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AN AMBROSETTI–PRODI-TYPE RESULT FOR A QUASILINEAR NEUMANN PROBLEM

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Abstract We study the problem $-\Delta_p u = f(x,u) + t$ in Ω with Neumann boundary condition $|\nabla u|^{p-2}(\partial u/\partial\nu) = 0$ on $\partial\Omega$. There exists a $t_0 \in \mathbb{R}$ such that for $t > t_0$ there is no solution. If $t \leqslant t_0$, there is at least a minimal solution, and for $t < t_0$ there are at least two distinct solutions. We use the sub–supersolution method, *a priori* estimates and degree theory.

Keywords: a priori estimates; degree theory; sub–supersolutions

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1. Introduction

We study the problem

$$\left.\begin{aligned} -\Delta_p u &= f(x,u) + t \quad &&\text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= 0 &&\text{on } \partial\Omega,\end{aligned}\right\} \tag{P_t}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$, $t \in \mathbb{R}$ is a parameter and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator for $1 < p < \infty$. We also assume that

$$\liminf_{s\to\infty} \frac{f(x,s)}{|s|^{p-2}s} > 0 \tag{1.1}$$

and

$$\limsup_{s\to-\infty} \frac{f(x,s)}{|s|^{p-2}s} < 0, \tag{1.2}$$

where the limits are uniform in $x \in \Omega$. Conditions (1.1) and (1.2) are a sort of eigenvalue crossing of f. These assumptions imply, respectively,

$$f(x,s) \geqslant \mu|s|^{p-2}s - C, \qquad s > 0, \tag{1.3}$$

and

$$f(x,s) \geqslant -\mu|s|^{p-2}s - C, \quad s < 0, \tag{1.4}$$

for some constants $C > 0$ and $\mu > 0$. The next hypothesis is standard in order to apply the sub–supersolution method. We shall assume that for all $M > 0$ there exists $\lambda > 0$ such that

$$f(x,u) + \lambda|u|^{p-2}u \quad \text{is non-decreasing in } u \text{ on } [-M,M]. \tag{1.5}$$

Hypothesis (1.5) is in fact not needed when one assumes that $p = 2$ and that f is C^1, since in this case the derivative is $f_u + \lambda > 0$ for u belonging to some interval $[-M,M]$ and λ large.

A function $u \in W^{1,p}(\Omega)$ is called a (weak) solution of (P_t) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \, \mathrm{d}x = \int_\Omega f(x,u)\phi \, \mathrm{d}x - t\int_\Omega u\phi \, \mathrm{d}x \quad \text{for all } \phi \in C^1(\bar{\Omega}),$$

where

$$F(x,t) = \int_0^t f(x,s) \, \mathrm{d}s.$$

Theorem 1.1. *Suppose (1.1), (1.2) and (1.5) hold and that there exists a constant c such that*

$$f(x,s) \leqslant c(1+|s|^{p-1}) \quad \text{for all } s \in \mathbb{R} \text{ and uniformly for } x \in \Omega. \tag{1.6}$$

Then there exists $t_0 \in \mathbb{R}$ such that

(i) *if $t > t_0$, then (P_t) has no solution and*

(ii) *if $t \leqslant t_0$, then (P_t) has at least a minimal solution.*

Moreover, assume f is locally Lipschitz continuous in s uniformly a.e. in $x \in \Omega$. Then

(iii) *there exists $t_1 \leqslant t_0$ such that for $t < t_1$ (P_t) has at least two distinct solutions and*

(iv) *if, moreover, $f \in C(\bar{\Omega} \times \mathbb{R})$, then $t_1 = t_0$.*

