrow \mathbb{P}^N_S \stackrel{v_m}{\hookrightarrow} \mathbb{P}^M_S$ (where $M = \binom{N+m}{m} - 1$); we call the resulting notion stable C-flatness.

We conjecture that stable C-flatness, K-flatness, local K-flatness, and formal K-flatness are equivalent, giving a very robust concept. This would show that our notion is truly about the singularities in families of divisors. The equivalence of C-flatness and K-flatness would be very helpful computationally, but does not seem to be theoretically significant.

Good Properties of K-flatness

K-flat families have several good properties. Some of them are needed for the moduli theory of stable pairs, but others, for example (7.5), come as a bonus.

The functoriality of K-flatness is not obvious. Indeed, let $T \subset S$ be a closed subscheme. Then a finite morphism $\pi_T\colon X_T \to \mathbb{P}^n_T$ need not extend to a finite morphism $\pi_S\colon X_S \to \mathbb{P}^n_S$. Thus flatness of all $\pi_S(X_S)$ does not directly imply that $\pi_T(X_T)$ is also flat.

Nonetheless, we prove in (7.40) and (7.50) that being K-flat is preserved by arbitrary base changes and it descends from faithfully flat base changes. Thus we get the functor $\mathcal{KDiv}(X/S)$ of K-flat, relative Mumford divisors on X/S. If we have a fixed relatively ample divisor H on X, then $\mathcal{KDiv}_d(X/S)$ denotes the functor of K-flat, relative Mumford divisors of degree d.

We have a disjoint union decomposition $\mathcal{KDiv}(X/S) = \cup_d \mathcal{KDiv}_d(X/S)$. The main result is the following, to be proved in (7.66).

Theorem 7.3 *Let $f\colon X \to S$ be a projective morphism of pure relative dimension n. Then the functor $\mathcal{KDiv}_d(X/S)$ of K-flat, relative Mumford divisors of degree d is representable by a separated S-scheme of finite type* $\mathrm{KDiv}_d(X/S)$.

Complement 7.3.1 If f is flat with normal fibers, then $\mathrm{KDiv}_d(X/S)$ is proper over S, but otherwise usually not. This is not a problem for us.

7.4 (Properties of K-flatness) We list a series of good properties of K-flatness. Let $f\colon X \to S$ be a projective morphism of pure relative dimension n and D or D_i relative Mumford divisors.

7.4.1 (Comparison with flatness) K-flatness is a generalization of flatness and it is equivalent to it for smooth morphisms and for normal divisors.

- If $f|_D\colon D \to S$ is flat, then D is K-flat; see (7.54).
- If $f\colon X \to S$ is smooth, then D is K-flat $\Leftrightarrow$ D is flat over S $\Leftrightarrow$ D is a relative Cartier divisor; see (7.53).
- Assume that D is K-flat, $D_s \subset X_s$ has multiplicity 1, and $\operatorname{red}(D_s)$ is normal for some $s \in S$. Then $f|_D\colon D \to S$ is flat along D_s by (7.67).

These properties also hold locally on X. Hence, the notion of K-flatness gives something new only at the points where f is not smooth and $f|_D$ is not flat.

7.4.2 (Reduced base schemes) If S is reduced then every relative Mumford divisor is K-flat; see (7.29). In retrospect, this is the reason why the moduli theory of pairs could be developed over reduced base schemes without the notion of K-flatness in Chapter 4.

7.4.3 (Artinian base schemes) A divisor $D \subset X$ is K-flat over S iff $D_A \subset X_A$ is K-flat over A for every Artinian subscheme $A \subset S$; see (7.44). Thus one can fully understand K-flatness by studying it over reduced bases (as in Chapter 4) and over Artinian base schemes.

7.4.4 (Push-forward) Let $g\colon Y \to S$ be another projective morphisms of pure relative dimension n, and $\tau\colon X \to Y$ a finite morphism. Assume that $D \subset X$ is K-flat and $\tau_* D$ is also a relative Mumford divisor. (That is, g is smooth at generic points of $\tau(D_s)$ for every s.) Then $\tau_* D$ is also K-flat, see (7.45). (See Section 7.3 for the definition of $\tau_* D$.) A similar property fails for flatness; combine (7.7.3) and (7.45).

7.4.5 (Additivity) If D_1, D_2 are K-flat, then so is $D_1 + D_2$, see (7.45). This again fails for flatness; see (7.7.3).

7.4.6 (Multiplicativity) Let $m > 0$ be relatively prime to the residue characteristics. Then D is K-flat iff mD is K-flat; see (7.45).

By contrast, if A is Artinian, nonreduced, with residue field k of characteristic $p > 0$, then the divisors D on $\mathbb{P}^2_A$ such that pD is K-flat (= relative Cartier), but D is not K-flat, span an infinite dimensional k-vectorspace; see (7.10.4–5). This is an extra difficulty in positive characteristic, see Section 8.8.

7.4.7 (Linear equivalence) K-flatness is preserved by linear equivalence; see (7.33). (Note that flatness is not preserved by linear equivalence (7.7.4).)

7.4.8 (K-flatness depends only on the divisor) It is well understood that in the theory of pairs (X, Δ) one cannot separate the underlying variety X from the divisorial part Δ. For example, if X is a surface with quotient singularities only and $D \subset X$ is a smooth curve, then the pair (X, D) is plt if $D \cap \operatorname{Sing} X = \emptyset$, but not even lc in some other cases. It really matters how exactly D sits inside X.

Thus it is unexpected that K-flatness depends only on the divisor D, not on the ambient variety X, though maybe this is less surprising if one thinks of K-flatness as a variant of flatness.

On the other hand, not all K-flat deformations (7.37) of D are realized on deformations of a given X. For example, for deformations of the pair $(\mathbb{A}^2, D_1 := (xy = 0))$, K-flatness is equivalent to flatness by (7.4.2). However, there are deformations of the pair $((xy = z^2), (z = 0))$ that induce a K-flat, but non-flat deformation of $D_2 := (xy - z^2 = z = 0) \simeq D_1$. A typical example is

$$((xy = z^2 - t^2), (x = z + t = 0) \cup (y = z - t) = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{A}^1_t.$$

Now we come to a property that is quite unexpected, but makes the whole theory much easier to use: K-flatness is essentially a property of surface pairs (S, D). Thus K-flatness is mostly about families of singular curves.

Theorem 7.5 (Bertini theorems, up and down) *Let $f\colon X \to S$ be a projective morphism of pure relative dimension n, and D a Mumford divisor on X. Assume that $n \geq 3$, and let $|H|$ be a linear system on X that is base point free in characteristic 0 and very ample in general. Then D is K-flat iff $D|_{H_\lambda}$ is K-flat for general $H_\lambda \in |H|$.*

This is established by combining (7.57–7.59) with (7.40). As a consequence, K-flatness is really a question about families of surfaces and curves on them.

This reduction to surfaces is very helpful both conceptually and computationally, since we have rather complete lists of singularities of log canonical surface pairs (X, Δ), at least when the coefficients of Δ are not too small.

Another variant of the phenomenon, that higher codimension points sometimes do not matter much, is the Hironaka-type flatness theorem (10.72).

7.6 (Problems and questions about K-flatness) There are also some difficulties with K-flatness. We believe that they do not effect the general moduli theory of stable pairs, but they make some of the proofs convoluted and explicit computations lengthy.

7.6.1 (The definition is not formal-local) One expects K-flatness to be a formal-local property on X, but there are some (hopefully only technical) problems with this. See (7.41) and (7.60) for partial results. This is probably the main open foundational question.

7.6.2 (Hard to compute) The definition of K-flatness is quite hard to check, since for $X \subset \mathbb{P}^N$ we need to check not just linear projections $\mathbb{P}^N_S \dashrightarrow \mathbb{P}^n_S$ (7.36), but all morphisms $X \to \mathbb{P}^n_S$ involving all linear systems on X.

It is, however, possible that checking general linear projections is in fact sufficient; see (7.47) and (7.42) for a precise formulation.

In the examples in Sections 7.8–7.9, the computation of the restrictions imposed by general linear projections is the hard part. From the resulting answers it is then easy to read off what happens for all morphisms $X \to \mathbb{P}^n_S$. It would be good to work out more examples of space curves $C \subset \mathbb{A}^3$.

7.6.3 (Tangent space and obstruction theory) We do not know how to write down the tangent space of $\mathrm{KDiv}(X/S)$. A handful of examples are computed in Sections 7.8–7.9, but they do not seem to suggest any general pattern. The obstruction theory of K-flatness is completely open.

7.6.4 (Universal deformations) Let D be a reduced, projective scheme over a field k. Is there a universal deformation space for its K-flat deformations?

Examples 7.7 The first example shows that the space of first order deformations of the smooth divisor $(x = 0) \subset \mathbb{A}^2$, that are Cartier away from the origin, is infinite dimensional. Thus working with generically flat divisors (3.26) does not give a sensible moduli space.

(7.7.1) Start with $X := \operatorname{Spec} k[x, y, \varepsilon]_{(x,y)}$ over $\operatorname{Spec} k[\varepsilon]$ and set $X^\circ := X \setminus (x = y = 0)$. Let $g(y^{-1}) \in y^{-1}k[y^{-1}]$ be a polynomial of degree n. Then

$$x + g(y^{-1})\varepsilon \in k[x, y, y^{-1}, \varepsilon]_{(x,y)}$$

defines a relative Cartier divisor D°_g, whose restriction to the closed fiber is $(x = 0)$. One can check (7.14) that, if $g_1 \neq g_2$, then $D^\circ_{g_1}$ and $D^\circ_{g_2}$ give different elements of $\mathrm{Pic}(X^\circ)$. Set

$$I_g := (x^2, xy^n + y^n g(y^{-1})\varepsilon, \varepsilon x) \subset k[x, y, \varepsilon]_{(x,y)}, \quad \text{and} \quad D_g := \operatorname{Spec} k[x, y, \varepsilon]/I_g.$$

Note that $y^n g(y^{-1})$ is invertible in $k[x, y, \varepsilon]_{(x,y)}$, hence

$$k[x, y, \varepsilon]_{(x,y)}/(x^2, xy^n + y^n g(y^{-1})\varepsilon, \varepsilon x) \simeq k[x, y]_{(x,y)}/(x^2).$$

Thus D_g is the scheme-theoretic closure of D°_g, $(I_g, \varepsilon)/(\varepsilon) = (x^2, xy^n)$, D_g has no embedded points, and $D_{g_1} \sim D_{g_2}$ iff $g_1 = g_2$. More general computations are done in (7.20).

(7.7.2) To illustrate (7.1.5), let C be a smooth projective curve and E a vector bundle over C of rank $n + 1 \geq 2$ and of degree 0. We claim that usually there is no finite morphism $\pi\colon \mathbb{P}_C(E) \to \mathbb{P}^n \times C$.

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Indeed, let $p_0, \ldots, p_{n+1} \in \mathbb{P}^n$ be the coordinate vertices plus $(1:\cdots:1)$. Then $C_i := \pi^{-1}(\{p_i\}\times C)$ are $n+2$ disjoint multi-sections of $\mathbb{P}_C(E) \to C$. Pick $p\colon D \to C$ that factors through all of the $C_i \to C$. Then $\mathbb{P}_D(p^*E)$ has $n + 2$ disjoint sections in linearly general position, hence $\mathbb{P}_D(p^*E) \simeq \mathbb{P}^n \times D$. Equivalently, $p^*D \simeq L \otimes \mathscr{O}_D^{n+1}$ for some line bundle L of degree 0.

This cannot happen for most line bundles. The simplest example is $E = \mathscr{O}_{\mathbb{P}^1}(1)\oplus\mathscr{O}_{\mathbb{P}^1}(-1)$. More generally, such a line bundle has to be semi-stable. If E is stable, hence comes from a representation $\pi_1(C) \to \mathrm{U}(n + 1)$, then its image in $\mathrm{PU}(n + 1)$ must be finite.

(7.7.3) As an example for (7.4.5), set $X := (xy = uv)$ and let $\pi\colon X \to \mathbb{A}^1_t$ be given by $t = x + y$. Then $D_1 := (x = u = 0)$ and $D_2 := (y = v = 0)$ are both flat over $\mathbb{A}^1_t$, but $D_1 \cup D_2$ is not flat.

(7.7.4) As an example for (7.4.7), let $A \subset \mathbb{P}^n$ be a projectively normal abelian variety of dimension ≥ 2 and $C_A \subset \mathbb{P}^{n+1}$ the cone over it. Let $\pi\colon \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$ be a general projection. Let $H \subset C_A$ be a hyperplane section. If H does not pass through the vertex then $H \simeq A$ is smooth and $\pi|_H\colon H \dashrightarrow \mathbb{P}^2$ is flat.

If H does pass through the vertex v, then $\operatorname{depth}_v H = 1$ by (2.35), hence $\pi|_H\colon H \dashrightarrow \mathbb{P}^2$ is not flat at v.

7.2 Infinitesimal Study of Mumford Divisors

In this section, we review the divisor theory of nonreduced schemes. The standard reference books treat Cartier divisors in detail, but for us the interesting cases are precisely when the divisors fail to be Cartier. We start with the general theory, and at the end give explicit formulas for some cases.

Definition 7.8 (Mumford class group) Let S be a scheme and $f\colon X \to S$ a morphism of pure relative dimension n. Two relative Mumford divisors (4.68) $D_1, D_2 \subset X$ are *linearly equivalent over* S if $\mathscr{O}_X(-D_1) \simeq \mathscr{O}_X(-D_2) \otimes f^*L$ for some line bundle L on S. The linear equivalence classes generate the *relative Mumford class group* $\mathrm{MCl}(X/S)$.

This is a higher dimensional version of the generalized Jacobians, worked out in Severi (1947), Rosenlicht (1954), and Serre (1959). It is slightly different from the theory of almost-Cartier divisors of Hartshorne (1986) and Hartshorne and Polini (2015).

By definition, if D is a Mumford divisor then there is a closed subset $Z \subset X$ such that $D|_{X\setminus Z}$ is Cartier and Z/S has relative dimension $\leq n-2$. This gives a natural identification

$$\mathrm{MCl}(X/S) = \lim_Z \mathrm{Pic}((X \setminus Z)/S), \tag{7.8.1}$$

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where the limit is over all closed subsets $Z \subset X$ such that Z/S has relative dimension $\leq n-2$.

As with the Picard group, it may be better to sheafify $\mathrm{MCl}(X/S)$ in the étale topology as in Bosch et al. (1990, chap.8). However, we use this notion mostly when S is local, so this is not important for our current purposes.

7.9 The infinitesimal method to study families of objects in algebraic geometry posits that we should proceed in three broad steps:

- Study families over Artinian schemes.
- Inverse limits then give families over complete local schemes.
- For arbitrary local schemes, descend properties from the completion.

This approach has been very successful for proper varieties and for coherent sheaves. One of the problems with general (possibly nonflat) families of divisors is that the global and the infinitesimal computations do not match up; in fact they say the opposite in some cases. We discuss two instances of this:

- Relative Cartier divisors on non-proper varieties.
- Generically flat families of divisors on surfaces.

The surprising feature is that the two behave quite differently. We state two cases where the contrast between Artinian and DVR bases is striking.

Claim 7.9.1 Let $\pi\colon X \to (s, S)$ be a smooth, affine morphism, S local.

(a) If S is Artinian, then the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_s)$ is an isomorphism by (7.10.2).

(b) If $S = \operatorname{Spec} k[[t]]$, then $\mathrm{Pic}(X)$ can be infinite dimensional by (7.13.3).

That is, there can be many nontrivial line bundles on X over $\operatorname{Spec} k[[t]]$, but we do not see them when working over $\operatorname{Spec} k[[t]]/(t^m)$.

The opposite happens for the Mumford class group of projective surfaces.

Claim 7.9.2 $\mathrm{MCl}(\mathbb{P}^2_{k[[t]]/(t^m)}) \simeq \mathbb{Z} + k^\infty$ for $m \geq 2$, but $\mathrm{MCl}(\mathbb{P}^2_{k[[t]]}) \simeq \mathbb{Z}$.

Proof Here $\mathbb{P}^2_{k[[t]]}$ is regular, so every Weil divisor on X is Cartier. The first part follows from (7.8.1) and (7.10.3), since $H^1(\mathbb{P}^2 \setminus Z, \mathscr{O}_{\mathbb{P}^2\setminus Z}) \simeq H^2_Z(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2})$ is infinite dimensional. □

7.10 (Picard group over Artinian schemes) Let (A, m, k) be a local Artinian ring and $X_A \to \operatorname{Spec} A$ a flat morphism. Let $(\varepsilon) \subset A$ be an ideal such that $I \simeq k$ and set $B = A/(\varepsilon)$. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_{X_k} \xrightarrow{\ e\ } \mathscr{O}^*_{X_A} \longrightarrow \mathscr{O}^*_{X_B} \longrightarrow 1, \tag{7.10.1}$$

where $e(h) = 1 + h\varepsilon$ is the exponential map. We use its long exact cohomology sequence and induction on length A to compute $\operatorname{Pic}(X_A)$. There are three cases that are especially interesting for us.

Claim 7.10.2 Let $X_A \to \operatorname{Spec} A$ be a flat, affine morphism. Then the restriction map $\operatorname{Pic}(X_A) \to \operatorname{Pic}(X_k)$ is an isomorphism.

Proof We use the exact sequence

$$H^1(X_k, \mathscr{O}_{X_k}) \to \operatorname{Pic}(X_A) \to \operatorname{Pic}(X_B) \to H^2(X_k, \mathscr{O}_{X_k}). \tag{7.10.2.a}$$

Since X is affine, the two groups at the ends vanish, hence we get an isomorphism in the middle. Induction completes the proof. □

Claim 7.10.3 Let $X_A \to \operatorname{Spec} A$ be a flat, proper morphism. If $H^0(X_k, \mathscr{O}_{X_k}) = k$, then the kernel of the restriction map $\operatorname{Pic}(X_A) \to \operatorname{Pic}(X_k)$ is a unipotent group scheme of dimension $\leq h^1(X_k, \mathscr{O}_{X_k}) \cdot (\operatorname{length} A - 1)$ and equality holds if $H^2(X_k, \mathscr{O}_{X_k}) = 0$. (In fact, if $\operatorname{char} k = 0$, then the kernel is a k-vector space and equality holds even if $H^2(X_k, \mathscr{O}_{X_k}) \neq 0$; see Bosch et al. (1990, chap.8).)

