Can. J. Math., Vol. XXVI, No. 6, 1974, pp. 1430–1441

# RANGES OF PRODUCTS OF OPERATORS

SANDY GRABINER

**1. Introduction.** Suppose that $T$ and $A$ are bounded linear operators. In this paper we examine the relation between the ranges of $A$ and $TA$, under various additional hypotheses on $T$ and $A$. We also consider the dual problem of the relation between the null-spaces of $T$ and $AT$; and we consider some cases where $T$ or $A$ are only closed operators. Our major results about ranges of bounded operators are summarized in the following theorem.

THEOREM 1. *Suppose that $T$ is a bounded operator on a Banach space $E$ and that $A$ is a non-zero bounded operator from some Banach space to $E$.*

(A) *If $T$ is quasi-nilpotent, then the range of $TA$ does not equal or properly contain the range of $A$.*

(B) *If $T$ is a Riesz operator and $A$ has infinite rank, and if $T$ and $A$ commute, then the range of $TA$ has infinite co-dimension in the range of $A$.*

(C) *If 0 is a boundary point of the spectrum of $T$, but is not a pole of $T$, and if $A$ is a non-negative power of $T$, then the range of $TA$ has infinite co-dimension in the range of $A$.*

Theorem 1(C) and its dual result for null-spaces can be viewed as a new characterization of those points in the spectrum of a bounded operator $T$ which are poles of $T$ (see Theorem (5.4) below and the abstract [**11**]). Many papers have considered characterizations of poles (see for instance [**2**; **20**; **24**; **25**], and the survey [**19**]). Our characterization of poles is similar to that of D. C. Lay [**20**, pp. 202–206], which contains a slightly weaker version of our Theorem 1(C). The main difference between Lay's results and our Theorem 1(C) is one of method. Lay's proofs are based on the rather deep perturbation theory of Kato [**16**], while our proofs use only the more elementary results on spectral boundaries due to Rickart and Yood [**22**, pp. 22 and 278–279; **29**, pp. 493–494].

There are elementary proofs known for parts of Theorem 1(C) in the special case that $T$ is quasi-nilpotent. In this case, Johnson [**14**, p. 913] and Kato [**17**, Theorem 5.30, p. 240] have shown that $T(E)$ has infinite co-dimension in $E$; and we have shown [**9**, Theorem 2, p. 150] that if $T$ is not nilpotent, then the spaces $\{T^n(E)\}$ are all distinct. Results intermediate in generality between the above special cases and the results in this paper were announced in the abstract [**8**].

Received May 23, 1973 and in revised form, August 13, 1973. This research was partially supported by NSF Grant GP-37817.



https://doi.org/10.4153/CJM-1974-137-7 Published online by Cambridge University Press

**2. Quasi-nilpotent operators.** In this section we prove Theorem 1(A) (restated as Theorem (2.1), below) together with some generalizations. We also apply Theorem (2.1) to the study of continuous or discontinuous linear operators which commute with a pair of bounded operators (related to results in [**13**; **14**; **15**]). Our proof of Theorem (2.1) is adapted from our earlier proof of a special case [**9**, Theorem 2, p. 150].

THEOREM (2.1). *If $T$ is a quasi-nilpotent operator on a Banach space $E$ and if $A$ is a non-zero bounded operator from some Banach space $F$ to $E$, then $TA(F)$ does not equal or properly contain $A(F)$.*

*Proof.* We assume that no $T^nA = 0$, since the theorem is obvious otherwise. If $\phi$ belongs to $E$ and if $f = \sum \lambda_n z^n$ is a complex formal power series in the indeterminate $z$, we denote the series $\sum \lambda_n T^n \phi$ by $\bar{f}(T)\phi$.

Let $c_n = \|T^nA\|$. We will construct a formal power series $f = \sum \lambda_n z^n$ for which:

$$(2.2) \quad \sum |\lambda_n c_{n+1}| \qquad \text{converges}$$

$$(2.3) \quad \lim |\lambda_n c_n| = \infty .$$

The construction of such a series $f$ will complete the proof. For Formula (2.2) implies that $\bar{f}(T)\phi$ converges for all $\phi$ in $TA(F)$; while Formula (2.3) implies that $\bar{f}(T)\phi$ diverges for some $\phi$ in $A(F)$, because of the Banach-Steinhaus uniform boundedness theorem.