Problems of this nature fit into a general framework devised in the pioneering paper by Ambrosetti and Prodi [**1**]. They studied the problem

$$\left.\begin{aligned} -\Delta u &= g(u) + h(x) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.7}$$

where g interacts with the spectrum $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4 \leqslant \cdots$ of $-\Delta$ in $H_0^1(\Omega)$ in such a way that $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$. Assuming that $g'' > 0$, they proved that the singular set S of the mapping $\Phi(u) = -\Delta u - g(u)$ from $U = \{u \in C^{2,\alpha}(\Omega)\colon u = 0 \text{ on } \partial\Omega\}$ to $V = \{u \in C^{0,\alpha}(\Omega)\colon u = 0 \text{ on } \partial\Omega\}$, $0 < \alpha < 1$, consists of a codimension-1 manifold parametrized over $\{u \in U\colon \int_\Omega u\varphi_1 = 0\}$, where φ_1 denotes the first eigenfunction corresponding to λ_1. Moreover, $\Phi(S)$ is a smooth codimension-1 manifold and $V - \Phi(S)$ has exactly two components V_0 and V_2. If $h \in V_2$, then (1.7) has two solutions. If $h \in V_0$, then (1.7) has no solution. If $h \in \Phi(S)$, then (1.7) has one solution. Subsequently, in [**5**] these results have been generalized by parametrizing $\Phi(S)$ in H_0^1. A complete characterization showing that Φ is globally diffeomorphic to the fold map is given in [**4**]. A rich structure of the mapping Φ is described in [**6–8**] in spatial dimension 1. In the present paper we do not give such a detailed description of (P_t), since we do not have the Hilbert space structure. Also, $-\Delta_p$ does not possesses the same regularizing properties of $-\Delta$.

Equation (1.7) with Neumann condition $\partial u/\partial\nu = 0$ on $\partial\Omega$ was studied in [**3, 13**]. A result similar to ours for Dirichlet boundary condition $u = 0$ on $\partial\Omega$ was addressed in [**2, 9**] using different techniques.

In § 2 we show some *a priori* estimates for solutions of (P_t). We use these lemmas to define adequate sets to apply degree theory to prove Theorem 1.1 in § 3.

2. Preliminaries

Throughout this section we assume (1.1), (1.2) and (1.6) hold. We begin by establishing an *a priori* bound for solutions of (P_t).

Lemma 2.1. *Let u be a weak solution of (P_t) in $W_0^{1,p}(\Omega)$. If t belongs to a bounded interval, then $\|u\|_{L^\infty} \leqslant c$, where $c > 0$ is a constant depending on t but not on u.*

Proof. First we shall prove that $\|u^-\|_{L^\infty}$ is bounded. Indeed, multiply (P_t) by $\varphi = \max(u^- - k, 0) \in W^{1,p}(\Omega)$, where $k > 0$. Defining $A_k = \{x \in \Omega\colon u^- > k\}$ and using (1.4), we obtain

$$
\begin{aligned}
\int_{A_k} |\nabla(u^- - k)|^p &= -\int_{A_k} (f(x,u) + t)\varphi \\
&\leqslant \int_{A_k} (c(u^-)^{p-1} + c + |t|)(u^- - k) \\
&= \int_{A_k} c(u^-)^{p-1}(u^- - k) + (c + |t|)\int_{A_k} (u^- - k) \\
&\leqslant \int_{A_k} C((u^- - k)^p + k^{p-1}(u^- - k)) + (c + |t|)\int_{A_k} (u^- - k) \\
&= C\int_{A_k} (u^- - k)^p + (Ck^{p-1} + c + |t|)\int_{A_k} (u^- - k). \qquad (2.1)
\end{aligned}
$$

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In the course of this proof, constant $C > 0$ may vary from line to line. By the Hölder and Sobolev inequalities,

$$\int_{A_k} (u^- - k)^p \leqslant |A_k|^{p/n} \biggl(\int_{A_k} (u^- - k)^{np/(n-p)} \biggr)^{(n-p)/n}$$
$$\leqslant |A_k|^{p/n} C \biggl(\int_{A_k} |\nabla(u^- - k)|^p + \int_{A_k} (u^- - k)^p \biggr), \tag{2.2}$$