Proof By Hartshorne (1977, III.12.11), $H^0(X_A, \mathscr{O}_{X_A}) \to H^0(X_B, \mathscr{O}_{X_B})$ is surjective and so is $H^0(X_A, \mathscr{O}^*_{X_A}) \to H^0(X_B, \mathscr{O}^*_{X_B})$. Thus we get the exactness of

$$0 \to H^1(X_k, \mathscr{O}_{X_k}) \to \operatorname{Pic}(X_A) \to \operatorname{Pic}(X_B) \to H^2(X_k, \mathscr{O}_{X_k}).$$ □

Claim 7.10.4 Let $X_A \to \operatorname{Spec} A$ be a flat morphism and $Z \subset X_A$ a closed subset of codimension ≥ 2. Set $X_A^\circ := X_A \setminus Z$. Assume that X_k is S_2. Then the kernel of the restriction map $\operatorname{Pic}(X_A^\circ) \to \operatorname{Pic}(X_k^\circ)$ is a unipotent group scheme of dimension $\leq h^1(X_k^\circ, \mathscr{O}_{X_k^\circ}) \cdot (\operatorname{length} A - 1)$.

Proof Since X_k is S_2, $H^0(X_k^\circ, \mathscr{O}_{X_k^\circ}) \simeq H^0(X_k, \mathscr{O}_{X_k})$ and similarly for X_A. Thus $H^0(X_A^\circ, \mathscr{O}^*_{X_A^\circ}) \to H^0(X_B^\circ, \mathscr{O}^*_{X_B^\circ})$ is surjective and the rest of the argument works as in (7.10.3). □

Remark 7.10.5. Although (7.10.4) is very similar to (7.10.3), a key difference is that in (7.10.4) the group $H^1(X_k^\circ, \mathscr{O}_{X_k^\circ})$ can be infinite dimensional. Indeed, $H^1(X_k^\circ, \mathscr{O}_{X_k^\circ}) \simeq H^2_Z(X_k, \mathscr{O}_{X_k})$ and it is

(a) infinite dimensional if $\dim X_k = 2$,

(b) finite dimensional if X_k is S_2 and $\operatorname{codim}_{X_k} Z \geq 3$, and

(c) 0 if X_k is S_3 and $\operatorname{codim}_{X_k} Z \geq 3$.

See, for example, Section 10.3 for these claims.

The following immediate consequence of (7.10.5.c) is especially useful.

Corollary 7.10.6 Let $X \to S$ be a smooth morphism, $D \subset X$ a closed subscheme, and $Z \subset X$ a closed subset. Assume that D is a relative Cartier divisor on $X \setminus Z$, D has no embedded points in Z, and $\operatorname{codim}_{X_s} Z_s \geq 3$ for every $s \in S$.

Then D is a relative Cartier divisor. □

The following is a special case of (4.28).

Lemma 7.11 *Let $X \to S$ be a flat morphism with S_2 fibers and D a divisorial subscheme. Let $U \subset X$ be an open subscheme such that $D|_U$ is relatively Cartier and* $\operatorname{codim}_{X_s}(X_s \setminus U_s) \geq 2$ *for every $s \in S$.*

Then D is relatively Cartier iff the generically Cartier pull-back $\tau^{[*]}D$ (4.2.7) is relatively Cartier for every Artinian subscheme $\tau\colon A \hookrightarrow S$. □

Relative Cartier divisors also have some unexpected properties over non-reduced base schemes. These do not cause theoretical problems, but it is good to keep them in mind.

Example 7.12 (Cartier divisors over $k[\varepsilon]$) Let R be an integral domain over a field k. Relative principal ideals in $R[\varepsilon]$ over $k[\varepsilon]$ are given as $(f + g\varepsilon)$ where $f, g \in R$ and $f \neq 0$. We list some properties of such principal ideals that hold for any integral domain R:

(7.12.1) $(f + g_1\varepsilon) = (f + g_2\varepsilon)$ iff $g_1 - g_2 \in (f)$.

(7.12.2) If $u \in R$ is a unit then so is $u + g\varepsilon$ since $(u + g\varepsilon)(u^{-1} - u^{-2}g\varepsilon) = 1$.

(7.12.3) If f is irreducible then so is $f + g\varepsilon$ for every g.

(7.12.4) $(f + g\varepsilon)(f - g\varepsilon) = f^2$ shows that there is no unique factorization.

(7.12.5) If R is a UFD and the f_i are pairwise relatively prime, then

$$\textstyle\prod_i(f_i + g_i\varepsilon) = \prod_i(f_i + g_i'\varepsilon) \quad \text{iff} \quad g_i - g_i' \in (f_i) \quad \forall i.$$

The following concrete example illustrates several of the above features.

Example 7.13 (Picard group of a constant elliptic curve) Let $(0, E)$ be a smooth, projective elliptic curve. Over any base S we have the constant family $\pi\colon E \times S \to S$ with the constant section $s_0\colon S \simeq \{0\} \times S$. Let L be a line bundle on $E \times S$. Then $L \otimes \pi^* s_0^* L^{-1}$ has a canonical trivialization along $\{0\} \times S$, hence it defines a morphism $S \to \operatorname{Pic}(E)$. Thus

$$\operatorname{Pic}(E \times S/S) \simeq \operatorname{Mor}(S, \operatorname{Pic}(E)). \tag{7.13.1}$$

Corollary 7.13.2 Let (R, m) be a complete local ring. Set $S = \operatorname{Spec} R$ and $S_n = \operatorname{Spec} R/m^n$. Then $\operatorname{Pic}(E \times S/S) = \varprojlim \operatorname{Pic}(E \times S_n/S_n)$. □

Corollary 7.13.3 Let $S = \operatorname{Spec} k[t]_{(t)}$ be the local ring of the affine line at the origin and $\widehat{S} = \operatorname{Spec} k[[t]]$ its completion. Then $\operatorname{Pic}(E \times S/S) \simeq \operatorname{Pic}(E)$, but $\operatorname{Pic}(E \times \widehat{S}/\widehat{S})$ is infinite dimensional. □

Next consider the affine elliptic curve $E^\circ = E \setminus \{0\}$ and the constant affine family $E^\circ \times S \to S$. Note that $\operatorname{Pic}(E^\circ) \simeq \operatorname{Pic}^\circ(E)$.

If S is smooth and D° is a Cartier divisor on $E^\circ \times S$ then its closure $D \subset E \times S$ is also Cartier. More generally, this also holds if S is normal, using (4.4). Thus (7.13.1–2) give the following.

Corollary 7.13.3 If S is normal then $\operatorname{Pic}(E^\circ \times S/S) \simeq \operatorname{Mor}(S, \operatorname{Pic}^\circ(E))$. □

Corollary 7.13.4 If $S = \operatorname{Spec} A$ is Artinian then $\operatorname{Pic}(E^\circ \times S/S) \simeq \operatorname{Pic}^\circ(E)$. So $\operatorname{Pic}(E^\circ \times S/S)$ has dimension 1, but $\dim_k \operatorname{Mor}(S, \operatorname{Pic}^\circ(E)) = \operatorname{length} A$. □

For the rest of the section we make some explicit computations about Mumford divisors on schemes that are smooth over an Artinian ring.

Proposition 7.14 *Let (A, k) be a local Artinian ring, $k \simeq (\varepsilon) \subset A$ an ideal, and $B = A/(\varepsilon)$. Let (R_A, m) be a flat, local, S_2, A-algebra and set $X_A := \operatorname{Spec}_A R_A$. Let $f_B \in R_B$ be a non-zerodivisor and set $D_B := (f_B = 0) \subset X_B$.*

Then the set of relative Mumford divisors $D_A \subset X_A$ such that $\operatorname{pure}((D_A)|_B) =$ *D_B, is a torsor under the k-vector space $H^1_m(D_k, \mathscr{O}_{D_k})$.*

Proof We can lift f_B to $f_A \in R_A$. Choose $y \in m$ that is not a zerodivisor on D_B and such that D_A is a principal divisor on $X_A \setminus (y = 0)$. After inverting y, we can write the ideal of D_A as

$$(I, y^{-1}) = (f_A + \varepsilon y^{-r} g_k), \quad \text{where} \quad g_k \in R_k, r \in \mathbb{N}. \tag{7.14.1}$$

We can multiply $f_A + \varepsilon y^{-r} g_k$ by $1 + \varepsilon y^{-s} v$. This changes $y^{-r} g_k$ to $y^{-r} g_k + v y^{-s} f_A$. By (7.15) the relevant information is carried by the residue class

$$\overline{y^{-r} g_k} \in H^0(D_k^\circ, \mathscr{O}_{D_k^\circ}), \tag{7.14.2}$$

where $D_k^\circ \subset D_k$ denotes the complement of the closed point.

If the residue class is in $H^0(D_k, \mathscr{O}_{D_k})$, then we get a Cartier divisor. Thus the non-Cartier divisors are parametrized by

$$H^0(D_k^\circ, \mathscr{O}_{D_k^\circ}) / H^0(D_k, \mathscr{O}_{D_k}) \simeq H^1_m(D_k, \mathscr{O}_{D_k}). \tag{7.14.3}$$

We get distinct divisors by (7.17.2). □

Lemma 7.15 *Let (A, k) be a local Artinian ring, $k \simeq (\varepsilon) \subset A$ an ideal, and $B = A/(\varepsilon)$. Let (R_A, m) be a flat, local, S_2, A-algebra. Let $f_A \in R_A$ and $g_k \in R_k$ be non-zerodivisors and y a non-zerodivisor modulo both f_A and g_k.*

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For $I := R_A \cap (f_A + \varepsilon y^{-r} g_k) R_A[y^{-1}]$ *the following are equivalent:*

(7.15.1) *I is a principal ideal.*

(7.15.2) *The residue class* $\overline{y^{-r} g_k}$ *lies in* $R_k/(f_k)$.

(7.15.3) $g_k \in (f_k, y^r)$.

Note that we can change $f_A + \varepsilon y^{-r} g_k$ to $(f_A + \varepsilon h_k) + \varepsilon y^{-r}(g_k - y^r h_k)$ for any $h_k \in R_k$, but $g_k \in (f_k, y^r)$ iff $g_k - y^r h_k \in (f_k, y^r)$.

Proof I is a principal ideal iff it has a generator of the form $f_A + \varepsilon h_k$ where $h_k \in R_k$. This holds iff

$$f_A + \varepsilon y^{-r} g_k = (1 + \varepsilon y^{-s} b_k)(f_A + \varepsilon h_k) \quad \text{for some} \quad b_k \in R_A.$$

Equivalently, iff $y^{-r} g_k = h_k + y^{-s} b_k f_k$. If $r > s$ then $g_k = y^r h_k + y^{r-s} b_k f_k$, which is impossible since y is not a zerodivisor modulo g_k. If $r < s$ then $y^{s-r} g_k = y^s h_k + b_k f_k$, which is impossible since y is not a zerodivisor modulo f_k. Thus $r = s$ and then $g_k = y^r h_k + b_k f_k$ is equivalent to $g_k \in (f_k, y^r)$. □

The next will be crucial in the proof of (7.60). To state it, let $\text{nil}(n_A)$ denote the smallest $r \geq 0$ such that $n_A^r = 0$, and for $f \in R_A[y^{-1}]$, let ord_y denote *pole order* in y, that is, the smallest $r \geq 0$ such that $y^r f \in R_A$.

Proposition 7.16 *Let* (A, n_A, k) *be a local Artinian ring and* (R_A, m_R) *a flat, local,* S_2, *A-algebra of dimension* ≥ 2. *Let* $f_k \in m_k$ *be a non-zerodivisor and* $y \in m_R$ *a non-zerodivisor modulo* f_k. *Let* $f_A, f'_A \in R_A[y^{-1}]$ *be two liftings of* f_k. *Assume that* $f_A - f'_A \in y^N R_A$, *where* $N = \text{nil}(n_A) \cdot \text{ord}_y f_A$.

Then $(f_A) \cap R_A$ *is a principal ideal iff* $(f'_A) \cap R_A$ *is.*

Proof Note first that $N \geq 0$, so $f_A - f'_A \in y^N R_A$ implies that $\text{ord}_y f_A = \text{ord}_y f'_A$, so the assumption is symmetric in f_A, f'_A. It is thus enough to prove that if $(f_A) \cap R_A$ is a principal ideal, then so is $(f'_A) \cap R_A$.

Assume that $(f_A) \cap R_A = (F_A)$. Then there is unit u_A in $R_A[y^{-1}]$ such that $f_A = u_A F_A$. Since $f_k = F_k$, we see that u_k is a unit in R_k.

We claim that $\text{ord}_y f_A = \text{ord}_y u_A$. Indeed, if $\text{ord}_y u_A = r$ then we get a nonzero remainder $\bar{u}_A \in y^{-r} R_A / y^{1-r} R_A \simeq R_A / y R_A$. Multiplication by F_A preserves the pole-order filtration, so

$$\overline{F_A u_A} = F_A \bar{u}_A \in y^{-r} R_A / y^{1-r} R_A \simeq R_A / y R_A.$$

Here $R_A / y R_A$ has a filtration whose successive quotients are $R_k / y R_k$ and F_A acts by multiplication by f_k an each graded piece. Since f_k is a non-zerodivisor modulo y, we see that $\overline{F_A u_A} \neq 0$. So $\text{ord}_y f_A = \text{ord}_y u_A$. Taylor expansion of the inverse shows that $\text{ord}_y(u_A^{-1}) \leq \text{nil}(n_A) \cdot \text{ord}_y f_A =: N$. Thus

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$$u_A^{-1} f_A' = u_A^{-1} f_A + u_A^{-1}(f_A' - f_A) = F_A + (y^N u_A^{-1})(y^{-N}(f_A' - f_A)) \in R_A. \qquad \square$$

The connection between (7.14) and (7.10) is given by the following.

7.17 Let X be an affine, S_2 scheme and $D := (s = 0) \subset X$ a Cartier divisor. Let $Z \subset D$ be a closed subset that has codimension ≥ 2 in X. Set $X^\circ := X \setminus Z$ and $D^\circ := D \setminus Z$. Restricting the exact sequence

$$0 \to \mathscr{O}_X \xrightarrow{s} \mathscr{O}_X \to \mathscr{O}_D \to 0$$

to X° and taking cohomologies we get

$$0 \to H^0(X^\circ, \mathscr{O}_{X^\circ}) \xrightarrow{s} H^0(X^\circ, \mathscr{O}_{X^\circ}) \to H^0(D^\circ, \mathscr{O}_{D^\circ}) \xrightarrow{\partial} H^1(X^\circ, \mathscr{O}_{X^\circ}).$$

Note that $H^0(X^\circ, \mathscr{O}_{X^\circ}) = H^0(X, \mathscr{O}_X)$ since X is S_2 and its image in $H^0(D^\circ, \mathscr{O}_{D^\circ})$ is $H^0(D, \mathscr{O}_D)$. Thus ∂ becomes the injection

$$\partial\colon H^1_Z(D, \mathscr{O}_D) \simeq H^0(D^\circ, \mathscr{O}_{D^\circ})/H^0(D, \mathscr{O}_D) \hookrightarrow H^2_Z(X, \mathscr{O}_X). \tag{7.17.1}$$

We are especially interested in the case when (x, X) is local, two-dimensional and $Z = \{x\}$. In this case (7.17.1) becomes

$$\partial\colon H^1_x(D, \mathscr{O}_D) \hookrightarrow H^2_x(X, \mathscr{O}_X). \tag{7.17.2}$$

The left side describes first order deformations of D by (7.14) and the right side the Picard group of the first order deformation of $X \setminus \{x\}$ by (7.10.4).

We can be especially explicit about first order deformations in the smooth case. Let us start with the description as in (7.14).

7.18 (Mumford divisors in $k[[u, v]][\varepsilon]$) Set $X = \operatorname{Spec} k[[u, v]][\varepsilon]$ with closed point $x \in X$. By (7.10), the Picard group of the punctured spectrum $X \setminus \{x\}$ is

$$H^2_x(X, \mathscr{O}_X) \simeq \bigoplus\nolimits_{i,j>0} \tfrac{1}{u^i v^j} \cdot k.$$

An ideal corresponding to $cu^{-i}v^{-j}$ (where $c \in k^\times$) can be given as

$$I(cu^{-i}v^{-j}) := (u^{2i}, u^i v^j + c\varepsilon, u^i\varepsilon);$$

a more systematic derivation of this is given in (7.20.1).

This is explicit, but we are more interested in the point of view of (7.10).

Lemma 7.19 *Let $f \in k[[u]][v]$ be a monic polynomial in v of degree n defining a curve $C_k \subset \widehat{\mathbb{A}}^2_{uv}$. Let $C \subset \widehat{\mathbb{A}}^2_{k[\varepsilon]}$ be a relative Mumford divisor such that*

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$\mathrm{pure}((C)_k) = C_k$. *Then the restriction of* C *to the complement of* $(u = 0)$ *can be uniquely written as*

$$(f + \varepsilon\textstyle\sum_{i=0}^{n-1} v^i\phi_i(u) = 0) \quad \textit{where} \quad \phi_i(u) \in u^{-1}k[u^{-1}].$$

Thus the set of all such C *is naturally isomorphic to the infinite dimensional* k*-vector space* $H^1_m(C_k, \mathscr{O}_{C_k}) \simeq \oplus_{i=0}^{n-1} u^{-1}k[u^{-1}]$.

Note that, by the Weierstrass preparation theorem, almost every curve in $\widehat{\mathbb{A}}_{uv}$ is defined by a monic polynomial in v.

Proof Note that $k[[u]][v]/(f) \simeq \oplus_{i=0}^{n-1} v^i k[[u]]$ as a $k[[u]]$-module, so

$$H^0(C_k, \mathscr{O}_{C_k}) \simeq \oplus_{i=0}^{n-1} v^i k[[u]] \quad \text{and} \quad H^0(C_k^\circ, \mathscr{O}_{C_k^\circ}) \simeq \oplus_{i=0}^{n-1} v^i k((u)). \tag{7.19.1}$$

That is, if $g \in k((u))[v]$ is a polynomial of degree $< n$ in v, then $g|_{C^\circ}$ extends to a regular function on C iff $g \in k[[u]][v]$. □

We can also restate (7.19.1) as

$$H^1_m(C_k, \mathscr{O}_{C_k}) \simeq \oplus_{i=0}^{n-1} v^i k((u))/k[[u]] \simeq \oplus_{i=0}^{n-1} v^i u^{-1}k[u^{-1}]. \tag{7.19.2}$$

Example 7.20 Consider next the special case of (7.19) when $f = v$. We can then write the restriction of C as $(v + \phi(u)\varepsilon = 0)$ where $\phi \in u^{-1}k[u^{-1}]$. Let r denote the pole-order of ϕ and set $q(u) := u^r\phi(u)$. By (7.7.1), the ideal of C is

$$I_C = (v^2, vu^r + q(u)\varepsilon, v\varepsilon). \tag{7.20.1}$$

Thus the fiber over the closed point is $k[[u, v]]/(v^2, vu^r)$. Its torsion submodule is isomorphic to $k[[u, v]]/(v, u^r) \simeq k[u]/(u^r)$.

The ideals of relative Mumford divisors in $k[[u, v]][\varepsilon]$ are likely to be more complicated in general. At least the direct generalization of (7.20.1) does not always give the correct generators.