Since $T$ is quasi-nilpotent, $\lim (c_n)^{1/n} = 0$; so that $\liminf c_n/c_{n-1} = 0$. Choose a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ for which $c_{n_k}/c_{n_k-1} < 1/k^3$. Then

$$(2.4) \quad f = \sum_k \frac{z^{nk-1}}{(k^2 c_{n_k})}$$

satisfies (2.2) and (2.3); so the proof is complete.

Theorem (2.1) yields the following easy corollary for certain non-quasi-nilpotent operators $T$.

COROLLARY (2.5). *Suppose that $T$ is a bounded linear operator on a Banach space $E$ and that $A$ is a bounded operator from a Banach space $F$ to $E$. If $0$ is an isolated point of the spectrum of $T$, if $P$ is the projection associated with $0$, and if $PA \neq 0$, then $TA(F)$ does not properly contain or equal $A(F)$.*

*Proof.* Let $M$ be the range of $P$, and let $T_r$ be the restriction of $T$ to $M$. Then $T_r$ is quasi-nilpotent [**4**, p. 574]. If $TA(F)$ contains $A(F)$, then $PTA(F) = TPA(F) = T_rPA(F)$ contains $PA(F)$. But this is impossible, by Theorem (2.1); so the proof is complete.

The above corollary, with essentially the same proof, remains true if $T$ is just a closed operator. We merely use the theory of spectral sets for closed operators [**23**, pp. 298–300] in place of the theory for bounded operators [**4**, pp. 574–575].

Suppose that $T$ and $V$ are bounded linear operators on Banach spaces $E$ and $F$, respectively. Following Johnson and Sinclair [**15**, p. 533], we say that a linear operator $A$ (not necessarily bounded) with domain equal to $F$ and range in $E$ *commutes with the pair* $\{V, T\}$, if $AV = TA$. They call a subspace $M$ of $E$, *$T$-divisible* [**15**, p. 534; **14**, p. 914] if $p(T)M = M$ for all polynomials $p$. Since $E$ becomes a module over the polynomials under the operation $p \cdot \phi = p(T)\phi$, $M$ is a $T$-divisible subspace if and only if it is a divisible polynomial module, and hence if and only if it is an injective polynomial module [**12**, pp. 5–6]. Notice that if $T$ is one-to-one and quasi-nilpotent, then $\bigcap_n T^n(E)$ is the maximum $T$-divisible subspace. Johnson and Sinclair show that a bounded operator $T$ with a non-zero $T$-divisible subspace has a discontinuous commuting operator [**15**, Lemma 2.4, p. 534; **14**, p. 914]. Using Theorem (2.1), above, we extend this result to a pair $\{V, T\}$ of quasi-nilpotent operators.

THEOREM (2.6). *Suppose that $T$ and $V$ are one-to-one quasi-nilpotent operators on Banach spaces $E$ and $F$, respectively. If there exist non-zero $T$-divisible subspaces and non-zero $V$-divisible subspaces, then there is a discontinuous linear operator $A$ commuting with the pair $\{V, T\}$.*

*Proof.* Let $x$ be a fixed non-zero element of some $T$-divisible subspace of $E$. Then the map; $r \to r(T)x$ is one-to-one from the space of quotients of polynomials into $E$. Let $Rx$ be the space of all $r(T)x$. Similarly choose some non-zero $y$ in a $V$-divisible subspace of $F$, and form $Ry$. Define the map $A$ from $Ry$ to $Rx$, by $A(r(V)y) = r(T)x$. $Ry$ and $Rx$ are polynomial submodules of $F$ and $E$, respectively, and $A$ is a module homomorphism. $Ry$ is a divisible module, and hence is an injective module, so $A$ can be extended to a module homomorphism from $F$ onto $Rx$ [**12**, pp. 5–6]. $A$ commutes with $\{V, T\}$, and $TA(F) = T(Rx) = Rx = A(F)$. So $A$ must be discontinuous, because of Theorem (2.1); and the proof is complete.

Most of [**13**; **14**], and [**15**] are devoted to proving that certain pairs $\{V, T\}$ have no discontinuous commuting operators. Combining their results with Theorem (2.1) allows us to prove that certain pairs have no non-zero commuting linear operators at all.

THEOREM (2.7). *Suppose that $T$ is an injective quasi-nilpotent operator on a Banach space $E$, that $V$ is a surjective bounded linear operator on a Banach space $F$, and that $A$ is a linear operator from $F$ to $E$. If $F$ has no non-zero $V$-divisible subspaces and if $AV = TA$, then $A = 0$.*

*Proof.* $A$ is continuous by [**15**, Theorem 3.3, p. 537]. Also $TA(F) = AV(F) = A(F)$. Hence $A = 0$, by Theorem (2.1).