Thus, (2.1) and (2.2) imply

$$(|A_k|^{-p/n} - C) \int_{A_k} (u^- - k)^p \leqslant C \int_{A_k} (u^- - k)^p + C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k); \tag{2.3}$$

hence,

$$(|A_k|^{-p/n} - C) \int_{A_k} (u^- - k)^p \leqslant C(k^{p-1} + 1 + |t|) \int_{A_k} (u^- - k).$$

Note that $|A_k| \to 0$ as $k \to \infty$. Indeed, by the proof of Lemma 2.2 (see below), one obtains

$$|A_k| = \int_{u^- > k} dx \leqslant \int_{A_k} \frac{(u^-)^{p-1}}{k^{p-1}} \leqslant C k^{1-p}.$$

Therefore, $|A_k|^{-p/n} - C > 0$ for every $k \geqslant k_0$, where k_0 is fixed, large enough and does not depend on u.

By the Hölder inequality and (2.3),

$$\int_{A_k} (u^- - k) \leqslant |A_k|^{(p-1)/p} \biggl(\int_{A_k} (u^- - k)^p \biggr)^{1/p}$$
$$\leqslant C|A_k|^{(p-1)/p} \biggl(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-p/n} - C} \int_{A_k} (u^- - k) \biggr)^{1/p}.$$

Thus,

$$\biggl(\int_{A_k} (u^- - k) \biggr)^{(p-1)/p} \leqslant C|A_k|^{(p-1)/p} \biggl(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-p/n} - C} \biggr)^{1/p}.$$

Consequently,

$$\int_{A_k} (u^- - k) \leqslant C|A_k| \biggl(\frac{k^{p-1} + 1 + |t|}{|A_k|^{-p/n} - C} \biggr)^{1/(p-1)}$$
$$= C|A_k|^{1+(p/n(p-1))} \biggl(\frac{k^{p-1} + 1 + |t|}{1 - |A_k|^{p/n} C} \biggr)^{1/(p-1)}.$$

We can assume that $1-|A_k|^{p/n}C \geqslant \frac{1}{2}$ for $k \geqslant k_0$, and then

$$\begin{aligned}\int_{A_k}(u^- - k) &\leqslant C|A_k|^{1+(p(p-1)/n)}(k^{p-1}+1+|t|)^{p-1}\\ &= C|A_k|^{1+(p(p-1)/n)}k\left(1+\frac{1+|t|}{k^{p-1}}\right)^{p-1}\\ &\leqslant C|A_k|^{1+(p(p-1)/n)}k\left(1+\frac{1+|t|}{k_0^{p-1}}\right)^{p-1}\\ &\leqslant C|A_k|^{1+(p(p-1)/n)}k.\end{aligned}$$

We are now in a position to apply [**10**, Lemma 5.1, p. 71], to conclude that $\|u^-\|_{L^\infty}$ is bounded by a constant that depends only on t, k_0, μ, p, n and $\|u^-\|_{L^1(A_{k_0})}$. As we said before, constant k_0 does not depend on u. We shall bound $\|u^-\|_{L^1(A_{k_0})}$ more accurately; this is done below.

Multiplying (P_t) by $-u^-$ and using (1.4), we obtain

$$\begin{aligned}0 &\leqslant \int_\Omega |\nabla u^-|^p\\ &= \int_\Omega -f(x,u)u^- - t\int_\Omega u^-\\ &\leqslant -\mu\int_\Omega (u^-)^p + C\int_\Omega u^- - t\int_\Omega u^-.\end{aligned}$$

Hence,

$$\int_\Omega |\nabla u^-|^p + \mu\int_\Omega (u^-)^p \leqslant (C+|t|)\int_\Omega u^- \leqslant \frac{\mu}{2}\int_\Omega (u^-)^p + C(1+|t|).$$

It follows that $\|u^-\|_{W^{1,p}}$ is bounded. Thus,

$$\int_{A_{k_0}}|u^-| \leqslant \int_\Omega |u^-| \leqslant C\left(\int_\Omega |u^-|^p\right)^{1/p} \leqslant C(1+|t|).$$

Therefore, $\|u^-\|_{L^\infty}$ is bounded by a constant depending only on t, μ, p and n.