For example, let $f = v^2 - u^3$ and consider the ideal $I \subset k[[u, v]][\varepsilon]$ extended from $((v^2 - u^3) + u^{-3}v\varepsilon)$. The formula (7.20.1) suggests the elements

$$(v^2 - u^3)^2,\ u^3(v^2 - u^3) + v\varepsilon,\ (v^2 - u^3)\varepsilon \in I.$$

However, $u^3(v^2 - u^3) + v\varepsilon = v^2(v^2 - u^3) + v\varepsilon$, giving that

$$I = ((v^2 - u^3)^2, v(v^2 - u^3) + \varepsilon, (v^2 - u^3)\varepsilon). \tag{7.20.2}$$

Using the isomorphism $R[\varepsilon]/(f^2, fg + \varepsilon, f\varepsilon) \simeq R/(f^2, -f^2g) \simeq R/(f^2)$, the examples can be generalized to the nonsmooth case as follows.

Claim 7.20.3 Let (R, m) be a local, S_2, k-algebra of dimension 2, and $f, g \in m$ a system of parameters. Then $J_{f,g} = (f^2, fg + \varepsilon, f\varepsilon)$ is (the ideal of) a relative Mumford divisor in $R[\varepsilon]$ whose central fiber is $R/(f^2, fg)$, with embedded subsheaf isomorphic to $R/(f, g)$. □

7.3 Divisorial Support

There are at least three ways to associate a divisor to a sheaf (7.22), but only one of them – the divisorial support – behaves well in flat families. In this Section we develop this notion and a method to compute it. The latter is especially important for the applications. First, we recall the definition of the Fitting ideal sheaf.

7.21 (Fitting ideal) Let R be a Noetherian ring, M a finite R-module, and

$$R^s \xrightarrow{A} R^r \to M \to 0$$

a presentation of M, where A is given by an $s \times r$-matrix with entries in R. The *Fitting ideal,* or, more precisely, the 0*th Fitting ideal* of M, denoted by $\mathrm{Fitt}_R(M)$, is the ideal generated by the determinants of $r\times r$-minors of A. For the following basic properties, see Fitting (1936) or (Eisenbud, 1995, Sec.20.2).

(7.21.1) $\mathrm{Fitt}_R(M)$ is independent of the presentation chosen.

(7.21.2) If R is regular and $M \simeq \oplus_i R/(g_i^{m_i})$ then $\mathrm{Fitt}_R(M) = (\prod g_i^{m_i})$.

(7.21.3) The Fitting ideal commutes with base change. That is, if S is an R-algebra then $\mathrm{Fitt}_S(M \otimes_R S)$ is generated by $\mathrm{Fitt}_R(M) \otimes_R S$.

The following is a special case of Lipman (1969, lem.1).

(7.21.4) Let M be a torsion module. Then $\mathrm{Fitt}_R(M)$ is a principal ideal generated by a non-zerodivisor iff the projective dimension of M is 1.

One direction is easy. If the projective dimension of M is 1, then M has a presentation

$$0 \to R^s \xrightarrow{A} R^r \to M \to 0.$$

Here $r = s$ since M is torsion, thus $\det(A)$ generates $\mathrm{Fitt}_R(M)$.

We prove the converse only in the following special case that we use later, which, however, captures the essence of the general proof.

(7.21.5) Let X be a smooth variety of dimension n and F a coherent sheaf of generic rank 0 on X. Then $\mathrm{Fitt}_X(F)$ is a principal ideal iff F is CM of pure dimension $n - 1$.

Proof This can be checked after localization and completion. Thus we have a module M over $S := k[[x_1, \dots, x_n]]$, and, after a coordinate change, we may

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assume that it is finite over $R := k[[x_1, \dots, x_{n-1}]]$ of generic rank say r. Using first (7.21.2) and then (7.21.3) we get that

$$\begin{aligned}\dim_k M \otimes_S k[[x_n]] &= \dim_k k[[x_n]]/\operatorname{Fitt}_{k[[x_n]]}(M \otimes_S k[[x_n]]) \\ &= \dim_k (S/\operatorname{Fitt}_S(M)) \otimes_S k[[x_n]].\end{aligned} \tag{7.21.6}$$

Next note that M is CM $\Leftrightarrow$ M is free over $R \Leftrightarrow \dim M \otimes_S k[[x_n]] = r$. Using (7.21.1) and the previous equivalences for $S/\operatorname{Fitt}_S(M)$, we get that these are equivalent to $S/\operatorname{Fitt}_S(M)$ being CM. This holds iff $\operatorname{Fitt}_S(M)$ is a height 1 unmixed ideal, hence principal. □

The following explicit formula is quite useful.

Computation 7.21.7 Let S be a smooth R-algebra and $v \in S$ such that $S/(v) \simeq R$. (The examples we use are $S = R[v]$ and $S = R[[v]]$.) Let M be an S-module that is free of finite rank as an R-module. Write $M = \oplus_{i=1}^r R m_i$ and $v m_i = \sum_{i=1}^r a_{ij} m_j$ for $a_{ij} \in R$. Then $\operatorname{Fitt}_S(M)$ is generated by $\det(v\mathbf{1}_r - (a_{ij}))$.

Proof A presentation of M as an S-module is given by

$$\oplus_{i=1}^r S e_i \xrightarrow{\phi} \oplus_{i=1}^r S f_i \xrightarrow{\psi} M \to 0,$$

where $\psi(f_i) = m_i$ and $\phi(e_i) = v f_i - \sum_{j=1}^r a_{ij} f_j$. Thus $\phi = v\mathbf{1}_r - (a_{ij})$ and so $\det(v\mathbf{1}_r - (a_{ij}))$ generates $\operatorname{Fitt}_S(M)$. □

Computation 7.21.8 Let T be a free S-algebra and $t \in T$ a non-zerodivisor. Then $\operatorname{Fitt}_S(T/tT)$ is generated by $\operatorname{norm}_{T/S}(t)$.

Proof We use $0 \to T \xrightarrow{t} T \to T/tT \to 0$ and the definition of the norm. □

Definition 7.22 (Divisorial support I) Let X be a scheme and F a coherent sheaf on X. One usually defines its *support* $\operatorname{Supp} F$ and its *scheme-theoretic support* $\operatorname{SSupp} F := \operatorname{Spec}_X(\mathscr{O}_X/\operatorname{Ann} F)$.

Assume next that $\operatorname{Supp} F$ is nowhere dense and X is regular at every generic point $x_i \in \operatorname{Supp} F$ that has codimension 1 in X. Then there is a unique divisorial sheaf (3.25) associated to the Weil divisor $\sum \operatorname{length}(F_{x_i}) \cdot [\bar{x}_i]$. We call it the *divisorial support* of F and denote it by $\operatorname{DSupp} F$. Equivalently,

$$\operatorname{DSupp}(F) = \operatorname{Spec}\left(\mathscr{O}_X/\operatorname{Fitt}_X(F)\right), \tag{7.22.1}$$

where pure denotes the pure codimension 1 part (10.1).

If every associated point of F has codimension 1 in X, then we have inclusions of subschemes

$$\operatorname{Supp} F \subset \operatorname{SSupp} F \subset \operatorname{DSupp} F. \tag{7.22.2}$$

In general, all three subschemes are different, though with the same support.

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Our aim is to develop a relative version of this notion and some ways of computing it in families. Let $X \to S$ be a morphism and F a coherent sheaf on X. Informally, we would like the relative divisorial support of F, denoted by $\operatorname{DSupp}_S F$, to be a scheme over S whose fibers are $\operatorname{DSupp}(F_s)$ for all $s \in S$. If S is reduced, this requirement uniquely determines $\operatorname{DSupp}_S F$, but in general there are two problems.

- Even in nice situations, this requirement may be impossible to meet.
- For nonreduced base schemes, the fibers do not determine $\operatorname{DSupp}_S F$.

In our main applications, X is smooth over some base scheme S that may well have nilpotent elements. As in (9.12), we need to allow embedded subsheaves that "come from" S, but not the others.

Definition 7.23 (Divisorial support II) Let $X \to S$ be a smooth morphism of pure relative dimension n. Let F be a coherent sheaf on X that is flat over S with CM fibers of pure dimension $n-1$. We define its divisorial support as

$$\operatorname{DSupp}_S(F) := \operatorname{Spec}(\mathscr{O}_X / \operatorname{Fitt}_X(F)).$$

Lemma 7.24 *Under the assumptions of (7.23),*

(7.24.1) $\operatorname{DSupp}_S(F)$ *is a relative Cartier divisor, and*

(7.24.2) $\operatorname{DSupp}_S(F)$ *commutes with base change. That is, let $h\colon S' \to S$ be a morphism. By base change we get $g'\colon X' \to S'$, $h_X\colon X' \to X$. Then $h_X^*(\operatorname{DSupp} F) = \operatorname{DSupp}(h_X^* F)$.*

Proof The first claim can be checked after localization and completion. We may thus assume that $S = \operatorname{Spec} B$ where (B, m) is local with residue field k, $X = \operatorname{Spec} B[[x_1, \dots, x_n]]$ and F is the sheafification of M. Since $M \otimes_B k$ has dimension $n-1$ over $k[[x_1, \dots, x_n]]$, after a general coordinate change we may assume that $M/(x_1, \dots, x_{n-1}, m)M$ is finite. Thus M is a finite $R := B[[x_1, \dots, x_{n-1}]]$-module. Set $R_k = R \otimes_B k \simeq k[[x_1, \dots, x_{n-1}]]$. Since M is flat over B, its generic rank over R equals the generic rank of $M \otimes_B k$ over R_k. By assumption, $M \otimes_B k$ is CM, hence free over R_k. Thus the generic rank of M over R equals $\dim_k M \otimes_R k$ and M is free as an R-module. The rest follows from (7.21.7). The second claim is immediate from (7.21.3). □

The following restriction property is also implied by (7.21.3).

Lemma 7.25 *Continuing with the notation and assumptions of (7.23), let $D \subset X$ be a relative Cartier divisor that is also smooth over S. Assume that D does not contain any generic point of* $\operatorname{Supp} F_s$ *for any $s \in S$. Then*

$$\operatorname{DSupp}(F|_D) = (\operatorname{DSupp} F)|_D. \qquad \square$$

Now we are ready to define the sheaves for which the relative divisorial support makes sense, but first we have to distinguish associated points that come from the base from the other ones.

Definition 7.26 Let $X \to S$ be a morphism and F a coherent sheaf on X. The *flat locus* of F is the largest open subset $U \subset X$ such that $F|_U$ is flat over S. We denote it by $\mathrm{Flat}_S(X, F)$.

It is usually more convenient to work with the *flat-CM locus* of F. It is the largest open subset $U \subset X$ such that $F|_U$ is flat with CM fibers over S. We denote it by $\mathrm{FlatCM}_S(X, F)$. If F is generically flat over S of relative dimension d, then $(\mathrm{Supp}\, F \setminus \mathrm{FlatCM}_S(X, F)) \to S$ has relative dimension $< d$.

Definition 7.27 Let $X \to S$ be a morphism. A coherent sheaf F is a *generically flat family of pure sheaves* of dimension d over S, if F is generically flat (3.26) and $\mathrm{Supp}\, F \to S$ has pure relative dimension d. This property is preserved by any base change $S' \to S$.

For our current purposes, we can harmlessly replace F by its vertically pure quotient vpure(F) (9.12). The generic fibers of vpure(F) are pure of dimension d, but special fibers may have embedded points outside the flat locus (7.26). Vertically purity is preserved by flat base changes.

Definition–Lemma 7.28 (Divisorial support III) Let $g\colon X \to S$ be a flat morphism of pure relative dimension n and $g^\circ\colon X^\circ \to S$ the smooth locus of g.

Let F be a coherent sheaf on X that is generically flat and pure over S of dimension $n - 1$. Assume that for every $s \in S$, every generic point of F_s is contained in X°.

Set $U := \mathrm{FlatCM}_S(X, F) \cap X^\circ)$ and $j\colon U \hookrightarrow X$ the natural injection. We define the *divisorial support* of F over S as

$$\mathrm{DSupp}_S(F) := \overline{\mathrm{DSupp}_S(F|_U)}, \tag{7.28.1}$$

the scheme-theoretic closure of $\mathrm{DSupp}_S(F|_U)$. This makes sense since the latter is already defined by (7.23).

Note that $\mathrm{Supp}\, \mathrm{DSupp}_S(F) = \mathrm{Supp}\, F$ and $\mathrm{DSupp}_S(F)$ is a generically flat family of pure subschemes of dimension $n - 1$ over S, whose restriction to U is relatively Cartier.

It is enough to check the following equalities at codimension 1 points, which follow from (7.24) and (7.21.3).

Claim 7.28.2 Let $g_i\colon X_i \to S$ be flat morphisms of pure relative dimension n and $\pi\colon X_1 \to X_2$ a finite morphism. Let $D \subset X_1$ be a relative Mumford divisor.

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Assume that red $D_s \to \mathrm{red}(\pi(D_s))$ is birational and π is étale at generic points of D_s. Then $\mathrm{DSupp}_S(\pi_*\mathscr{O}_D) = \pi(D)$, the scheme-theoretic image of D. □

Claim 7.28.3 (Divisorial support commutes with push-forward) Let $g_i\colon X_i \to S$ be flat morphisms of pure relative dimension n and $\pi\colon X_1 \to X_2$ a finite morphism. Let F be a coherent sheaf on X_1 that is generically flat and pure over S of relative dimension $n-1$. Assume that g_1 (resp. g_2) is smooth at every generic point of F_s (resp. π_*F_s) for every $s \in S$. Then

$$\mathrm{DSupp}_S(\pi_*F) = \mathrm{DSupp}_S(\pi_*\mathscr{O}_{\mathrm{DSupp}_S(F)}). \qquad \square$$

Claim 7.28.4 Let $g_i\colon X_i \to S$ be flat morphisms of pure relative dimension n and $\pi_1\colon X_1 \to X_2$, $\pi_2\colon X_2 \to X_3$ finite morphisms. Let F be a coherent sheaf on X_1 that is generically flat and pure over S of relative dimension $n-1$. Assume that g_1 (resp. g_2, g_3) is smooth at every generic point of F_s (resp. $\pi_{1*}F_s$, $(\pi_2 \circ \pi_1)_*F_s$) for every $s \in S$. Then

$$\mathrm{DSupp}_S((\pi_2 \circ \pi_1)_*F) = \mathrm{DSupp}_S(\pi_{2*}\mathscr{O}_{\mathrm{DSupp}_S(\pi_{1*}F)}). \qquad \square$$

Lemma 7.29 *Let $X \to S$ be a smooth morphism of pure relative dimension n. Let F be a coherent sheaf on X that is generically flat over S with pure fibers of dimension $n-1$. Assume that either F is flat over S, or S is reduced.*

Then $\mathrm{DSupp}_S F$ *is a relative Cartier divisor.*

Proof Assume first that F is flat over S. If $x \in X_s$ is a point of codimension ≤ 2, then F_s is CM at x, hence $\mathrm{DSupp}_S F$ is a relative Cartier divisor at x by (7.23). Since $X \to S$ is smooth, $\mathrm{DSupp}_S F$ is a relative Cartier divisor everywhere by (7.10.6).

For the second claim, our argument gives only that $\mathrm{DSupp}_S F$ is a relative, generically Cartier divisor. By (4.34), it is then enough to check the conclusion after base change $T \to S$, where T is the spectrum of a DVR. Then X_T is regular, so $\mathrm{DSupp}_T F_T$ is Cartier. □

7.30 (Restriction to divisors) Let (s, S) be a local scheme and $g\colon X \to S$ a flat morphism of pure relative dimension n. Let F be a generically flat family of pure sheaves of relative dimension $n-1$ such that g is smooth at every generic point of $\mathrm{Supp}\, F_s$. Let $D \subset X$ be a relative Cartier divisor.

(7.30.1) Assume that $g|_D$ is smooth and F is flat with CM fiber, at every generic point of $D \cap \mathrm{Supp}\, F_s$. Then

$$\mathrm{DSupp}_S(F|_D) = \mathrm{vpure}((\mathrm{DSupp}_S\, F)|_D).$$

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(7.30.2) Assume in addition, that D contains neither a generic point of $\operatorname{Supp} F_s \setminus \operatorname{FlatCM}_S(X, F)$, nor a codimension ≥ 2 point of $\operatorname{Supp} F_s$ where $\operatorname{DSupp}_S F$ is not S_2, then

$$\operatorname{DSupp}_S(F|_D) = (\operatorname{DSupp}_S F)|_D.$$

Corollary 7.31 (Bertini theorem for divisorial support) *Let $g\colon X \to S$ be a flat morphism of pure relative dimension n and F a generically flat family of pure sheaves of dimension $n-1$ over S. Fix $s \in S$ such that g is smooth at every generic point of* $\operatorname{Supp} F_s$*. Let D be a general member of a linear system on X, that is base point free in characteristic 0 and very ample in general. Then there is an open neighborhood $s \in S^\circ \subset S$ such that* $\operatorname{DSupp}_S(F|_D) = (\operatorname{DSupp}_S F)|_D$ *holds over S°.*

Lemma 7.32 (Divisorial support commutes with base change) *Let $g\colon X \to S$ be a flat morphism of pure relative dimension n and F a generically flat family of pure sheaves of dimension $n-1$ over S. Assume that g is smooth at every generic point of* $\operatorname{Supp} F_s$*, for every $s \in S$. Let $h\colon S' \to S$ be a morphism. By base change, we get $g'\colon X' \to S'$ and $h_X\colon X' \to X$. Then*

$$h_X^{[*]}(\operatorname{DSupp}_S F) = \operatorname{DSupp}_{S'}(h_X^* F),$$

where $h_X^{[*]}$ is the generically Cartier pull-back (4.2.7).

Proof Set $U := \operatorname{FlatCM}_S(X, F) \subset X$ with injection $j\colon U \hookrightarrow X$. Set $U' := h_X^{-1}(U)$ and $h_U\colon U' \to U$ the restriction of h_X. Then (7.24) shows the equality $h_U^*(\operatorname{DSupp}_S F|_U) = \operatorname{DSupp}_{S'}(h_U^*(F|_U))$.

By (7.27.4), $h_X^{[*]}(\operatorname{DSupp}_S F)$ is a generically flat family of pure divisors and it agrees with $\operatorname{DSupp}_{S'}(h_X^* F)$ over U'. Thus the two are equal. □

7.33 (Proof of 7.4.7) Assume that we have $f\colon X \to (s, S)$ of relative dimension n and relative Mumford divisors $D_1, D_2 \subset X$, where (s, S) is local. Let $\operatorname{FlatCM}_S(X) \subset X$ be the largest open subset where f has CM fibers and $Z = X \setminus \operatorname{FlatCM}_S(X)$. Note that $Z \to S$ has relative dimension $\leq n-2$.