**3. Operator ranges.** A linear subspace $M$ of a Banach space $E$ will be called an *operator range* if it is the range of a bounded linear operator from some Banach space to $E$. $M$ is an operator range if and only if it can be given a norm under which it becomes a Banach space continuously embedded in $E$ [**5**, pp. 255–257]. By the closed graph theorem, all such norms on $M$ are equivalent. If $M$ is the range of the bounded operator $A$, a convenient formula for the norm on $M$ is:

$$(3.1)\quad ||y||' = \inf \{||x|| : Ax = y\}.$$

The operator ranges in $E$ can also be characterized as the domains of closed operators from $E$ to some Banach space; or as the ranges of closed operators [**5**, pp. 255–257]. If $M$ is the domain of the closed operator $A$, a convenient norm for $M$ is the graph norm:

$$(3.2)\quad ||x||' = ||x|| + ||Ax||$$

In Sections 4 and 5, we will obtain Theorem 1(B) and (C), and the dual results about null-spaces, by studying the restrictions to suitable operator ranges of certain bounded linear operators between Banach spaces. The restrictions are continuous, by the closed graph theorem. In this section, we prove the dual results, for null-space, to Theorem (2.1) and Corollary (2.5), and we obtain some general results about operator ranges.

THEOREM (3.3). *Suppose that $T$ is a quasi-nilpotent operator on a Banach space $E$ and that $A$ is a non-zero bounded linear operator from $E$ into a Banach space $F$. If $AT(E)$ is closed, then $N(AT)$ is not a subspace (proper or improper) of $N(A)$.*

*Proof.* Since $A(E)$ is itself a Banach space, there is no loss of generality in assuming that $A(E) = F$. In this case, both $A$ and $AT$ have closed range. Hence, by [**7**, Theorem IV.1.2, p. 95], the orthogonal complement of $N(A)$ is $A^*(F^*)$, and the orthogonal complement of $N(AT)$ is $T^*A^*(F^*)$. But the range of $T^*A^*$ cannot contain the range of $A^*$, by Theorem (2.1); so the proof is complete.

Since the following result follows from Corollary (2.5) in the same way that Theorem (3.3) followed from Theorem (2.1), we omit the proof.

COROLLARY (3.4). *Suppose that $T$ is a bounded operator on a Banach space $E$, that $0$ is an isolated point of the spectrum of $T$, and that $P$ is the projection associated with $0$. If $A$ is a bounded operator from $E$ to a Banach space $F$, if $AP \neq 0$, and if $AT(E)$ is closed in $F$, then $N(AT)$ is not a subspace of $N(A)$.*

As we pointed out above, not only are all operator ranges in a Banach space $E$ Banach spaces continuously embedded in $E$ (called *$BT$-subspaces* in [**1**, Definition 5.I, p. 64]); but also all $BT$-subspaces are operator ranges. Many properties known for $BT$-subspaces, or more generally for Banach and

Fréchet spaces embedded in a Hausdorff vector space, have apparently not been explicitly recognized as properties of arbitrary operator ranges. Very useful discussions of these properties of $BT$-subspaces can be found in [**27**, pp. 202–207; **28**, pp. 62–69 and 78–79; **1**, pp. 64–65].

We mention a few of the more important properties of operator ranges which follow from viewing them as $BT$-subspaces. The collection of operator ranges form a lattice [**27**, Theorem 3, p. 205; **1**, p. 64]. No operator range can have countably infinite co-dimension in another operator range [**27**, Corollary 6, p. 205; **1**, p. 65; **18**]. Also, two algebraically complementary operator ranges must both be closed subspaces [**28**, p. 78]. The above facts have long been known for Hilbert space operator ranges [**21**; **3**, pp. 42–44; **5**, pp. 260–262].

The following corollary rephrases Theorem (2.1) as a condition which guarantees that a space is not an operator range.

COROLLARY (3.5). *If $T$ is a quasi-nilpotent operator on a Banach space $E$, and if $M$ is a non-zero linear subspace of $E$ for which $T(M) \supseteq M$, then $M$ is not an operator range in $E$.*

The above corollary is useful for showing that certain subspaces of a Banach space are not operator ranges, even when they can be continuously embedded as Fréchet spaces. For instance, using a quasi-nilpotent weighted shift, one can show that the space of rapidly decreasing sequences is not an operator range in any $l_p$ space. Similarly, using the Volterra integral operator, one can show that $C^\infty[0, 1]$ is not an operator range in any $C^n[0, 1]$ or $L_p[0, 1]$.