Now we prove that $\|u^+\|_{L^\infty}$ is bounded. We only need to prove that $\|u^+\|_{L^p}$ is bounded, since the computations above to prove the boundedness of $\|u^-\|_{L^\infty}$ can be performed in a similar manner to conclude that $\|u^+\|_{L^\infty}$ is bounded.

Assume by contradiction that there exist $a,b\in\mathbb{R}$ such that $\|u_{t_n}^+\|_{L^p}\to\infty$ with $t_n\in[a,b]$. Define $w_n = u_{t_n}^+/(\|u_{t_n}^+\|_{L^p})$. Note that w_n is bounded in $W^{1,p}$. Indeed, using (1.6),

$$\begin{aligned}\int_\Omega |\nabla u_{t_n}|^p &= \int_\Omega f(x,u_{t_n})u_{t_n} + t\int_\Omega u_{t_n}\\ &\leqslant c\int_\Omega |u_{t_n}|^p + c(1+|t|)\int_\Omega |u_{t_n}|\\ &\leqslant c(1+|t|)\int_\Omega |u_{t_n}|^p.\end{aligned}$$

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Thus, w_n is bounded in $W^{1,p}$. So we can assume that $w_n \rightharpoonup w$ in $W^{1,p}$, $w_n \to w$ in L^p and $w_n \to w$ a.e. $x \in \Omega$. Moreover, $\|w\|_{L^p} = 1$ and $w \geqslant 0$, since $\|u_{t_n}^+\|_{L^\infty}$ is bounded. Note that $w_n^- \to 0$ a.e. $x \in \Omega$.

Let $\varphi \in W^{1,p}(\Omega)$ and $\varphi \geqslant 0$. Then

$$\begin{aligned}\int |\nabla w_n|^{p-2}\nabla w_n \nabla\varphi &= \int \frac{f(x,u_{t_n})}{\|u_{t_n}^+\|}\varphi + t_n \int \frac{\varphi}{\|u_{t_n}^+\|} \\ &\geqslant \mu \int |w_n|^{p-1}\varphi + o(n).\end{aligned}$$

Letting $n \to \infty$, we get

$$\int |\nabla w|^{p-2}\nabla w \nabla\varphi \geqslant \mu \int w^{p-1}\varphi.$$

Taking $\varphi \equiv 1$, we obtain $\mu \int w^{p-1} \leqslant 0$: a contradiction of the fact that $w \geqslant 0$ and $w \not\equiv 0$. The proof is complete. □

Since weak solutions of (P_t) are bounded, by a result from [**10**], these solutions belong to $C(\bar{\Omega})$. The $C^{1,\alpha}(\bar{\Omega})$ regularity follows from [**12**].

Lemma 2.2. *Problem (P_t) has no solution for sufficiently large $t > 0$.*

Proof. By (1.3) and (1.4) we have that $f(x,s) \geqslant \mu|s|^{p-1} - C$ for every $s \in \mathbb{R}$. Integrating (P_t), we obtain

$$\begin{aligned}0 &= \int_\Omega f(x,u_t) + t|\Omega| \\ &\geqslant \mu \int_\Omega |u_t|^{p-1} - C|\Omega| + t|\Omega|.\end{aligned}$$