Let $\pi\colon X \to \mathbb{P}^n_S$ be a finite morphism. Set $P^\circ := \mathbb{P}^n_S \setminus \pi(Z)$ and $X^\circ := \pi^{-1}(P^\circ)$. Then $\pi\colon X^\circ \to P^\circ$ is finite and flat. If $(f) = D_1 - D_2$ then, by (7.21.8),

$$(\operatorname{norm}_{X^\circ/X^\circ}(f)) = \operatorname{DSupp}_S(D_1)|_{P^\circ} - \operatorname{DSupp}_S(D_2)|_{P^\circ}.$$

Since $Z \to S$ has relative dimension $\leq n-2$, this implies that $\operatorname{DSupp}_S(D_1)$ and $\operatorname{DSupp}_S(D_2)$ are linearly equivalent. Thus, if one of them is relatively Cartier, then so is the other. □

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7.4 Variants of K-Flatness

We introduce five versions of K-flatness, which may well be equivalent to each other. From the technical point of view, Cayley–Chow-flatness (or C-flatness) is the easiest to use, but a priori it depends on the choice of a projective embedding. Then most of the work in the next two sections goes to proving that a modified version (stable C-flatness) is equivalent to K-flatness, hence independent of the projective embedding.

7.34 (Projections of $\mathbb{P}^n$) Let S be an affine scheme. Projecting $\mathbb{P}^n_S$ from the section $(a_0 : \cdots : a_n)$ (where $a_i \in \mathscr{O}_S$) to the $(x_n = 0)$ hyperplane is given by

$$\pi\colon (x_0 : \cdots : x_n) \to (a_n x_0 - a_0 x_n : \cdots : a_n x_{n-1} - a_{n-1} x_n). \tag{7.34.1}$$

It is convenient to normalize $a_n = 1$ and then we get

$$\pi\colon (x_0 : \cdots : x_n) \to (x_0 - a_0 x_n : \cdots : x_{n-1} - a_{n-1} x_n). \tag{7.34.2}$$

Similarly, a Zariski open set of projections of $\mathbb{P}^n_S$ to $L^r = (x_n = \cdots = x_{r+1} = 0)$ is given by

$$\pi\colon (x_0 : \cdots : x_n) \to \big(x_0 - \ell_0(x_{r+1}, \dots, x_n) : \cdots : x_r - \ell_r(x_{r+1}, \dots, x_n)\big), \tag{7.34.3}$$

where the ℓ_i are linear forms.

Note that in affine coordinates, when we set $x_0 = 1$, the projections become

$$\pi\colon (x_1, \dots, x_n) \to \Big(\tfrac{x_1 - \ell_1}{1 - \ell_0}, \dots, \tfrac{x_r - \ell_r}{1 - \ell_0}\Big), \tag{7.34.4}$$

where again the ℓ_i are (homogeneous) linear forms in the $x_{r+1}, \dots, x_n$. If $\ell_0 \equiv 0$, then we recover the linear projections, but in general the coordinate functions have a non-linear expansion

$$\tfrac{x_i - \ell_i}{1 - \ell_0} = (x_i - \ell_i)(1 + \ell_0 + \ell_0^2 + \cdots). \tag{7.34.5}$$

Finally, formal projections are given as

$$\pi\colon (x_1, \dots, x_n) \to \big(x_1 - \phi_1(x_1, \dots, x_n), \dots, x_r - \phi_r(x_1, \dots, x_n)\big), \tag{7.34.6}$$

where ϕ_i are power series such that $\phi_i(x_1, \dots, x_r, 0, \dots, 0) \equiv 0$ for every i.

7.35 (Approximation of formal projections) Let $v_m\colon \mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$ (where $N = \binom{n+m}{n} - 1$) be the mth Veronese embedding. Pulling back the linear coordinates on $\mathbb{P}^N_S$ we get all the monomials of degree m. In affine coordinates $x_1, \dots, x_n$ as above, we get all monomials of degree $\le m$.

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In particular, we see that given a formal projection π as in (7.34.6) and $m > 0$, there is a unique linear projection π_m of $\mathbb{P}^N_S$ such that $\pi_m \circ v_m$ is

$$\begin{aligned} &(x_1, \dots, x_n) \to (x_1 - \psi_1, \dots, x_r - \psi_r), \quad \text{where} \\ &\psi_i \equiv \phi_i \mod (x_1, \dots, x_n)^{m+1}, \quad \text{and} \quad \deg \psi_i \le m \quad \forall i. \end{aligned} \tag{7.35.1}$$

That is, we can approximate formal projections by linear projections composed with a Veronese embedding. Thus it is reasonable to expect that K-flatness is very close to C-flatness for all Veronese images; this leads to the notion of stable C-flatness in (7.37.2).

The uniqueness of this approximation is not always an advantage. In practice we would like π_m to be in general position away from the chosen point. This is easy to achieve if we increase m a little. In particular, we get the following obvious result.

Claim 7.35.2 Let (s, S) be a local scheme and $Y \subset \mathbb{P}^n_S$ a closed subset of pure relative dimension d. Let $p \in Y_s$ be a closed point with maximal ideal $\mathbf{m}_p$ such that $x_0(p) \neq 0$. Fix $m \in \mathbb{N}$ and let $(\widehat{g}_1 : \cdots : \widehat{g}_e) : \widehat{Y}_p \to \widehat{\mathbb{A}}^e_S$ be a finite morphism. Then for every $M \geq m + 1$ there are $g_1, \dots, g_e \in H^0(\mathbb{P}^n_S, \mathscr{O}_{\mathbb{P}^n_S}(M))$ such that $\pi\colon (x_0^M : g_1 : \cdots : g_e) : Y \to \mathbb{P}^e_S$ is a finite morphism, $\pi^{-1}(\pi(p)) \cap Y = \{p\}$, and $\widehat{g}_i \equiv g_i / x_0^M \mod \mathbf{m}_p^m$ for every i. □

Despite having good approximations, the equivalence of K-flatness and stable C-flatness is not clear. The problem is the following.

Assume for simplicity that S is the spectrum of an Artinian ring A. For sheaves of dimension d, using the notation of (7.21.7), we can write the equation of $\mathrm{DSupp}(\widehat{\pi_* F})$ in the form $\det(v\mathbf{1}_r - M)) = 0$, where the entries of the matrix M involve rational functions in the power series ϕ_i. The problem is that inverses of power series usually do not have good approximations by rational functions. For example, there is no rational function $g(x_1, x_2)$ such that

$$(x_2 - \sin x_1)^{-1} - g(x_1, x_2) \in k[[x_1, x_2]].$$

The exception is the one-variable case, where truncations of Laurent series give good approximations. This is what we exploit in (7.60) to prove that K-flatness is equivalent to stable C-flatness for curves.

Definition 7.36 Let E be a vector bundle over a scheme S and $F \subset E$ a vector subbundle. This induces a natural *linear projection* map $\pi\colon \mathbb{P}_S(E) \dashrightarrow \mathbb{P}_S(F)$. If S is local, then E, F are free. After choosing bases, π is given by a matrix of constant rank with entries in $\mathscr{O}_S$. We call these $\mathscr{O}_S$*-projections* if we want to emphasize this. If S is over a field k, we can also consider *k-projections,* given

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by a matrix with entries in k. These, however, only make good sense if we have a canonical trivialization of E; this rarely happens for us.

We can now formulate various versions of K-flatness.

Definition 7.37 Let (s, S) be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of relative dimension d on $\mathbb{P}^n_S$ (7.27), with scheme-theoretic support $Y := \operatorname{SSupp} F$.

(7.37.1) F is *C-flat* over S iff $\operatorname{DSupp}(\pi_* F)$ is Cartier over S for every $\mathscr{O}_S$-projection $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ (7.36) that is finite on Y.

(7.37.2) F is *stably C-flat* iff $(v_m)_* F$ is C-flat for every Veronese embedding $v_m\colon \mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$ (where $N = \binom{n+m}{n} - 1$).

(7.37.3) F is *K-flat* over S iff $\operatorname{DSupp}(\varrho_* F)$ is Cartier over S for every finite morphism $\varrho\colon Y \to \mathbb{P}^{d+1}_S$.

(7.37.4) F is *locally K-flat* over S at $y \in Y$ iff $\operatorname{DSupp}(\varrho_* F)$ is Cartier over S at $\varrho(y)$ for every finite $\varrho\colon Y \to \mathbb{P}^{d+1}_S$ for which $\{y\} = \operatorname{Supp} \varrho^{-1}(\varrho(y))$.

(7.37.5) F is *formally K-flat* over S at a closed point $y \in Y$ iff $\operatorname{DSupp}(\varrho_* \widehat{F})$ is Cartier over $\widehat{S}$ for every finite morphism $\varrho\colon \widehat{Y} \to \widehat{\mathbb{A}}^{d+1}_{\widehat{S}}$, where $\widehat{S}$ (resp. $\widehat{Y}$) denotes the completion of S at s (resp. Y at y).

7.37.6 (Base change properties) We see in (7.50) that being C-flat is preserved by arbitrary base changes and the property descends from faithfully flat base changes. This then implies the same for stable C-flatness. Once we prove that the latter is equivalent to K-flatness, the latter also has the same base change properties. Most likely the same holds for formal K-flatness.

7.37.7 (General base schemes) We say that any of the above notions (7.37.1–5) holds for a local base scheme (s, S) (with finite residue field) if it holds after some faithfully flat base change $(s', S') \to (s, S)$, where $k(s')$ is infinite. Property (7.37.6) assures that this is independent of the choice of S'.

Finally, we say that any of the notions (7.37.1–5) holds for an arbitrary base scheme S, if it holds for all of its localizations.

Variants 7.38 These definitions each have other versions and relatives. I believe that each of the five are natural and maybe even optimal, though they may not be stated in the cleanest form. Here are some other possibilities and equivalent versions.

(7.38.1) It could have been better to define C-flatness using the Cayley–Chow form; the equivalence is proved in (7.47). The Cayley–Chow form version matches better with the study of Chow varieties; the definition in (7.37.1) emphasizes the similarity with the other four.

(7.38.2) In (7.37.2), it would have been better to say that F is *stably C-flat for* $L := \mathscr{O}_Y(1)$. However, we see in (7.62) that this notion is independent of the choice of an ample line bundle L, so we can eventually drop L from the name.
(7.38.3) In (7.37.3), we get an equivalent notion if we allow all finite morphisms $\varrho\colon Y \to W$, where $W \to S$ is any smooth, projective morphism of pure relative dimension $d+1$ over S. Indeed, let $\pi\colon W \to \mathbb{P}^{d+1}_S$ be a finite morphism. If F is K-flat then $\operatorname{DSupp}((\pi \circ \varrho)_* F)$ is a relative Cartier divisor, hence $\operatorname{DSupp}(\varrho_* F)$ is K-flat by (7.28.3). Since $W \to S$ is smooth, $\operatorname{DSupp}(\varrho_* F)$ is a relative Cartier divisor by (7.53).
(7.38.4) It would be natural to consider an affine version of C-flatness: We start with a coherent sheaf F on $\mathbb{A}^n_S$ and require that $\operatorname{DSupp}(\pi_* F)$ be Cartier over S for every projection $\pi\colon \mathbb{A}^n_S \to \mathbb{A}^{d+1}_S$ that is finite on Y.

The problem is that the relative affine version of Noether's normalization theorem does not hold, thus there may not be any such projections (10.47), though one can try to go around this using (10.46.2). This is why (7.37.4) is stated for projective morphisms only.

Although a more local version is defined in (7.51), we did not find a truly local theory. Nonetheless, the notions (7.37.1–4) are étale local on X, and most likely the following Henselian version of (7.37.5) does work.
(7.38.5) Assume that $f\colon (y, Y) \to (s, S)$ is a local morphism of pure relative dimension d of Henselian local schemes such that $k(y)/k(s)$ is finite. Let F be a coherent sheaf on X that is pure of relative dimension d over S. Then F is *K-flat* over S iff $\operatorname{DSupp}(\varrho_* F)$ is Cartier over S for every finite morphism $\varrho\colon Y \to \operatorname{Spec} \mathscr{O}_S\langle x_0, \dots, x_d\rangle$ (where $R\langle \mathbf{x}\rangle$ denotes the Henselization of $R[\mathbf{x}]$).

It is possible that in fact all five versions (7.37.1–5) are equivalent to each other, but for now we can prove only 13 of the 20 possible implications. Four of them are easy to see.

Proposition 7.39 *Let F be a generically flat family of pure, coherent sheaves of relative dimension d on $\mathbb{P}^n_S$. Then*

$$\textit{formally K-flat} \Rightarrow \textit{K-flat} \Rightarrow \textit{locally K-flat} \Rightarrow \textit{stably C-flat} \Rightarrow \textit{C-flat}.$$

Proof A divisor D on a scheme X is Cartier iff its completion $\widehat{D}$ is Cartier on $\widehat{X}$ for every $x \in X$ by (7.11). Thus formally K-flat $\Rightarrow$ K-flat.

K-flat $\Rightarrow$ locally K-flat is clear, and locally K-flat $\Rightarrow$ stably C-flat follows from (7.52). Finally stably C-flat $\Rightarrow$ C-flat is clear; see also (7.56). □

A key technical result of the chapter is the following, to be proved in (7.63).

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Theorem 7.40 *K-flatness is equivalent to local K-flatness and to stable C-flatness.*

It is quite likely that our methods will prove the following.

Conjecture 7.41 *Formal K-flatness is equivalent to K-flatness.*

We prove the special case of relative dimension 1 in (7.60); this is also a key step in the proof of (7.40).

The remaining question is whether C-flat implies stably C-flat. This holds in the examples computed in Sections 7.8–7.9, but we do not have any conceptual argument why these two notions should be equivalent.

Question 7.42 Is C-flatness equivalent to stable C-flatness and K-flatness?

Next we show that K-flatness is automatic over reduced schemes and can be checked on Artinian subschemes.

Proposition 7.43 *Let S be a reduced scheme and F a generically flat family of pure, coherent sheaves on $\mathbb{P}^n_S$. Then F is K-flat over S.*

Proof This follows from (7.29.2). □

Proposition 7.44 *Let S be a scheme and F a generically flat family of pure, coherent sheaves on $\mathbb{P}^n_S$. Then F satisfies one of the properties (7.37.1–5) iff τ^*F satisfies the same property for every Artinian subscheme $\tau\colon A \hookrightarrow S$.*

Proof Set $Y := \operatorname{SSupp} F$ and let $\pi\colon Y \to \mathbb{P}^{d+1}_S$ be a finite morphism. By (7.11), $\operatorname{DSupp}_S(\pi_*F)$ is Cartier iff $\operatorname{DSupp}_A((\pi_A)_*\tau^*F)$ is Cartier for every Artinian subscheme $\tau\colon A \hookrightarrow S$. Thus the Artinian versions imply the global ones.

To check the converse, we may localize at $\tau(A)$. The claim is clear if every finite morphism $\pi_A\colon Y_A \to \mathbb{P}^{d+1}_A$ can be extended to $\pi\colon Y \to \mathbb{P}^{d+1}_S$. This is obvious for C-flatness, stable C-flatness, and formal K-flatness, but it need not hold for K-flatness and local K-flatness.

These cases will be established only after we prove (7.40) in (7.63). Thus we have to be careful not to use this direction in Section 7.5. □

7.45 (Push-forward, additivity and multiplicativity) First, as a generalization of (7.4.4), let $f\colon X \to S$ and $g\colon Y \to S$ be projective morphisms of pure relative dimension n and $\tau\colon X \to Y$ a finite morphism. Let F be a coherent sheaf on X that is generically flat and pure over S of dimension $n-1$ such that

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g is smooth at generic points of $f_*(F_s)$ for every $s \in S$. Let $\pi\colon Y \to \mathbb{P}^n_S$ be any finite morphism. Then

$$\operatorname{DSupp}_S((\pi \circ \tau)_* F) = \operatorname{DSupp}_S(\pi_*(\tau_* F)) = \operatorname{DSupp}_S(\pi_* \mathscr{O}_{\operatorname{DSupp}_S(\tau_* F)}),$$

where the first equality follows from the identity $\pi_*(\tau_* F) = (\pi \circ \tau)_* F$ and for the second we apply (7.28.3) to $\tau_* F$. This proves (7.4.4).

Additivity (7.4.5) is essentially a special case of this. Let $f\colon X \to S$ be a projective morphism of pure relative dimension n and $D_1, D_2 \subset X$ K-flat, relative Mumford divisors. Next take two copies $X' := X_1 \cup X_2$ of X, mapping to X by the identity map $\tau\colon X' \to X$. Let $D' \subset X'$ be the union of the divisors $D_i \subset X_i$. Then $\operatorname{DSupp}_S(\tau_* \mathscr{O}_{D'}) = D_1 + D_2$. Thus if the D_i are K-flat, then so is $D_1 + D_2$.

Finally, consider (7.4.6). If D is K-flat, then so is every mD by additivity, the interesting claim is the converse. Let $\pi\colon Y \to \mathbb{P}^n_S$ be any finite morphism. Set $E := \operatorname{DSupp}_S(\pi_* D)$. Then $mE = \operatorname{DSupp}_S(\pi_*(mD))$, thus we need to show that if mE is Cartier and $\operatorname{char} k \nmid m$, then E is Cartier. This was treated in (4.37). □

7.5 Cayley–Chow Flatness

Let $Z \subset \mathbb{P}^n$ be a subvariety of dimension d. Cayley (1860, 1862) associates to it the Cayley–Chow hypersurface

$$\operatorname{Ch}(Z) := \{L \in \operatorname{Gr}(n{-}d{-}1, \mathbb{P}^n) \colon Z \cap L \neq \emptyset\} \subset \operatorname{Gr}(n{-}d{-}1, \mathbb{P}^n).$$

We extend this definition to coherent sheaves on $\mathbb{P}^n_S$ over an arbitrary base scheme. We use two variants, but the proof of (7.47) needs two other versions as well. All of these are defined in the same way, but $\operatorname{Gr}(n{-}d{-}1, \mathbb{P}^n)$ is replaced by other universal varieties.

Definition 7.46 (Cayley–Chow hypersurfaces) Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$ (7.27). We define four versions of the Cayley–Chow hypersurface associated to F. In all four versions the left-hand side map σ is a smooth fiber bundle.

7.46.1 (Grassmannian version) Consider the diagram

$$\mathbb{P}^n_S \xleftarrow{\ \sigma_{gr}\ } \operatorname{Flag}_S(\text{point}, n{-}d{-}1, \mathbb{P}^n) \xrightarrow{\ \tau_{gr}\ } \operatorname{Gr}_S(n{-}d{-}1, \mathbb{P}^n)$$

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where the flag variety parametrizes pairs (point) $\in L^{n-d-1} \subset \mathbb{P}^n$. Set

$$\mathrm{Ch}_{gr}(F) := \mathrm{DSupp}_S\big((\tau_{gr})_* \sigma_{gr}^* F\big).$$

7.46.2 (Product version) Consider the diagram

$$\begin{array}{ccc} & \mathrm{Inc}_S\big(\text{point}, (\check{\mathbb{P}}^n)^{d+1}\big) & \\ \swarrow^{\sigma_{pr}} & & \searrow^{\tau_{pr}} \\ \mathbb{P}^n_S & & (\check{\mathbb{P}}^n)^{d+1}_S, \end{array}$$

where the incidence variety parametrizes $(d + 2)$-tuples $\big((\text{point}), H_0, \ldots, H_d\big)$ satisfying (point) $\in H_i$ for every i. Set

$$\mathrm{Ch}_{pr}(F) := \mathrm{DSupp}_S\big((\tau_{pr})_* \sigma_{pr}^* F\big).$$

7.46.3 (Flag version) Consider the diagram

$$\begin{array}{ccc} & \mathrm{PFlag}_S(0, n{-}d{-}2, n{-}d{-}1, \mathbb{P}^n) & \\ \swarrow^{\sigma_{fl}} & & \searrow^{\tau_{fl}} \\ \mathbb{P}^n_S & & \mathrm{Flag}_S(n{-}d{-}2, n{-}d{-}1, \mathbb{P}^n), \end{array}$$

where PFlag parametrizes triples $\big((\text{point}), L^{n-d-2}, L^{n-d-1}\big)$ such that (point) $\in L^{n-d-1}$ and $L^{n-d-2} \subset L^{n-d-1}$ (but the point need not lie on L^{n-d-2}). Set

$$\mathrm{Ch}_{fl}(F) := \mathrm{DSupp}_S\big((\tau_{fl})_* \sigma_{fl}^* F\big).$$

7.46.4 (Incidence version) Consider the diagram

$$\begin{array}{ccc} & \mathrm{Inc}_S\big(\text{point}, L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}\big) & \\ \swarrow^{\sigma_{in}} & & \searrow^{\tau_{in}} \\ \mathbb{P}^n_S & & \mathrm{Inc}_S\big(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}\big), \end{array}$$

where the $(d+3)$-tuples $\big((\text{point}), L^{n-d-1}, H_0, \ldots, H_d\big)$ satisfy (point) $\in L^{n-d-1} \subset H_i$ for every i. Set

$$\mathrm{Ch}_{in}(F) := \mathrm{DSupp}_S\big((\tau_{in})_* \sigma_{in}^* F\big).$$

Theorem 7.47 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. The following are equivalent:*

(7.47.1) $\mathrm{Ch}_{pr}(F) \subset (\check{\mathbb{P}}^n)^{d+1}_S$ *is Cartier over S.*

(7.47.2) $\mathrm{Ch}_{gr}(F) \subset \mathrm{Gr}_S(n-d-1, \mathbb{P}^n)$ *is Cartier over S.*

If S is local with infinite residue field, then these are also equivalent to

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(7.47.3) $\mathrm{DSupp}(\pi_* F)$ *is Cartier over* S *for every* $\mathscr{O}_S$*-projection* $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ *(7.36) that is finite on* $\operatorname{Supp} F$.

(7.47.4) $\mathrm{DSupp}(\pi_* F)$ *is Cartier over* S *for a dense set of* $\mathscr{O}_S$*-projections* $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$.

Proof The extreme cases $d = 0$ and $d = n-1$ are somewhat exceptional, so we deal with them first.

If $d = n-1$, then $\mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S) = \mathrm{Gr}_S(0, \mathbb{P}^n_S) \simeq \mathbb{P}^n_S$ and the only projection is the identity. Furthermore, $\mathrm{Ch}_{gr}(F) = \mathrm{DSupp}_S(F)$ by definition, so (7.47.2) and (7.47.3) are equivalent. If these hold, then $\mathrm{Ch}_{pr}(F) = \mathrm{Ch}_{pr}(\mathrm{DSupp}_S(F))$ is also flat by (7.23). For (7.47.1) $\Rightarrow$ (7.47.2), the argument in (7.48) works.

If $d = 0$, then F is flat over S and (7.47.1–3) hold by (7.29).

We may thus assume from now on that $0 < d < n-1$. These cases are discussed in (7.48–7.49). □

7.48 (Proof of 7.47.1 $\Leftrightarrow$ 7.47.2) To go between the product and the Grassmannian versions, the basic diagram is the following.

$$
\begin{array}{ccc}
 & \mathrm{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) & \\
\swarrow & & \searrow {\scriptstyle (\mathbb{P}^d)^{d+1}\text{-bundle}} \\
(\check{\mathbb{P}}^n)^{d+1}_S & & \mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S).
\end{array}
$$

The right-hand side projection

$$
\pi_2\colon\ \mathrm{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \to \mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S)
$$

is a $(\mathbb{P}^d)^{d+1}$-bundle. Therefore $\mathrm{Ch}_{in}(F) = \pi_2^* \mathrm{Ch}_{gr}(F)$. Thus $\mathrm{Ch}_{gr}(F)$ is Cartier over S iff $\mathrm{Ch}_{in}(F)$ is Cartier over S. It remains to compare $\mathrm{Ch}_{in}(F)$ and $\mathrm{Ch}_{pr}(F)$.

The left-hand side projection

$$
\pi_1\colon\ \mathrm{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \to (\check{\mathbb{P}}^n)^{d+1}_S
$$

is birational. It is an isomorphism over $(H_0, \dots, H_d) \in (\check{\mathbb{P}}^n)^{d+1}_S$ iff $\dim(H_0 \cap \cdots \cap H_d) = n-d-1$, the smallest possible. That is, when the rank of the matrix formed from the equations of the H_i is $d+1$. Thus π_1^{-1} is an isomorphism outside a subset of codimension $n+1-d$ in each fiber of $(\check{\mathbb{P}}^n)^{d+1}_S \to S$.

Therefore, if $\mathrm{Ch}_{in}(F)$ is Cartier over S then $\mathrm{Ch}_{pr}(F)$ is Cartier over S, outside a subset of codimension $n+1-d \geq 3$ on each fiber of $(\check{\mathbb{P}}^n)^{d+1}_S \to S$. Then $\mathrm{Ch}_{pr}(F)$ is Cartier over S everywhere by (7.10.6).

Conversely, let E be the support of the π_1-exceptional divisor. If $\mathrm{Ch}_{pr}(F)$ is a relative Cartier divisor, then so is $\pi_1^* \mathrm{Ch}_{pr}(F)$, which agrees with $\mathrm{Ch}_{in}(F)$ outside E.

Note that E consists of those $(L^{n-d-1}, H_0, \dots, H_d)$ for which $H_0, \dots, H_d$ are linearly dependent. This is easiest to describe using π_2, which is a $(\mathbb{P}^d)^{d+1}$-bundle over $\mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S)$. In a local trivialization, the points in the ith copy of $\mathbb{P}^d$ have coordinates $(a_{i,0}{:}\cdots{:}a_{i,d})$. Then the equation of E is $\det(a_{i,j}) = 0$. Thus E is irreducible and the restriction of π_2

$$\mathrm{Inc}_S(L^{n-d-1}, (\check{\mathbb{P}}^n)^{d+1}) \setminus E \to \mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S)$$

is surjective. Since $\mathrm{Ch}_{in}(F) = \pi_2^* \mathrm{Ch}_{gr}(F)$, this implies that $\mathrm{Ch}_{gr}(F)$ is relative Cartier (2.92.1). □

7.49 (Proof of 7.47.2 ⇒ 7.47.3 ⇒ 7.47.4 ⇒ 7.47.2) To go between the Grassmannian version and the projection versions, the basic diagram is the following:

$$\begin{array}{ccc} & \mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) & \\ {\scriptstyle \mathbb{P}^{n-d-1}\text{-bundle}} \swarrow & & \searrow {\scriptstyle \mathbb{P}^{d+1}\text{-bundle}} \\ \mathrm{Gr}_S(n-d-1, \mathbb{P}^n) & & \mathrm{Gr}_S(n-d-2, \mathbb{P}^n). \end{array}$$

The left-hand side projection

$$\varrho_1 \colon \mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) \to \mathrm{Gr}_S(n-d-1, \mathbb{P}^n_S)$$

is a $\mathbb{P}^{n-d-1}$-bundle and $\mathrm{Ch}_{fl}(X) = \varrho_1^* \mathrm{Ch}_{gr}(X)$. Thus $\mathrm{Ch}_{gr}(F)$ is Cartier over S iff $\mathrm{Ch}_{fl}(F)$ is Cartier over S.

The right-hand side projection

$$\varrho_2 \colon \mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) \to \mathrm{Gr}_S(n-d-2, \mathbb{P}^n_S)$$

is a $\mathbb{P}^{d+1}$-bundle, but $\mathrm{Ch}_{fl}(X)$ is not a pull-back from $\mathrm{Gr}_S(n-d-2, \mathbb{P}^n_S)$.

Let $L \subset \mathbb{P}^n_S$ be a flat family of $(n-d-2)$-planes over S. The preimage of $[L]$ is the set of all $n-d-1$-planes that contain L; we can identify this with sections of the target of the projection $\pi_L \colon \mathbb{P}^n \dashrightarrow L^\perp$. Thus the restriction of $\mathrm{Ch}_{fl}(X)$ to the preimage of L is $\mathrm{DSupp}((\pi_L)_*(F))$.

So, if $\mathrm{Ch}_{fl}(F)$ is Cartier over S, then $\mathrm{DSupp}((\pi_L)_*(F)) = \mathrm{Ch}_{fl}(F)|_{L^\perp}$ is also Cartier over S. Thus (7.47.2) ⇒ (7.47.3) and (7.47.3) ⇒ (7.47.4) is obvious.

Conversely, assume that $\mathrm{DSupp}((\pi_L)_*(F))$ is Cartier over S for general L. By (7.10.6) it is enough to show that $\mathrm{Ch}_{fl}(F)$ is flat over S, outside a subset of codimension ≥ 3.

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Let $U_F \subset \mathrm{Gr}_S(n-d-2, \mathbb{P}^n_F)$ be the open subset consisting of those L^{n-d-2} that are disjoint from DSupp(F). The restriction of the projection π_f to $\mathrm{Supp}\, \sigma_f^* F$ is finite over $\varrho_2^{-1} U_F$, thus $\mathrm{Ch}_f(F) = \mathrm{DSupp}_S\big((\pi_f)_* \sigma_f^* F\big)$ is flat over S, outside a codimension ≥ 2 subset of each fiber of $\varrho_2^{-1} U_F \to U_F$ by (7.29). By assumption, the non-flat locus is disjoint from the generic fiber, hence the non-flat locus has codimension ≥ 3 over U_F.

It remains to understand what happens over $Z_F := \mathrm{Gr}_S(n-d-2, \mathbb{P}^n_S) \setminus U_F$. Note that $\varrho_2^{-1}(Z_F)$ has codimension 2 in $\mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S)$, so it is enough to show that $\mathrm{Ch}_{fl}(F)$ is flat over S at a general point of a general fiber over Z_F.

Thus let L^{n-d-2} be a general point of Z_F. Then $\mathrm{DSupp}(F) \cap L^{n-d-2}$ is a single point p and F is flat over S at p. Furthermore, a general $L^{n-d-1} \supset L^{n-d-2}$ still intersects DSupp(F) only at p. Thus $\sigma_{fl}^*(F)$ is flat over S at

$$(p, L^{n-d-2}, L^{n-d-1}) \in \mathrm{PFlag}_S(0, n-d-2, n-d-1, \mathbb{P}^n),$$

and $\mathrm{Supp}\, \sigma_{fl}^* F$ is finite over $(L^{n-d-2}, L^{n-d-1}) \in \mathrm{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S)$.

Since $\mathrm{Ch}_{fl}(F) = \mathrm{DSupp}_S\big((\pi_{fl})_* \sigma_{fl}^* F\big)$ by (7.46.3), it is flat over S at the point (L^{n-d-2}, L^{n-d-1}) by (7.29). □

Corollary 7.50 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. Let $h\colon S' \to S$ be a morphism. By base change, we get $g'\colon X' \to S'$ and $F' = \mathrm{vpure}(h_X^* F)$ (9.12).*

(7.50.1) *If F is C-flat, then so is F'.*

(7.50.2) *If F' is C-flat and h is scheme-theoretically dominant, then F is C-flat.*

Proof We may assume that S is local with infinite residue field. Being C-flat is exactly (7.47.3), which is equivalent to (7.47.1). $F \mapsto \mathrm{Ch}_{pr}(F)$ commutes with base change by (7.32) and, if h is scheme-theoretically dominant, then, by (4.28), a divisorial sheaf is Cartier iff its divisorial pull-back is. □

Definition 7.51 Let S be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension d over S (7.27). F is *locally C-flat* over S at $y \in Y :=$ SSupp F iff $\mathrm{DSupp}(\pi_* F)$ is Cartier over S at $\pi(y)$ for every $\mathscr{O}_S$-projection $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ that is finite on Y for which $\{y\} = \mathrm{Supp}(\pi^{-1}(\pi(y)) \cap Y)$.

Lemma 7.52 *Let S be a local scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. Then F is C-flat iff it is locally C-flat at every point.*

Proof It is clear that C-flat implies locally C-flat. Conversely, assume that F is locally C-flat. Set $Z_s := \operatorname{Supp}(F_s) \setminus \operatorname{FlatCM}_S(X, F)$ and pick points $\{y_i : i \in I\}$, one in each irreducible component of Z_s. If $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ is a general $\mathscr{O}_S$-projection, then $\{y_i\} = \pi^{-1}(\pi(y_i)) \cap Y$ for all $i \in I$.

Note that $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor along $\mathbb{P}^{d+1}_s \setminus \pi(Z_s)$ by (7.23) and it is also relative Cartier at the points $\pi(y_i)$ for $i \in I$ since F is locally C-flat. Thus $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor outside a codimension ≥ 3 subset of $\mathbb{P}^{d+1}_s$, hence a relative Cartier divisor everywhere by (7.10.6). □

Corollary 7.53 *Let (s, S) be a local scheme and $X \subset \mathbb{P}^n_S$ a closed subscheme that is flat over S of pure relative dimension $d + 1$. Let $D \subset X$ be a relative Mumford divisor. Let $x \in X_s$ be a smooth point. Then $\mathscr{O}_D$ is locally C-flat at x iff D is a relative Cartier divisor at x.*

Proof We may assume that S has infinite residue field. A general linear projection $\pi\colon X \to \mathbb{P}^{d+1}_S$ is étale at x, and $D \cap \pi^{-1}(\pi(x)) = \{x\}$. Thus $\pi|_D : D \to \pi(D)$ is a local isomorphism at x, hence D is a relative Cartier divisor at x iff $\pi(D)$ is a relative Cartier divisor at $\pi(x)$. By (7.28.2) $\operatorname{DSupp}_S(\pi_* \mathscr{O}_D) = \pi(D)$, thus D is a relative Cartier divisor at x iff $\operatorname{DSupp}_S(\pi_* \mathscr{O}_D)$ is a relative Cartier divisor at $\pi(x)$. That is, iff $\mathscr{O}_D$ is locally C-flat at x. □

Corollary 7.54 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d over S. If F is flat at $y \in Y :=$ SSupp F then it is also locally C-flat at y.*

Proof We may assume that (s, S) is local. By (10.17), F_s is CM outside a subset $Z_s \subset Y_s$ of dimension $\leq d - 2$. Let $W_s \subset Y_s$ be the set of points where F is not flat. Let $\pi\colon Y \to \mathbb{P}^{d+1}_S$ be a general linear projection. By (7.23), $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor outside $\pi(Z_s \cup W_s)$, so we may assume that $\pi(y) \notin \pi(W_s)$. Thus, in a neighborhood of $\pi(y)$, $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor outside $\pi(Z_s)$, which has dimension $\leq d - 2$. Thus $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor at y by (7.10.6). □

Lemma 7.55 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. Let $g_m\colon Y \hookrightarrow \mathbb{P}^N_S$ be an embedding such that $g_m^* \mathscr{O}_{\mathbb{P}^N_S}(1) \simeq \pi^* \mathscr{O}_{\mathbb{P}^{d+1}_S}(m)$. If $(g_m)_* F$ is C-flat then F is C-flat.*

Proof We may assume that S is local with infinite residue field. Let $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ be a general linear projection. We need to show that $\operatorname{DSupp}(\pi_* F)$ is a relative Cartier divisor.

Choosing $d+2$ general sections of $\mathscr{O}_{\mathbb{P}^{d+1}_S}(m)$ gives a morphism $w_m\colon \mathbb{P}^{d+1}_S \to \mathbb{P}^{d+1}_S$. There is a linear projection $\varrho\colon \mathbb{P}^N_S \dashrightarrow \mathbb{P}^{d+1}_S$ such that $w_m \circ \pi = \varrho \circ g_m$. By assumption $\operatorname{DSupp}((\varrho \circ g_m)_* F)$ is a relative Cartier divisor, hence so is

$$\operatorname{DSupp}((w_m \circ \pi)_* F) = \operatorname{DSupp}((w_m)_* \mathscr{O}_{\operatorname{DSupp}(\pi_* F)}),$$

where the equality follows from (7.30.2).

Pick a point $x \in \operatorname{DSupp}(\pi_* F)$. Then a general w_m is étale at x and also $\{x\} = w_m^{-1}(w_m(x)) \cap \operatorname{DSupp}(\pi_* F)$. Thus $w_m\colon\ \operatorname{DSupp}(\pi_* F) \to \operatorname{DSupp}((w_m \circ \pi)_* F)$ is étale at x. Thus $\operatorname{DSupp}(\pi_* F)$ is Cartier at x. □

Corollary 7.56 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. Let $v_m\colon \mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$ be the mth Veronese embedding. If $(v_m)_* F$ is C-flat then so is F.* □

There are very useful Bertini theorems for C-flatness. The going-down versions are straightforward.

Lemma 7.57 *Let (s, S) be a local scheme and F a C-flat family of pure, coherent sheaves of dimension $d \geq 1$ on $\mathbb{P}^n_S$ (7.27). Then there is a finite set of points $\Sigma \subset \operatorname{Supp} F_s$ with the following property.*

Let $H \subset \mathbb{P}^n_S$ be a hyperplane that does not contain any point in Σ and H_s is smooth at generic points of $H \cap \operatorname{Supp} F_s$. Then $F|_H$ is C-flat.

Proof We may assume that the residue field is infinite. Every projection $H \dashrightarrow \mathbb{P}^d_S$ is obtained as the restriction of a projection $\mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$. The rest follows from (7.30.2). □

Corollary 7.58 *Let (s, S) be a local scheme and F a stably C-flat family of pure, coherent sheaves of dimension $d \geq 1$ on $\mathbb{P}^n_S$. Set $Y := \operatorname{SSupp} F$. Let $D \subset Y$ be a relative Cartier divisor that does not contain any point in Σ (7.57) and D_s is smooth at generic points of $D \cap \operatorname{Supp} F_s$. Then $F|_D$ is also stably C-flat.*

Proof We may assume that the residue field is infinite. By (7.52) it is sufficient to prove that $F|_D$ is locally C-flat. Pick a point $y \in D$ and let $H \supset \mathbb{P}^n_S$ be a general hypersurface such that $H \cap Y$ equals D in a neighborhood of y. After a Veronese embedding, H becomes a hyperplane section, and then (7.57) implies that $F|_H$ is stably C-flat. Hence $F|_H$ is locally C-flat by (7.52) and so $F|_D$ also locally C-flat at y. □

The going-up version needs a little more care.