Then

$$\mu \int_\Omega |u_t|^{p-1} + t|\Omega| \leqslant C|\Omega|,$$

which gives a contradiction for $t > 0$ large enough. □

Lemma 2.3. *Problem (P_t) has a subsolution for all t.*

Proof. There is a constant z_t satisfying

$$\left.\begin{aligned}-\Delta_p z \leqslant f(x,z) + t \quad & \text{in } \Omega, \\ z \leqslant 0 \quad & \text{in } \bar{\Omega}, \\ |\nabla z|^{p-2}\frac{\partial z}{\partial \nu} = 0 \quad & \text{on } \partial\Omega.\end{aligned}\right\} \tag{2.4}$$

In fact, since (1.2) reads as $f(x,u) \geqslant -\mu|u|^{p-2}u - C$ for some constants $\mu, C > 0$, then clearly

$$z_t = -\left(\frac{|t| + C}{\mu}\right)^{1/(p-1)}$$

satisfies the requirements we need. □

Remark 2.4. It is easy to see that every constant $z < z_t$ is a strict subsolution.

Lemma 2.5. *If u_t is a solution of (P_t), then $u_t \geqslant z_t$.*

Proof. It is enough to show that $(z_t - \varepsilon - u_t)^+ = 0$ for all $\varepsilon > 0$. Suppose by contradiction that there exists $\varepsilon_0 > 0$ such that $(z_t - \varepsilon_0 - u_t)^+$ is non-trivial. Then $\Omega_\varepsilon = \{x \in \Omega \colon u_t(x) < z_t - \varepsilon\} \neq \emptyset$ for all $0 < \varepsilon < \varepsilon_0$. Multiplying (P_t) by $(z_t - \varepsilon - u_t)^+ \not\equiv 0$ and using (1.4) yields

$$\begin{aligned}\int_{\Omega_\varepsilon} &|\nabla u_t|^{p-2}\nabla u_t \nabla(z_t - \varepsilon - u_t)^+ \\ &= \int_{\Omega_\varepsilon} f(x, u_t)(z_t - \varepsilon - u_t)^+ + t\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+ \\ &\geqslant -\mu\int_{\Omega_\varepsilon}|u_t|^{p-2}u_t(z_t - \varepsilon - u_t)^+ - (C - t)\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+.\end{aligned}$$

Note that for $x \in \Omega_\varepsilon$ we have $\nabla(z_t - \varepsilon - u_t)^+ = -\nabla u_t$. Then one obtains

$$\begin{aligned}\int_{\Omega_\varepsilon}|\nabla u_t|^p &\leqslant \mu\int_{\Omega_\varepsilon}|u_t|^{p-2}u_t(z_t - \varepsilon - u_t)^+ + (C - t)\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+ \\ &= -\mu\int_{\Omega_\varepsilon}|u_t|^{p-1}(z_t - \varepsilon - u_t)^+ + (C - t)\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+ \\ &\leqslant -\mu\int_{\Omega_\varepsilon}|z_t - \varepsilon|^{p-1}(z_t - \varepsilon - u_t)^+ + (C - t)\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+ \\ &= (-\mu|z_t - \varepsilon|^{p-1} + C - t)\int_{\Omega_\varepsilon}(z_t - \varepsilon - u_t)^+.\end{aligned}$$

This is a contradiction, since $-\mu|z_t - \varepsilon|^{p-1} + C - t < 0$ for ε small enough. □

3. Proof of Theorem 1.1

The proof is divided into three parts.

For the first step we show that there exists t_0 such that for $t \leqslant t_0$ there is a solution and no solution exists for $t > t_0$; we then show that there is a minimal solution for $t \leqslant t_0$. Finally, we show that there exists t_1 such that for $t < t_1$ there are two distinct solutions.

Step 1. First, we shall prove that there exists a t' such that (P_t) has a solution for all $t \leqslant t'$. Actually, we take $t' = \inf\{-f(x, 0) \colon x \in \Omega\}$, so 0 is a supersolution for (P_t) for all $t \leqslant t'$. In fact, $-\Delta_p 0 = 0 \geqslant f(x, 0) + t'$ if and only if $t' \leqslant -f(x, 0)$ for all x. Since, by Lemma 2.3, for each t there is a negative constant subsolution z_t, the claim follows by the method of sub–supersolution [**11**].