Lemma 7.59 *Let (s, S) be a local Artinian scheme with infinite residue field and F a generically flat family of pure, coherent sheaves of dimension $d \geq 2$ on $\mathbb{P}^n_S$. Then F is C-flat iff $F|_H$ is C-flat for a dense set of hyperplanes $H \subset \check{\mathbb{P}}^n_S$.*

Proof The hyperplanes are parametrized by $H^0(\mathbb{P}^n_S, \mathscr{O}_{\mathbb{P}^n_S}(1)) \simeq \mathscr{O}_S^{n+1}$. Since $\mathscr{O}_S$ is Artinian, it makes sense to talk about a dense set of hyperplanes. (This is the only reason why the lemma is stated for Artinian schemes.)

One direction follows from (7.57). Conversely, if $F|_H$ is C-flat for a dense set of hyperplanes H, then there is a dense set of projections $\pi\colon \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ such that, for a dense set of hyperplanes $L \subset \mathbb{P}^{d+1}_S$, the restriction of F to $\pi^{-1}(L)$ is C-flat. Thus $\mathrm{DSupp}(\pi_* F)$ is a relative Cartier divisor in an open neighborhood of such an L by (7.31). Since $d \geq 2$, this implies that $\mathrm{DSupp}(\pi_* F)$ is a relative Cartier divisor everywhere by (7.10.6). Thus F is C-flat by (7.47). □

Now we come to the key result.

Proposition 7.60 *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension 1 on $\mathbb{P}^n_S$. Then F is stably C-flat ⇔ K-flat ⇔ formally K-flat.*

Proof By (7.39) formally K-flat ⇒ K-flat ⇒ stably C-flat.

Thus assume that F is stably C-flat. Set $Y := \mathrm{SSupp}\, F$ and pick a closed point $p \in Y$. We need to show that F is formally K-flat at p. By the already proved parts of (7.44), it is enough to prove this for Artinian base schemes with infinite residue field. We may thus assume that $S = \mathrm{Spec}\, A$ for a local Artinian ring (A, n_A, k) with k infinite, and $p \in Y(k)$ is the origin $(1{:}0{:}\cdots{:}0)$.

Let $\pi\colon \widehat{Y} \to \widehat{\mathbb{A}}^2_S = \mathrm{Spec}\, A[[u, v]]$ be a finite morphism. We need to show that $\mathrm{DSupp}(\pi_* \widehat{F})$ is Cartier.

Let m_0 be as in (7.61). By (7.35.2), for $m \gg m_0$ we can choose homogeneous polynomials $g_1, g_2 \in H^0(\mathbb{P}^n_A, \mathscr{O}_{\mathbb{P}^n_A}(m))$ such that

$$\tau\colon Y \to \mathbb{P}^2_S \quad \text{given by} \quad (x_0^m{:}g_1{:}g_2) \tag{7.60.1}$$

is a finite morphism, p is the only point of Y that maps to $(1{:}0{:}0)$,

$$g_1/x_0^m \equiv \pi^* u \mod n_R^{m_0}, \quad \text{and} \quad g_2/x_0^m \equiv \pi^* v \mod n_R^{m_0}, \tag{7.60.2}$$

where n_R is the ideal sheaf of $p \in Y$.

Since F is stably C-flat, $\mathrm{DSupp}(\tau_* F)$ is a Cartier divisor and so is its completion at the image of p. Then $\mathrm{DSupp}(\pi_* \widehat{F})$ is Cartier by (7.61). □

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Proposition 7.61 *Let (A, n_A, k) be an Artinian k-algebra, (R, n_R) a local, S_1, generically flat A-algebra of dimension 1, and F a generically free, finite R-module. Let $\pi: \operatorname{Spec} R \to \operatorname{Spec} A[[u, v]]$ be a projection such that R is finite over $A[[u]]$ and $\pi^* u, \pi^* v$ are non-zerodivisors. Then there is an m_0 such that*

(7.61.1) *if $\tau : \operatorname{Spec} R \to \operatorname{Spec} A[[u, v]]$ satisfies $\tau^* u \equiv \pi^* u \mod n_R^{m_0}$ and $\tau^* v \equiv \pi^* v \mod n_R^{m_0}$, then $\operatorname{DSupp}(\pi_* F)$ is Cartier iff $\operatorname{DSupp}(\tau_* F)$ is.*

Proof We follow the computation of $\operatorname{DSupp}(\pi_* F)$ as in (7.21) and show that the formula for $\operatorname{DSupp}(\tau_* F)$ is very similar. Then we finish using (7.16).

Set $s := \pi^* u$. Since R is finite over $A[[u]]$, (s) is n_R-primary, hence $n_R^e \subset (s)$ for some $e \geq 1$. Since F is generically free over $A[[s]]$, it contains a free $A[[s]]$-module $G = \oplus_j A[[s]]e_j$ of the same generic rank $= r$. Since R is a finite $A[[s]]$-algebra, $RG \subset s^{-c}G$ for some $c \geq 0$. Hence $\operatorname{DSupp}(\pi_* F)$ agrees with $\operatorname{DSupp}(\pi_* G)$ on the open set $(u \neq 0)$.

We can thus compute $\operatorname{DSupp}(\pi_* F)$ using multiplication by $\pi^* v$ on G, which is given by a meromorphic matrix

$$M_\pi(s) : \oplus_j A[[s]]e_j \simeq G \xrightarrow{\pi^* v} s^{-d}G \simeq \oplus_j s^{-d} A[[s]]e_j$$

for some $d \geq 0$. Our bound on m_0 depends on r, c, d, e, and $\operatorname{nil}(n_A)$.

Claim 7.61.2 If $s_1 \equiv s \mod (s^m)$ and $m \geq c + 1$, then $s_1^r G = s^r G$ for $r \geq 0$.

Proof Note that $s_1 G \subset sG + s^{m-c}(s^c RG) \subset sG + s^{m-c}G \subset sG$. Also, $s_1^c RG = R s_1^c G \subset R s^c G = s^c RG \subset G$, thus we can interchange s, s_1 in the previous argument to get that $s_1 G = sG$. □

In particular, if $t := \tau^* u \equiv \pi^* u \mod (s^m)$ and $m \geq c + 1$, then $G = \oplus_j A[[t]]e_j$. Thus we can use the same G for computing the divisorial support of $\tau_* F$. Multiplication by $\tau^* v$ is given by another meromorphic matrix $M_\tau(t) : G \to t^{-d}G$. Next we compare M_π and M_τ.

Claim 7.61.3 Assume that $\tau^* v \equiv \pi^* v \mod (s^{m+c+d})$ and $t \equiv s \mod (s^{m+c})$. Then $M_\pi(u) \equiv M_\tau(u) \mod u^m A[[u]]$.

Proof The assumptions imply that $G/s^m G = G/t^m G$, $s^{-d}G/s^m G = t^{-d}G/s^m G$, and $\tau^* v, \pi^* v$ induce the same map $G/s^m G \to s^{-d}G/s^m G$. □

Claim 7.61.4 Assume that $M_\pi(u) \equiv M_\tau(u) \mod u^{m+rd-d} A[[u]]$. Then

$$\det(v\mathbf{1}_r - M_\pi) \equiv \det(v\mathbf{1}_r - M_\tau) \mod u^m A[[u]].$$

Proof The difference of the two sides involves terms that contain at most $r - 1$ entries of M_π and at least one entry of $M_\pi - M_\tau$. □

Putting these together, we get that if (7.61.1) holds and m_0 is large enough, then $\det(v\mathbf{1}_r - M_\pi) \equiv \det(v\mathbf{1}_r - M_\tau) \mod u^m A[[u]]$ and $m \geq \operatorname{nil}(n_A) \cdot d$. The proposition now follows from (7.16). □

Corollary 7.62 *Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ on $\mathbb{P}^n_S$. Let L, M be relatively ample line bundles on $Y :=$ SSupp F. Then F is stably C-flat for L (as in (7.38.2)) iff it is stably C-flat for M.*

Proof We already proved (7.44) for stable C-flatness, thus it is enough to prove our claim when S is Artinian with infinite residue field.

Assume that F is stably C-flat for M. By (7.56), we may assume that L is very ample. Repeatedly using (7.58) we get that, for general $L_i \in |L|$, the restriction of F to the complete intersection curve $L_1 \cap \cdots \cap L_{d-1} \cap Y$ is stably C-flat for M. Thus the restriction of F to $L_1 \cap \cdots \cap L_{d-1} \cap Y$ is formally K-flat by (7.60). Using (7.60) in the other direction for L, we get that the restriction of F to $L_1 \cap \cdots \cap L_{d-1} \cap Y$ is stably C-flat for L. Now we can use (7.59) to conclude that F is stably C-flat for L. □

7.63 (Proof of 7.40 and 7.44) We already noted in (7.39) that K-flat ⇒ stably C-flat.

To see the converse, assume that F is stably C-flat. We aim to prove that it is K-flat. By the already established directions of (7.44), it is enough to prove this over Artinian rings. Thus assume that S is the spectrum of an Artinian ring and let $\pi\colon X \to \mathbb{P}^{d+1}_S$ be a finite projection. Set $L := \pi^* \mathscr{O}_{\mathbb{P}^{d+1}_S}(1)$. By (7.62) F is stably C-flat for L, hence DSupp$(\pi_* F)$ is a relative Cartier divisor by (7.55). This proves (7.40).

We already proved (7.44) for stable C-flatness. By the just established (7.40), stable C-flatness is equivalent to K-flatness and local C-flatness, hence (7.44) also holds for these. □

7.6 Representability Theorems

Definition 7.64 Let S be a scheme and F a generically flat family of pure, coherent sheaves on $\mathbb{P}^n_S$. As in (3.16.1), the *functor of K-flat pull-backs* is

$$\mathcal{K}\mathit{flat}_F(q\colon T \to S) = \begin{cases} \{\emptyset\} & \text{if } q^{[*]}_{\mathbb{P}} F \to T \text{ is K-flat, and} \\ \emptyset & \text{otherwise,} \end{cases}$$

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where $q_{\mathbb{P}} \colon \mathbb{P}^n_T \to \mathbb{P}^n_S$ is the induced morphism and $q_{\mathbb{P}}^{[*]}F := \operatorname{vpure}(q_{\mathbb{P}}^*F)$ is the divisorial pull-back as in (4.2.7) or (9.12). If $Y \subset \mathbb{P}^n_S$ is a generically flat family of pure subschemes then we write $\mathcal{K}\mathit{flat}_Y$ instead of $\mathcal{K}\mathit{flat}_{\mathcal{O}_Y}$.

If $\mathcal{K}\mathit{flat}_F$ is representable by a morphism, we denote it by $j_F^{\mathrm{kflat}} \colon S_F^{\mathrm{kflat}} \to S$. Note that j_F^{kflat} is necessarily a monomorphism.

One defines analogously the *functor of C-flat pull-backs* $\mathcal{C}\mathit{flat}_F$ and the *functor of stably C-flat pull-backs* $\mathcal{SC}\mathit{flat}_F$. The monomorphisms representing them are denoted by $j_F^{\mathrm{cflat}} \colon S_F^{\mathrm{cflat}} \to S$ and $j_F^{\mathrm{scflat}} \colon S_F^{\mathrm{scflat}} \to S$.

In our cases, several of the monomorphisms are subschemes $S^* \hookrightarrow S$ such that $\operatorname{red} S = \operatorname{red} S^*$. (In particular, $S^* \subset S$ is both open and closed.) We call such a subscheme *full.*

Proposition 7.65 *Let S be a scheme and F a generically flat family of pure, coherent sheaves of dimension d on $\mathbb{P}^n_S$. Then the functors of C-flat, stably C-flat or K-flat pull-backs of F are represented by full subschemes*

$$S_F^{\mathrm{kflat}} = S_F^{\mathrm{scflat}} \subset S_F^{\mathrm{cflat}} \subset S.$$

Proof By (7.47), $j_F^{\mathrm{cflat}} \colon S_F^{\mathrm{cflat}} \to S$ is the same as $j^{\mathrm{car}}_{\mathrm{Ch}_{pr}(F)} \colon S^{\mathrm{car}}_{\mathrm{Ch}_{pr}(F)} \to S$, with the Cayley–Chow hypersurface $\mathrm{Ch}_{pr}(F)$ as defined in (7.46.2). Thus (4.28) gives $S_F^{\mathrm{cflat}} \subset S$.

We can apply this to each Veronese embedding $v_m \colon \mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$, to get full subschemes $S^{\mathrm{cflat}}_{v_m(F)} \subset S$. Their intersection gives $S_F^{\mathrm{scflat}} \subset S$. (An intersection of closed subschemes is a subscheme.) Finally $S_F^{\mathrm{kflat}} = S_F^{\mathrm{scflat}}$ by (7.40). □

7.66 (Proof of 7.3) Fix an embedding $X \hookrightarrow \mathbf{P}_S$. By (4.76), there is a universal family of generically flat Mumford divisors $\mathrm{Univ}_d^{\mathrm{md}} \to \mathrm{MDiv}_d(X \subset \mathbf{P}_S)$. By (7.65), we get $\mathrm{KDiv}_d(X)$ as a full subscheme

$$j^{\mathrm{kflat}} \colon \ \mathrm{KDiv}_d(X) = \mathrm{MDiv}_d(X \subset \mathbf{P}_S)^{\mathrm{kflat}} \hookrightarrow \mathrm{MDiv}_d(X \subset \mathbf{P}_S). \qquad \square$$

7.7 Normal Varieties

In the next three sections, we aim to give explicit descriptions of K-flat deformations of certain varieties. First, we show that every K-flat deformation of a normal variety is flat. Then we consider K-flat deformations of planar curves and of seminormal curves. In both cases, we give a complete answer for first order deformations only.

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Theorem 7.67 *Let $g\colon Y \to (s, S)$ be a projective morphism. Assume that* $\operatorname{red}(Y_s)$ *is normal, g is K-flat, g is smooth at the generic points of Y_s, and $\mathscr{O}_Y$ is vertically pure. Then g is flat along Y_s.*

Proof If $\dim Y_s = 1$, then the claim follows from (7.68). In general, there is a smallest, closed subset $Z \subset Y_s$ such that g is flat along $Y_s \setminus Z$. Using the Bertini-type theorem (7.5), we see that the codimension of Z is ≥ 2. In this case, flatness holds even without K-flatness by (10.71). □

Lemma 7.68 *Let $g\colon (y, Y) \to (s, S)$ be a local morphism of pure relative dimension 1, that is, essentially of finite type. Assume that g is smooth along $Y \setminus \{y\}$, g is formally K-flat at y, and* $\operatorname{pure}(Y_s)$ *is smooth at y. Then g is smooth at y.*

Proof By (7.44), we may assume that S is Artinian. Then we can reduce it further to the case when Y is complete and $k(y) = k(s) =: k$; see (10.57) and (7.50). Write $Y = \operatorname{Spec} R_A$.

By induction on the length of A, we may assume that there is an ideal $A \supset (\varepsilon) \simeq k$ such that $\operatorname{pure}(R_A/\varepsilon R_A) \simeq (A/\varepsilon)[[\bar{x}]]$.

Let $x \in R_A$ be a lifting of $\bar{x}$. Set $J := \ker[R_A \to \operatorname{pure}(R_A/\varepsilon R_A)]$. Then J is a rank 1 R_k-module, hence free; let $y \in J$ be a generator. We have $x^r y = \varepsilon g_k(x)$, where $g_k \in k[[x]]$ is a unit and $r = \dim_k(J/\varepsilon R_A)$. These determine a projection of R_A whose image in $\operatorname{Spec} A[[x, y]]$ is given by the ideal

$$A[[x, y]] \cap (y - \varepsilon x^{-r} g_k(x))A[[x, x^{-1}, y]].$$

By (7.15), this is a principal ideal iff $g_k(x) \in (y, x^r)$, that is, when $r = 0$. Thus $R_A = A[[x]]$. □

7.8 Hypersurface Singularities

In this section we give a detailed description of K-flat deformations of hypersurface singularities over $k[\varepsilon]$.

7.69 (Non-flat deformations) Let $X \subset \mathbb{A}^n$ be a reduced subscheme of pure dimension d. We aim to describe nonflat deformations of X that are flat outside a subset $W \subset X$. Choose equations $g_1, \dots, g_{n-d}$ such that

$$(g_1 = \cdots = g_{n-d} = 0) = X \cup X',$$

where $Z := X \cap X'$ has dimension $< d$. Let h be an equation of $X' \cup W$ that does not vanish on any irreducible component of X. Thus X is a complete intersection in $\mathbb{A}^n \setminus (h = 0)$ with equation $g_1 = \cdots = g_{n-d} = 0$. Its flat deformations over an Artinian ring (A, m, k) are then given by

$$g_i(\mathbf{x}) = \Psi_i(\mathbf{x}), \quad \text{where} \quad \Psi_i \in m[x_1, \dots, x_n, h^{-1}]. \tag{7.69.1}$$

Note that we can freely change the Ψ_i by an element of the ideal $(g_i - \Psi_i)$. For $A = k[\varepsilon]$ the equations can be written as

$$g_i(\mathbf{x}) = \Phi_i(\mathbf{x})\varepsilon, \quad \text{where} \quad \Phi_i \in k[x_1, \dots, x_n, h^{-1}]. \tag{7.69.2}$$

Now we can freely change the Φ_i by any element of the ideal $\varepsilon(g_1, \dots, g_{n-d})$. Thus the relevant information is carried by $\phi_i := \Phi_i|_X$. So, generically, first order flat deformations can be given in the form

$$g_i = \phi_i\varepsilon, \quad \text{where} \quad \phi_i \in H^0(X, \mathscr{O}_X)[h^{-1}]. \tag{7.69.3}$$

Set $X^\circ := X \setminus (Z \cup W)$. By varying h, we see that in fact

$$g_i = \phi_i\varepsilon, \quad \text{where} \quad \phi_i \in H^0(X^\circ, \mathscr{O}_{X^\circ}). \tag{7.69.4}$$

This shows that the choice of h is largely irrelevant.

If the deformation is flat then the equations defining X lift, that is, $\phi_i \in H^0(X, \mathscr{O}_X)$. In some simple cases, for example if X is a complete intersection, this is equivalent to flatness. In the examples that we compute, the most important information is carried by the polar parts

$$\bar{\phi}_i \in H^0(X^\circ, \mathscr{O}_{X^\circ})/H^0(X, \mathscr{O}_X). \tag{7.69.5}$$

We study first order K-flat deformations of hypersurface singularities. Plane curves turn out to be the most interesting ones.

7.70 Consider a hypersurface singularity $X := (f = 0) \subset \mathbb{A}^n_{\mathbf{x}}$ and a generically flat deformation of it

$$\mathbf{X} \subset \mathbb{A}^{n+r}_{\mathbf{x},\mathbf{z}}[\varepsilon] \to \operatorname{Spec} k[\varepsilon]. \tag{7.70.1}$$

Aiming to work inductively, we assume that the deformation is flat outside the origin. Choose coordinates such that the x_i do not divide f.