Now, we shall see that if (P_t) has a solution for some t, then it also has a solution for all $s \leqslant t$. Indeed, let u be a solution of (P_t) corresponding to t. Clearly, u is a supersolution of (P_t) corresponding to s for every $s < t$, since

$$-\Delta_p u = f(x, u) + t \geqslant f(x, u) + s.$$

Again, the assertion follows by the method of sub–supersolution.

Hence, the set $S = \{t\colon (P_t) \text{ has a solution}\}$ is non-empty and bounded from above by Lemma 2.2. In particular, we have $(-\infty, t_0) \subset S$, where t_0 is the supremum of S. Let $\{t_n\}$ be such that $t_n \nearrow t_0$. By virtue of the *a priori* estimates for solutions u_t corresponding to t of Lemma 2.1, there exists a subsequence t_{n_k} such that $u_{t_{n_k}} \to u_{t_0}$ in $C^1(\bar{\Omega})$, and so u_{t_0} is a solution of (P_t). Thus, $S = (-\infty, t_0]$, and Step 1 is proved. This proves (i).

Step 2. Let $t \leqslant t_0$. Then (P_t) has a minimal solution if $t \leqslant t_0$. Indeed, by Lemmas 2.3 and 2.5, (P_t) has a subsolution and every solution satisfies

$$u \geqslant -\frac{1}{\mu}(|t| + C).$$

But

$$z = -\frac{1}{\mu}(|t| + C)$$

is a subsolution of (P_t) and $z \leqslant u$. Thus, (P_t) has a minimal solution in the set of functions satisfying $w \geqslant z$ in $\bar{\Omega}$, since all solutions satisfy the property $w \geqslant z$ in $\bar{\Omega}$. The proof of (ii) is complete.

Step 3. Define

$$t_1 = \sup_{\mathbb{R}} \inf_{\Omega} \{-f(x, s)\}.$$

By (1.3) and (1.4), $f(x, s)$ is bounded from below. Hence, t_1 is well defined. If $t < t_1$, then there is a σ such that $t < \inf\{-f(x, \sigma)\colon x \in \Omega\}$. Thus, $w_t = \sigma$ is a supersolution for (P_t) corresponding to t. Since for all t (P_t) has a constant subsolution z_t, $z_t < w_t$, we can solve (P_t) for t. Thus, there is a solution u_t for (P_t), corresponding to t, obtained by the sub–supersolution method, so $z_t \leqslant u_t \leqslant w_t$. In particular, we have $t_1 \leqslant t_0$. We shall apply degree theory to find a second solution for $t < t_1$.

First assume that $f(x, s)$ is locally Hölder continuous in s, uniformly in $x \in \Omega$. Then we can choose a constant w such that $w > \sigma$ and $t < \inf\{-f(x, w)\colon x \in \Omega\}$. Thus, $w > \sigma$ is also a supersolution. Moreover, for a fixed constant z with $z < z_t$ one concludes that z is also a subsolution. Define the open set

$$\Lambda = \{v \in C(\bar{\Omega})\colon z < v < w \text{ in } \bar{\Omega}\}.$$

Thus, $u_t \in \Lambda$. There exists a $\lambda > 0$ such that $f(x, u) + \lambda |u|^{p-2} u$ is non-decreasing in u on $[z, w]$ for $x \in \bar{\Omega}$ (see (1.5)). Clearly, u is a solution of (P_t) if and only if u is a fixed point of the compact operator $K_t\colon C(\bar{\Omega}) \to C(\bar{\Omega})$ defined by $K_t v = u$, where u is a solution of

$$\left.\begin{aligned} -\Delta_p u + \lambda |u|^{p-2} u &= f(x, v) + \lambda |v|^{p-2} v + t && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.1}$$