As in (7.69.3), any such deformation can be given as

$$f(\mathbf{x}) = \psi(\mathbf{x})\varepsilon \quad \text{and} \quad z_j = \phi_j(\mathbf{x})\varepsilon, \tag{7.70.2}$$

where $\psi, \phi_j \in \cap_i H^0(X, \mathscr{O}_X)[x_n^{-1}]$. If $n \geq 3$, then $\cap_i \mathscr{O}_X[x_i^{-1}] = \mathscr{O}_X$ and we get the following special case of (10.73).

Claim 7.70.3 Let $X := (f = 0) \subset \mathbb{A}^n$ be a hypersurface singularity and $\mathbf{X} \subset \mathbb{A}^{n+r}[\varepsilon]$ a first order deformation of X that is flat outside the origin. If $n \geq 3$ then $\mathbf{X}$ is flat over $k[\varepsilon]$. □

For $n = 2$, we use the following:

Notation 7.70.4 Let $B = (f(x, y) = 0) \subset \mathbb{A}^2$ be a reduced curve singularity. Set $B^\circ := B \setminus \{(0, 0)\}$. A nonflat deformation $\mathbf{B}$ over $k[\varepsilon]$ is written as

$$f(x, y) = \Psi(x, y)\varepsilon \quad \text{and} \quad z_j = \Phi_j(x, y)\varepsilon.$$

As in (7.69), we set $\psi := \Psi|_B, \phi_j := \Phi_j|_B$ and $\bar{\psi}, \bar{\phi}_j \in H^0(B^\circ, \mathscr{O}_{B^\circ})/H^0(B, \mathscr{O}_B)$ denote their polar parts.

We say that a (flat, resp. generically flat) deformation over $k[\varepsilon]$ *globalizes* if it is induced from a (flat, resp. generically flat) deformation over $k[[t]]$.

Theorem 7.71 *Consider a generically flat deformation* $\mathbf{B}$ *of the plane curve singularity* $B := (f = 0) \subset \mathbb{A}^2_{xy}$ *given in (7.70.4).*

(7.71.1) *If* $\mathbf{B}$ *is C-flat, then* $\psi \in H^0(B, \mathscr{O}_B)$.

(7.71.2) *If* $\psi \in H^0(B, \mathscr{O}_B)$, *then the deformation is*

(a) *flat iff* $\phi_j \in H^0(B, \mathscr{O}_B)$ *and*

(b) *C-flat iff* $f_x\phi_j, f_y\phi_j \in H^0(B, \mathscr{O}_B)$.

(7.71.3) *If B is reduced and* $\psi = 0$, *then the deformation globalizes iff* $\phi_j \in H^0(\bar{B}, \mathscr{O}_{\bar{B}})$, *where* $\bar{B} \to B$ *is the normalization.*

Remark 7.71.4 Note that Ω^1_B is generated by $dx|_B, dy|_B$, while ω_B is generated by $f_y^{-1}dx = -f_x^{-1}dy$.

If B is reduced, then Ω^1_B and ω_B are naturally isomorphic over the smooth locus B°. This gives a natural inclusion $\operatorname{Hom}(\Omega^1_B, \omega_B) \hookrightarrow \mathscr{O}_{B^\circ}$. Then (7.71.2.b) says that $\bar{\phi}_j \in \operatorname{Hom}(\Omega^1_B, \omega_B)/\mathscr{O}_B$. See (7.72) for monomial curves.

Proof For simplicity, we compute with one z coordinate. If $\psi, \phi \in H^0(B, \mathscr{O}_B)$ then we can assume that Ψ, Φ are regular, so the deformation is flat. The converse in (7.71.2.a) is clear.

As for (7.71.2.b), we write down the equation of image of the projection

$$(x, y, z) \mapsto (\bar{x}, \bar{y}) = \big(x - \alpha(x, y, z)z, y - \gamma(x, y, z)z\big),$$

where α, γ are constants for linear projections and power series that are nonzero at the origin in general. Since $z^2 = \phi^2\varepsilon^2 = 0$, Taylor expansion gives that

$$f(\bar{x}, \bar{y}) = f(x, y) - \alpha(x, y, z)f_x(x, y)z - \gamma(x, y, z)f_y(x, y)z.$$

Similarly, for any polynomial $F(x, y)$, we get that $F(\bar{x}, \bar{y}) \equiv F(x, y) \mod \varepsilon\mathscr{O}_{\mathbf{B}}$, hence $F(\bar{x}, \bar{y})z = F(x, y)z$ in $\mathscr{O}_{\mathbf{B}}$ since $z\varepsilon = 0$. Thus the equation is

$$f(\bar{x}, \bar{y}) - \big(\psi(\bar{x}, \bar{y}) - \alpha(\bar{x}, \bar{y}, 0)f_x(\bar{x}, \bar{y})\phi - \gamma(\bar{x}, \bar{y}, 0)f_y(\bar{x}, \bar{y})\phi\big) \cdot \varepsilon = 0. \quad (7.71.5)$$

By (7.15.2), this defines a relative Cartier divisor for every α, γ iff $\psi, f_x\phi, f_y\phi \in \mathscr{O}_B$, proving (7.71.2.b). (Thus linear and formal projections give the same restrictions, hence C-flatness implies formal K-flatness in this case.)

If **B** globalizes then $\phi \in H^0(\bar{B}, \mathscr{O}_{\bar{B}})$, this is the $n = 1$ case of (7.73.1). To prove the converse assertion in (7.71.3), we would like to write the global deformation as

$$(f(x,y) = 0, z = \phi(x,y)s) \subset \mathbb{A}^4_{xyzs}.$$

The problem with this is that ϕ has a pole at the origin. Thus we write $\phi = \phi_1 h^{-r}$ where ϕ_1 is regular at the origin and h is a general linear form in x, y. Then the correct equations are

$$(f(x,y) = 0, zh^r = \phi_1(x,y)s) \subset \mathbb{A}^4_{xyzs}.$$

Note that typically $\phi_1(0,0) = 0$, hence the two-plane $(x = y = 0) \subset \mathbb{A}^4_{xyzs}$ appears as an extra irreducible component. We need one more equation to eliminate it.

If $\phi \in H^0(\bar{B}, \mathscr{O}_{\bar{B}})$, then it satisfies an equation

$$\phi^m + \textstyle\sum_{j=0}^{m-1} r_j\phi^j = 0, \quad \text{where} \quad r_j \in H^0(B, \mathscr{O}_B).$$

Thus $z = \phi s$ satisfies the equation $z^m + \sum_{j=0}^{m-1} r_j z^j s^{m-j} = 0$. Now the three equations

$$f(x,y) = zh^r - \phi_1(x,y)s = z^m + \textstyle\sum_{j=0}^{m-1} r_j z^j s^{m-j} = 0$$

define the required globalization of the infinitesimal deformation. □

7.71.6 (Nonreduced curves) Consider $B = (y^2 = 0)$ with deformations

$$y^2 = (y\psi_1(x) + \psi_0(x))\varepsilon \quad \text{and} \quad z = (y\phi_1(x) + \phi_0(x))\varepsilon,$$

where $\psi_i, \phi_i \in k[x, x^{-1}]$. If this is C-flat, then $\psi_i \in k[x]$ by (7.71.1). Since $f_x \equiv 0$, (7.71.2.b) gives only one condition, that $y(y\phi_1(x) + \phi_0(x))$ be regular. Since $y^2 = 0$, we get that $\phi_0 \in k[x]$, but no condition on ϕ_1. So it can have a pole of arbitrary high order. Note that if ϕ_1 has a pole of order m, then regularizing the second equation we get $zx^m = y\varepsilon +$(other terms). This suggests that if these deformations lie on a family of surfaces, the total space must have more and more complicated singularity at the origin as $m \to \infty$.

Example 7.72 (Monomial curves) We can be quite explicit if B is the irreducible monomial curve $B := (x^a = y^c) \subset \mathbb{A}^2$ where $(a, c) = 1$. Its miniversal space of flat deformations is given as

$$x^a - y^c + \textstyle\sum_{i=0}^{a-2}\sum_{j=0}^{c-2} s_{ij}x^i y^j = 0.$$

Its dimension is $(a-1)(c-1)$.

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In order to compute C-flat deformations, we parametrize B as $t \mapsto (t^c, t^a)$. Thus $\mathscr{O}_B = k[t^c, t^a]$. Let $E_B = \mathbb{N}a + \mathbb{N}c \subset \mathbb{N}$ denote the semigroup of exponents. Then the condition (7.71.2.b) becomes

$$t^{ac-c}\phi(t), t^{ac-a}\phi(t) \in k[t^a, t^c]. \tag{7.72.1}$$

This needs to be checked one monomial at a time.

For $\phi = t^m$ and $m \geq 0$ the conditions (7.72.1) are automatic, and the deformation is nonflat iff $m \notin E_B$. These give a space of dimension $\frac{1}{2}(a-1)(c-1)$. (This is an integer since one of a, c must be odd.)

For $\phi = t^{-m}$ and $m \geq 0$, we get the conditions $ac-c-m \in E_B$ and $ac-a-m \in E_B$. By (7.72.4), these are equivalent to $ac-a-c-m \in E_B$. The largest value of m satisfying this gives the deformation

$$(x^a - y^c = z - t^{-ac+a+c}\varepsilon = 0) \quad \text{over} \quad k[\varepsilon]. \tag{7.72.2}$$

Note also that for $0 \leq m \leq ac-a-c$, we have that $ac-a-c-m \in E_B$ iff $m \notin E_B$. These again have $\frac{1}{2}(a-1)(c-1)$ solutions.

Thus we see that the space of C-flat deformations that are nonflat has $(a-1)(c-1)$ extra dimensions; the same as the space of flat deformations. This looks very promising, but the next example shows that we get different answers for non-monomial curve singularities.

7.72.3 (Non-monomial example) Consider the curve singularity $B = (x^4 + y^5 + x^2y^3 = 0)$. Blowing up the origin, we get $(x/y)^4 + y + (x/y)^2 y = 0$. Thus B is irreducible, it can be parametrized as $x = t^5 + \cdots, y = t^4 + \cdots$, and it is an equisingular deformation of the monomial curve $(x^4 + y^5 = 0)$.

In the monomial case we have the deformation (7.72.2) where $z - t^{-11}\varepsilon = 0$. We claim that B does not have a C-flat deformation $z - \phi\varepsilon = 0$ where $\phi = t^{-11} + \cdots$. Indeed, such a deformation would satisfy

$$f_x\phi = y \cdot (\text{local unit}) \quad \text{and} \quad f_y\phi = x \cdot (\text{local unit}).$$

Eliminating ϕ gives that $(xf_x)/(yf_y) = $ (local unit). We can compute the left-hand side as

$$\frac{4x^4 + 2x^2y^3}{5y^5 + 3x^2y^3} = \frac{-4y^5 - 4x^2y^3 + 2x^2y^3}{5y^5 + 3x^2y^3} = -\frac{4}{5} \cdot \frac{1 + (1/2)(x/y)^2}{1 + (3/5)(x/y)^2}.$$

This is invertible at the origin of the normalization of B, but it is not regular on B since $\frac{x}{y} = t + \cdots$. □

The following is left as an exercise.

Claim 7.72.4 For $(a, c) = 1$, set $E = \mathbb{N}a + \mathbb{N}c \subset \mathbb{N}$. Then

(a) If $0 \leq m \leq \min\{ac-a, ac-c\}$ then $ac-a-m, ac-c-m \in E$ iff $ac-a-c-m \in E$.

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(b) If $0 \le m \le ac - a - c$ then $ac - a - c - m \in E$ iff $m \notin E$. □

7.73 (Normalization of a deformation) Let T be the spectrum of a DVR with maximal ideal (t) and residue field k. Let $g\colon X \to T$ be a flat morphism of pure relative dimension d with generically reduced fibers. Set $Z\colon = \operatorname{Supp} \operatorname{tors}(X_0)$ and let $\pi\colon \bar{X} \to X$ be the normalization.

By composition, we get $\bar{g}\colon \bar{X} \to T$. Note that $\pi_0\colon \bar{X}_0 \to X_0$ is an isomorphism over $X_0 \setminus Z$ and $\bar{X}_0$ is S_1. In particular, $\bar{X}_0$ is dominated by the normalization X_0^{nor} of X_0.

Note that $t^n \mathscr{O}_X$ usually has some embedded primes contained in Z. The intersection of its height 1 primary ideals (also called the nth symbolic power of $t\mathscr{O}_X$) is $(t\mathscr{O}_X)^{(n)} = \mathscr{O}_X \cap t^n \mathscr{O}_{\bar{X}}$. In particular, we have injections

$$(t\mathscr{O}_X)^{(n)}/(t\mathscr{O}_X)^{(n+1)} \hookrightarrow t^n \mathscr{O}_{\bar{X}}/t^{n+1}\mathscr{O}_{\bar{X}} \simeq \mathscr{O}_{\bar{X}_k}. \tag{7.73.1}$$

A closely related computation is the following.

Example 7.74 Kollár (1999, 4.8) Using (7.34.1), we see that the ideal of Chow equations of the codimension 2 subvariety $(x_{n+1} = f(x_0, \dots, x_n) = 0) \subset \mathbb{P}^{n+1}$ is generated by the forms

$$f(x_0 - a_0 x_{n+1} : \cdots : x_n - a_n x_{n+1}) \quad \text{for all} \quad a_0, \dots, a_n. \tag{7.74.1}$$

If the characteristic is 0, then Taylor's theorem gives that

$$f(x_0 - a_0 x_{n+1} : \cdots : x_n - a_n x_{n+1}) = \textstyle\sum_I \frac{(-1)^I}{I!} a^I \frac{\partial^I f}{\partial x^I} x_{n+1}^{|I|}, \tag{7.74.2}$$

where $I = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$. The $a^{|I|}$ are linearly independent, hence we get that the ideal of Chow equations is

$$I^{\mathrm{ch}}(f(x_0, \dots, x_n), x_{n+1}) = (f, x_{n+1}D(f), \dots, x_{n+1}^m D^m(f)), \tag{7.74.3}$$

where we can stop at $m = \deg f$. Here we use the usual notation

$$D(f) := \left(f, \tfrac{\partial f}{\partial x_0}, \dots, \tfrac{\partial f}{\partial x_n}\right) \tag{7.74.4}$$

for derivative ideals.

If we want to work locally at the point $p = (x_1 = \cdots = x_n = 0)$, then we can set $x_0 = 1$ to get the local version

$$I^{\mathrm{ch}}(f(1, x_1, \dots, x_n), x_{n+1}) = (f, x_{n+1}D(f), \dots, x_{n+1}^m D^m(f)), \tag{7.74.5}$$

where we can now stop at $m = \operatorname{mult}_p f$. This also holds if f is an analytic function, though this needs to be worked out using the more complicated formulas (7.34.6) that for us become

$$\pi\colon (x_1, \dots, x_{n+1}) \to (x_1 - x_{n+1}\psi_1, \dots, x_n - x_{n+1}\psi_n), \tag{7.74.6}$$

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where $\psi_i = \psi_i(x_0, \dots, x_{n+1})$ are analytic functions. Expanding as in (7.74.2) we see that

$$f(x_1 - x_{n+1}\psi_1, \dots, x_n - x_{n+1}\psi_n) \in I^{\mathrm{ch}}(f(x_1, \dots, x_n), x_{n+1}). \qquad (7.74.7)$$

Thus we get the same ideal if we compute I^{ch} using analytic projections.

7.9 Seminormal Curves

Over an algebraically closed field k, every seminormal curve singularity is formally isomorphic to

$$B_n := \operatorname{Spec} k[x_1, \dots, x_n]/(x_i x_j : i \neq j) \subset \mathbb{A}^n_{\mathbf{x}},$$

formed by the union of the n coordinate axes. In this section, we study deformations of B_n over $k[\varepsilon]$ that are flat outside the origin.

A normal form is worked out in (7.75.4), which shows that the space of these deformations is infinite dimensional. Then we describe the flat deformations (7.76) and their relationship to smoothings (7.77).

We compute C-flat and K-flat deformations in (7.79); these turn out to be quite close to flat deformations.

The ideal of Chow equations of B_n is computed in Kollár (1999, 4.11). For $n = 3$, these are close to C-flat deformations, but the difference between the two classes increases rapidly with n.

7.75 (Generically flat deformations of B_n) Let $\mathbf{B}_n \subset \mathbb{A}^m_{\mathbf{x}}[\varepsilon]$ be a generically flat deformation of $B_n \subset \mathbb{A}^m_{\mathbf{x}}$ over $k[\varepsilon]$.

If $\mathbf{B}_n$ is flat over $k[\varepsilon]$, then we can assume that $n = m$, but a priori we only know that $n \leq m$. Following (7.69), we can describe $\mathbf{B}_n$ as follows.

Along the x_j-axis and away from the origin, the deformation is flat. Thus, in the $(x_j \neq 0)$ open set, $\mathbf{B}_n$ can be given as

$$x_i = \Phi_{ij}(x_1, \dots, x_m)\varepsilon, \text{ where } i \neq j \text{ and } \Phi_{ij} \in k[x_1, \dots, x_m, x_j^{-1}]. \qquad (7.75.1)$$

Note that $(x_1, \dots, \widehat{x_j}, \dots, x_m, \varepsilon)^2$ is identically 0 on $\mathbf{B}_n \cap (x_j \neq 0)$, so the terms in this ideal can be ignored. Thus along the x_j-axis we can change (7.75.1) to the simpler form

$$x_i = \phi_{ij}(x_j)\varepsilon, \quad \text{where } i \neq j \text{ and } \phi_{ij} \in k[x_j, x_j^{-1}]. \qquad (7.75.2)$$

There is one more simplification that we can make. Write

$$\phi_{ij} = \phi'_{ij} + \gamma_{ij} \quad \text{where} \quad \phi'_{ij} \in k[x_j^{-1}], \gamma_{ij} \in (x_j) \subset k[x_j],$$

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and set $x_i' = x_i - \sum_{j\neq i}\gamma_{ij}(x_j)$. Then we get the description

$$x_i' = \phi_{ij}'(x_j')\varepsilon \quad \text{where} \quad i \neq j \quad \text{and} \quad \phi_{ij}' \in k[x_j'^{-1}]. \tag{7.75.3}$$

For most of our computations, the latter coordinate change is not very important. Thus we write our deformations as

$$\mathbf{B}_n : \{x_i = \phi_{ij}(x_j)\varepsilon \quad \text{along the } x_j\text{-axis}\}, \tag{7.75.4}$$

where $\phi_{ij}(x_j) \in k[x_j, x_j^{-1}]$, but we keep in mind that we can choose $\phi_{ij}(x_j) \in k[x_j^{-1}]$ if it is convenient. Writing $\mathbf{B}_n$ as in (7.75.4) is almost unique; see (7.76.3) for one more coordinate change that leads to a unique normal form.