Since $u_t \in \Lambda$, we can suppose that $\deg(I - K_t, \Lambda, 0)$ is well defined, i.e. $0 \notin (I - K_t)(\partial\Lambda)$. Otherwise, the proof is complete. We claim that

$$\deg(I - K_t, \Lambda, 0) = 1.$$

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In fact, take $\varphi \in \Lambda$ and define $T_\eta\colon C(\bar{\Omega}) \to C(\bar{\Omega})$ by $T_\eta v = \eta K_t v + (1-\eta)\varphi$ for $\eta \in [0,1]$. Hence, for $v \in \Lambda$ we have that

$$-\Delta_p z + \lambda |z|^{p-2} z \leqslant -\Delta_p(K_t v) + \lambda |K_t v|^{p-2} K_t v \leqslant -\Delta_p w + \lambda |w|^{p-2} w.$$

It follows by the Weak Comparison Principle that $K_t v \in \bar{\Lambda}$, i.e. $z \leqslant K_t v \leqslant w$. Since Λ is convex and $\varphi \in \Lambda$, we obtain $T_\eta\colon \Lambda \to \Lambda$ for all $\eta \in (0,1]$. Thus, $0 \notin (I - T_\eta)(\partial\Lambda)$ for all $\eta \in [0,1]$. In this way, $\deg(I - T_\eta, \Lambda, 0)$ is well defined and independent of η. Hence,

$$\deg(I - K_t, \Lambda, 0) = \deg(I - T_\eta, \Lambda, 0) = \deg(I - T_0, \Lambda, 0).$$

The map $T_0 u = \varphi$ for every $u \in \Lambda$ and $\varphi \in \Lambda$, then

$$\deg(I - T_0, \Lambda, 0) = 1.$$

The claim is proved.

We also claim that for large enough $M > 0$ we have

$$\deg(I - K_t, B_M, 0) = 0,$$

where $B_M = \{u \in C(\bar{\Omega})\colon \|u\|_{C(\bar{\Omega})} < M\}$.

By Lemma 2.1, one concludes that all fixed points u_s of K_s, that is, $K_s u_s = u_s$, satisfy $\|u\|_{C(\bar{\Omega})} < M$ independently of $s \in [t, t_0 + 1]$. Note that t is kept fixed and $t < t_1 \leqslant t_0$. We can assume that $\Lambda \subset B_M$. Thus,

$$\deg(I - K_t, B_M, 0) = \deg(I - K_{t_0+1}, B_M, 0) = 0,$$

since K_{t_0+1} does not have fixed points. The claim is proved.

In conclusion,

$$\deg(I - K, B_M - \Lambda, 0) = -1.$$

Therefore, (P_t) has a solution which is not in Λ.

Finally, assume that f is continuous on $\bar{\Omega} \times \mathbb{R}$. For $t < t_0$ we have that u_{t_0}, the minimal solution corresponding to t_0, is a supersolution for (P_t) corresponding to t. Moreover, $z_t \leqslant u_{t_0}$ and we have a solution u_t such that $z_t \leqslant u_t \leqslant u_{t_0}$. In order to apply the ideas from the previous case (where f is only Hölder continuous), we need a subsolution z and a supersolution w such that $z < u_t < w$ in $\bar{\Omega}$. We can choose z as a fixed number less than z_t. We claim that $w = u_{t_0} + \theta$ is a supersolution if $\theta > 0$ is small enough. Actually, we have

$$-\Delta_p w = f(x, u_{t_0}) + t_0 = f(x, u_{t_0} + \theta) + t + (f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0).$$

It is a consequence of the continuity of f that $f(x, u_{t_0}) - f(x, u_{t_0} + \theta) + t - t_0 \geqslant 0$ for $\theta > 0$ small enough. Thus, $-\Delta_p w \geqslant f(x, u_{t_0} + \theta) + t$, and the claim is proved.

The rest of the proof follows as in the previous case.

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