Writing x_ix_j in two ways using (7.75.4) we get that

$$x_ix_j = (x_i\phi_{ji}(x_i)1_i + x_j\phi_{ij}(x_i)1_j)\varepsilon, \tag{7.75.5}$$

where 1_ℓ denotes the function that is 1 on the x_ℓ-axis and 0 on the others.

In order to deal with the cases when $m > n$, we make the following:

Convention 7.75.6 We set $\phi_{ij} \equiv 0$ for $j > n$.

We get the same result (7.75.4) if we work with the analytic or formal local scheme of B_n: we still end up with $\phi_{ij}(x_j) \in k[x_j^{-1}]$.

Proposition 7.76 *For $n \geq 3$, the generically flat deformation $\mathbf{B}_n \subset \mathbb{A}^n_{\mathbf{x}}[\varepsilon]$ as in (7.75.4) is flat iff*

(7.76.1) *either $n \geq 3$ and the ϕ_{ij} have no poles,*

(7.76.2) *or $n = 2$ and ϕ_{12}, ϕ_{21} have only simple poles with the same residue.*

Proof $\mathbf{B}_n$ is flat iff the equations $x_ix_j = 0$ of B_n lift to equations of $\mathbf{B}_n$. We computed in (7.75.5) that $x_ix_j = (x_i\phi_{ji}(x_i)1_i + x_j\phi_{ij}(x_i)1_j)\varepsilon$, thus x_ix_j lifts to an equation iff $x_i\phi_{ji}(x_i)1_i + x_j\phi_{ij}(x_i)1_j$ is regular. Thus the ϕ_{ij} have only simple poles and the residues must agree along all the axes. $x_i\phi_{ji}(x_i)1_i + x_j\phi_{ij}(x_i)1_j$ vanishes along the other $n-2$ axes for $n \geq 3$, so the residues must be 0. □

Corollary 7.76.3 The first order flat deformation space $T^1_{B_n}$ has dimension $n(n-1) - n = n(n-2)$.

Proof By (7.75.3) and (7.76), flat deformations can be given as

$$\mathbf{B}_n : \{x_i = e_{ij}\varepsilon \quad \text{along the } x_j\text{-axis, where } e_{ij} \in k\}.$$

The constants e_{ij} are not yet unique, $x_i \mapsto x_i - a_i$ changes $e_{ij} \mapsto e_{ij} - a_j$. □

Strangely, (7.76.3) says that every flat first order deformation of B_n is obtained by translating the axes independently of each other. These deformations all globalize in the obvious way, but the globalization is not a flat deformation of B_n unless the translated axes all pass through the same point. If this point is $(a_1\varepsilon, \dots, a_n\varepsilon)$, then $e_{ij} = a_j$ and applying (7.76.3) we get the trivial deformation. See (7.77) for smoothings of B_n.

If $n = 2$, then the universal deformation is $x_1x_2 + \varepsilon = 0$. One may ask why this deformation does not lift to a deformation of B_3: smooth two of the axes to a hyperbola and just move the third axis along. If we use $x_1x_2 + t = 0$, then the x_3-axis should move to the line $(x_1 - \sqrt{t} = x_2 - \sqrt{t} = 0)$. This gives the flat deformation given by equations

$$x_1x_2 + t = x_3(x_1 - \sqrt{t}) = x_3(x_2 - \sqrt{t}) = 0.$$

Of course this only makes sense if t is a square. Thus setting $\varepsilon = \sqrt{t} \mod t$ the $t = \varepsilon^2 \mod t$ term becomes 0 and we get

$$x_1x_2 = x_3x_1 - x_3\varepsilon = x_3x_2 - x_3\varepsilon = 0,$$

which is of the form given in (7.76.1).

Example 7.77 (Smoothing B_n) Rational normal curves $R_n \subset \mathbb{P}^n$ have a moduli space of dimension $(n + 1)(n + 1) - 1 - 3 = n^2 + 2n - 3$. The $B_n \subset \mathbb{P}^n$ have a moduli space of dimension $n + n(n - 1) = n^2$. Thus the smoothings of B_n have a moduli space of dimension $n^2 + 2n - 3 - n^2 = 2n - 3$. We can construct these smoothings explicitly as follows.

Fix distinct $p_1, \dots, p_n \in k$ and consider the map

$$(t, z) \mapsto \left(\tfrac{t}{z-p_1}, \dots, \tfrac{t}{z-p_n}\right).$$

Eliminating z gives the equations

$$(p_i - p_j)x_ix_j + (x_i - x_j)t = 0\colon\ 1 \le i \ne j \le n \tag{7.77.1}$$

for the closure of the image, which is an affine cone over a degree n rational normal curve $R_n \subset \mathbb{P}^n_{t,\mathbf{x}}$. So far this is an $(n - 1)$-dimensional space.

Applying the torus action $x_i \mapsto \lambda_i^{-1}x_i$, we get new smoothings given by

$$(p_i - p_j)x_ix_j + (\lambda_jx_i - \lambda_ix_j)t = 0\colon\ 1 \le i \ne j \le n. \tag{7.77.2}$$

Writing it in the form (7.75.4), we get

$$x_i = \tfrac{\lambda_i}{p_i-p_j}\varepsilon \quad \text{along the } x_j\text{-axis.} \tag{7.77.3}$$

This looks like a $2n$-dimensional family, but $\mathbf{Aut}(\mathbb{P}^1)$ acts on it, reducing the dimension to the expected $2n-3$. The action is clear for $z \mapsto \alpha z+\beta$, but $z \mapsto z^{-1}$ also works out since

$$\tfrac{\lambda_i}{p_i^{-1}-p_j^{-1}} = \tfrac{-\lambda_ip_i^2}{p_i-p_j} + \lambda_ip_i.$$

Claim 7.77.4 For distinct $p_i \in k$ and $\lambda_j \in k^*$, the vectors

$$\left(\tfrac{\lambda_j}{p_i - p_j} : i \neq j\right) \quad \text{span} \quad T^1_{B_n} \simeq k^{\binom{n}{2}}.$$

So the flat infinitesimal deformations determined in (7.76.3) form the Zariski tangent space of the smoothings.

Proof Assume that there is a linear relation $\sum_{ij} m_{ij} \frac{\lambda_j}{p_i - p_j} = 0$. If we let $p_i \to p_j$ and keep the others fixed, we get that $m_{ij} = 0$. □

Remark 7.77.5 If $n = 3$, then the Hilbert scheme of degree 3 reduced space curves with $p_a = 0$ is smooth; see Piene and Schlessinger (1985).

Example 7.78 (Simple poles) Among nonflat deformations, the simplest ones are given by $\phi_{ij}(x_j) = c_{ij} x_j^{-1} + e_{ij}$. By (7.75.5), $x_i x_j = (c_{ji} 1_i + c_{ij} 1_j)\varepsilon$. For $n \geq 3$ and general choices of the c_{ij}, the rational functions $c_{ji} 1_i + c_{ij} 1_j$ span $\mathscr{O}_{\bar{B}_n} / \mathscr{O}_{B_n}$. Thus we get an exact sequence

$$0 \to \varepsilon \cdot \mathscr{O}_{\bar{B}_n} \to \mathscr{O}_{\mathbf{B}_n} \to \mathscr{O}_{B_n} \to 0. \tag{7.78.1}$$

The main result is the following.

Theorem 7.79 *For a first order deformation of $B_n \subset \mathbb{A}^m$ specified by*

$$\mathbf{B}_n : \{x_i = \phi_{ij}(x_j)\varepsilon \quad \text{along the } x_j\text{-axis}\}, \tag{7.79.1}$$

the following are equivalent:

(7.79.2) $\mathbf{B}_n$ *is C-flat.*

(7.79.3) $\mathbf{B}_n$ *is K-flat.*

(7.79.4) *The ϕ_{ij} have only simple poles and ϕ_{ij}, ϕ_{ji} have the same residue.*

Recall that $\phi_{ij} \equiv 0$ for $j > n$ by (7.75.6), hence (4) implies that ϕ_{ij} has no poles for $i > n$.

Proof The proof consist of two parts. First, we show in (7.80) that (7.79.2) and (7.79.4) are equivalent by explicitly computing linear projections.

We see in (7.81) that if the ϕ_{ij} have only simple poles, then there is only one term of the equation of a nonlinear projection that could have a pole. This term is the same for the linearization of the projection. Hence it vanishes iff it vanishes for linear projections. This shows that (7.79.4) ⇒ (7.79.3). □

Remark 7.79.5 If $j > n$ then $\phi_{ij} \equiv 0$ by (7.75.6), so ϕ_{ji} is regular by (7.79.4). Evaluating them at the origin gives the vector $\mathbf{v}_j \in k^n$. If $\sum_{j>n} \lambda_j \mathbf{v}_j = 0$ then

$$\textstyle\sum_{j>n} \lambda_j (x_j - \sum_{i=1}^n \phi_{ji} x_\ell \varepsilon)$$

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is regular and identically 0 on $\mathbf{B}_n$. We can thus eliminate some of the x_j for $j > n$ and obtain that every K-flat deformation of $\mathbf{B}_n$ lives in $\mathbb{A}^{2n-1}$.

7.80 (Linear projections) Recall that by our convention (7.75.6), $\phi_{ij} \equiv 0$ for $j > n$. Extending this, in the following proof all sums/products involving i go from 1 to m and sums/products involving j go from 1 to n.

With $\mathbf{B}_n$, as in (7.79.1), consider the special projections

$$\pi_{\mathbf{a}} \colon \mathbb{A}^n_{\mathbf{x}}[\varepsilon] \to \mathbb{A}^2_{uv}[\varepsilon] \quad \text{given by} \quad u = \textstyle\sum x_i, v = \sum a_i x_i, \tag{7.80.1}$$

where $a_i \in k[\varepsilon]$. Write $a_i = \bar{a}_i + a'_i\varepsilon$. (One should think that $a'_i = \partial a_i/\partial\varepsilon$.)

In order to compute the projection, we follow the method of (7.21.7). Since we compute over $k[u, u^{-1}, \varepsilon]$, we may as well work with the $k[u, \varepsilon]$-module $M := \oplus_j k[x_j, \varepsilon]$ and write $1_j \in k[x_j, \varepsilon]$ for the jth unit. Then multiplication by u and v are given by

$$\begin{aligned} u \cdot 1_j &= \quad (\textstyle\sum_i x_i)1_j = \quad x_j + \sum_i \phi_{ij}\varepsilon, \quad \text{and} \\ v \cdot 1_j &= (\textstyle\sum_i a_i x_i)1_j = a_j x_j + \sum_i a_i \phi_{ij}\varepsilon. \end{aligned} \tag{7.80.2}$$

Thus $v \cdot 1_j = (a_j u + \sum_i (a_i - a_j)\phi_{ij}(u)\varepsilon) \cdot 1_j$. and the v-action on M is given by the diagonal matrix

$$\operatorname{diag}(a_j u + \textstyle\sum_i (a_i - a_j)\phi_{ij}(u)\varepsilon).$$

By (7.21.7), the equation of the projection is its characteristic polynomial

$$\textstyle\prod_j (v - a_j u - \sum_i (a_i - a_j)\phi_{ij}(u)\varepsilon) = 0. \tag{7.80.3}$$

Expanding it, we get an equation of the form

$$\begin{aligned} &\textstyle\prod_j (v - \bar{a}_j u) - E(u, v, a, \phi)\varepsilon = 0, \quad \text{where} \\ &E(u, v, a, \phi) = \textstyle\sum_j (\prod_{i \neq j} (v - \bar{a}_j u)) \cdot (a'_j u + \sum_i (\bar{a}_i - \bar{a}_j)\phi_{ij}(u)). \end{aligned} \tag{7.80.4}$$

This is a polynomial of degree $\leq n - 1$ in v, hence by (7.19) its restriction to the curve $(\prod_j (v - \bar{a}_j u) = 0)$ is regular iff $E(u, v, a, \phi)$ is a polynomial in u as well. Let r be the highest pole order of the ϕ_{ij} and write

$$\phi_{ij}(u) = c_{ij} u^{-r} + \text{(higher terms)}. \tag{7.80.5}$$

Then the leading part of the coefficient of v^{n-1} in $E(u, v, a, \phi)$ is

$$\textstyle\sum_j \sum_i (\bar{a}_i - \bar{a}_j) c_{ij} u^{-r} = u^{-r} \sum_i \bar{a}_i (\sum_j (c_{ij} - c_{ji})). \tag{7.80.6}$$

Since the $\bar{a}_i$ are arbitrary, we get that

$$\textstyle\sum_j (c_{ij} - c_{ji}) = 0 \quad \text{for every } i. \tag{7.80.7}$$

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Next we use a linear reparametrization of the lines $x_i = \lambda_i^{-1} y_i$ and then apply a projection $\pi_{\mathbf{a}}$ as in (7.80.1). The equations $x_i = \phi_{ij}(x_j)\varepsilon$ become

$$y_i = \lambda_i \phi_{ij}(\lambda_j^{-1} y_j)\varepsilon$$

and c_{ij} changes to $\lambda_i \lambda_j^r c_{ij}$. Thus the equations (7.80.7) become

$$\textstyle\sum_j (\lambda_i \lambda_j^r c_{ij} - \lambda_j \lambda_i^r c_{ji}) = 0 \quad \forall i. \tag{7.80.8}$$

If $r \geq 2$, this implies that $c_{ij} = 0$ and if $r = 1$ then we get that $c_{ij} = c_{ji}$.

This completes the proof of (7.79.2) $\Leftrightarrow$ (7.79.4).

Remark 7.80.9 Note that if we work over $\mathbb{F}_2$, then necessarily $\lambda_i = 1$, hence (7.80.8) does not exclude the $r \geq 2$ cases.

7.81 (Non-linear projections) Consider a general non-linear projection

$$(x_1, \ldots, x_n) \mapsto (\Phi_1(x_1, \ldots, x_n), \Phi_2(x_1, \ldots, x_n)).$$

After a formal coordinate change, we may assume that $\Phi_1 = \sum_i x_i$. Note that the monomials of the form $x_i x_j x_k, x_i^2 x_j^2, x_i x_j \varepsilon$ vanish on $\mathbf{B}_n$, so we can discard these terms from Φ_2. Thus, in suitable local coordinates, a general nonlinear projection can be written as

$$u = \textstyle\sum_i x_i, \quad v = \sum_i \alpha_i(x_i) + \sum_{i \neq j} x_i \beta_{ij}(x_j), \tag{7.81.1}$$

where $\alpha_i(0) = \beta_{ij}(0) = 0$. Note that $\alpha_i'(0) = a_i$ in the notation of (7.80). Now

$$\begin{aligned} u \cdot 1_j &= x_j + \textstyle\sum_i \phi_{ij}(x_j)\varepsilon, \quad \text{and} \\ v \cdot 1_j &= \alpha_j(x_j) + \textstyle\sum_{i \neq j} \alpha_i(\phi_{ij}(x_j)\varepsilon) + \sum_{i \neq j} \phi_{ij}(x_j)\beta_{ij}(x_j)\varepsilon. \end{aligned} \tag{7.81.2}$$

Note further that $\alpha_i(\phi_{ij}(x_j)\varepsilon) = \alpha_i'(0)\phi_{ij}(x_j)\varepsilon$ and

$$\alpha_j(x_j) = \alpha_j(u - \textstyle\sum_i \phi_{ij}(x_j)\varepsilon) = \alpha_j(u) - \alpha_j'(u)\sum_i \phi_{ij}(x_j)\varepsilon.$$

Thus, as in (7.80.4), the projection is defined by the vanishing of

$$\begin{aligned} &\textstyle\prod_j \Big(v - \alpha_j(u) - \sum_i (\beta_{ij}(u) + \alpha_i'(0) - \alpha_j'(u))\phi_{ij}(u)\varepsilon\Big) \\ &\quad =: \textstyle\prod_j (v - \bar\alpha_j(u)) - E(u, v, \alpha, \beta, \phi)\varepsilon. \end{aligned} \tag{7.81.3}$$

Let $\bar\beta_{ij}, \bar\alpha_j'$ denote the residue of β_{ij}, α_j' modulo ε and write $\alpha_j(u) = \bar\alpha_j(u) + \partial_\varepsilon \alpha_j(u)\varepsilon$. As in (7.80.5), expanding the product gives that $E(u, v, \alpha, \beta, \phi)$ equals

$$\textstyle\sum_j (\prod_{i \neq j} (v - \bar\alpha_i(u))) \cdot (\partial_\varepsilon \alpha_j(u) + \sum_i (\bar\beta_{ij}(u) + \bar\alpha_i'(0) - \bar\alpha_j'(u))\phi_{ij}). \tag{7.81.4}$$

We already know that $\phi_{ij}(u) = c_{ij} u^{-1}$ + (higher terms), hence $E(u, v, \alpha, \beta, \phi)$ has at most simple pole along $(u = 0)$. Computing its residue gives that

$$v^{n-1} \textstyle\sum_j \sum_i (\bar\beta_{ij}(0) + \bar\alpha_i'(0) - \bar\alpha_j'(0))c_{ij} = v^{n-1} \sum_{ij} (\bar a_i - \bar a_j)c_{ij}. \tag{7.81.5}$$

These are the same as in (7.80.6). Thus $E(u, v, \alpha, \beta, \phi)$ is regular iff it is regular for the linearization. This completes the proof of (7.79.4) ⇒ (7.79.3).

Example 7.82 The image of a general linear projection of $B_n \subset \mathbb{A}^n$ to $\mathbb{A}^2$ is n distinct lines through the origin. A general nonlinear projection to $\mathbb{A}^2$ gives n smooth curve germs with distinct tangent lines through the origin.

As a typical example, the miniversal deformation of $(x^n + y^n = 0)$ is

$$(x^n + y^n + \textstyle\sum_{i,j\le n-2} t_{ij}x^i y^j = 0) \subset \mathbb{A}^2_{xy} \times \mathbb{A}^{(n-1)^2}_{\mathbf{t}}. \tag{7.82.1}$$

Deformations with tangent cone $(x^n + y^n = 0)$ form the subfamily

$$(x^n + y^n + \textstyle\sum_{i+j>n} t_{ij}x^i y^j = 0) \subset \mathbb{A}^2_{xy} \times \mathbb{A}^{\binom{n-3}{2}}_{\mathbf{t}}. \tag{7.82.2}$$

For $n \le 4$, there is no such pair (i, j), thus, for $n \le 4$, every analytic projection $\widehat{B}_n \to \widehat{\mathbb{A}}^2$ is obtained as the composite of an automorphism of $\widehat{B}_n$, followed by a linear projection and an automorphism of $\widehat{\mathbb{A}}^2$.

For $n = 5$, we get the deformations $(x^5 + y^5 + tx^3y^3 = 0) \subset \mathbb{A}^2_{xy} \times \mathbb{A}_t$. For $t \neq 0$, these give curve germs that are images of $\widehat{B}_n$ by a nonlinear projection, but cannot be obtained as the image of a linear projection, up to automorphisms.