We conclude this section with an analogue of a result of Foias [**6**, Corollary 4, p. 888] about ranges of bounded operators on a Hilbert space.

THEOREM (3.6). *Suppose that $S$ is a uniformly closed algebra of bounded operators on a Banach space $E$ and that $S$ has no proper non-zero invariant operator ranges. Then $S$ is algebraically $n$-transitive for every $n$.*

*Proof.* The set of $x$ in $E$ for which $Sx = 0$ is a closed invariant subspace and must therefore be 0. Hence for each $x$ in $E$, $Sx$ is a non-zero invariant operator range. Thus $S$ is *algebraically* transitive, and hence is *algebraically* $n$-transitive for every $n$ [**22**, Theorem (2.4.6), p. 62].

## 4. Riesz operators and spectra of restrictions.

In this section and the next section we prove Theorems 1(B) and 1(C) (restated as Theorem (4.2) and Theorem (5.2), respectively) together with various generalizations. We break the proofs of these results into two parts. First, in Theorem (4.1), we show how information about the images of operator ranges under a bounded operator $T$ can be obtained from information about the spectra of the restrictions of $T$ to the ranges of various operators related to $T$. Then we concentrate on obtaining as much information as possible about the spectral properties of restrictions of various classes of operators $T$ to ranges of operators $A$ which are closely related to $T$. The example summarized in Theorem (4.6), below, indicates the need to place some restrictions on $A$.

Theorem (4.1)(A) is essentially the result announced in the abstract [**8**].

THEOREM (4.1). *Suppose that $T$ is a bounded linear operator on a Banach space $E$ and that* 0 *is in the boundary of the spectrum of $T$.*

(A) *If $T$ is not algebraically nilpotent, and if, for all non-negative integers $n$,* 0 *is in the spectrum of the restriction of $T$ to $T^n(E)$, then $T^{n+1}(E)$ always has infinite co-dimension in $T^n(E)$.*

(B) *If* 0 *belongs to the spectrum of $T$ restricted to $A(E)$ whenever $A$ is a bounded operator, of infinite rank, which commutes with $T$, then $TA(E)$ has infinite co-dimension in $A(E)$ for all such $A$.*

*Proof.* (B) will follow from applying (A) to the restriction of $T$ to $A(E)$, so we need only prove (A).

It is clear that the sequence $\{\dim(T^n(E)/T^{n+1}(E))\}$ is non-increasing. Also, 0 belongs to the *boundary* of the spectrum of the restriction of $T$ to $T^n(E)$, for each non-negative integer $n$. Hence the restriction of $T$ to $T^n(E)$ is a two-sided topological divisor of zero in the space of bounded operators on $T^n(E)$ [**22**, Theorem (1.5.4)(iii), p. 22]. The fact that each of these restrictions is a right topological divisor of zero implies that none of these restrictions can be surjective (see [**22**, p. 279] or [**29**, Theorem 3.6, p. 494]); in other words, $T^{n+1}(E)$ always has co-dimension at least one in $T^n(E)$.

Hence, if Theorem (4.1)(A) were false, we could find a positive integer $m$, a Banach space $E$, and a bounded operator $T$ on $E$, satisfying:

(i) 0 belongs to the boundary of the spectrum of $T$;

(ii) for all non-negative integers $n$, $T^{n+1}(E)$ has co-dimension exactly equal to $m$ in $T^n(E)$.

By (ii), each $T^n(E)$ has finite co-dimension in $E$, and is therefore a closed subspace of $E$ [**7**, Corollary IV.1.13, p. 101]. Hence $\bigcap T^n(E)$ is closed; so, by passing to a quotient space, if necessary, we can assume that $T$ also satisfies:

(iii) $\bigcap_1^\infty T^n(E) = \{0\}$.

We complete the proof by showing that no bounded operator, $T$, can satisfy (i), (ii), and (iii). As we pointed out above, if $T$ satisfies (i) it is a left topological divisor of zero in an algebra of bounded operators on $E$. Hence $T$ cannot be bounded below (see [**22**, pp. 278–279] or ([**9**, Theorem 3.5, p. 493]). On the other hand, $T$ has closed range, because of (ii), so if we show that $T$ is injective we will obtain a contradiction.

Suppose therefore that $x$ is a non-zero element of $E$. By (iii), there is a fixed non-negative integer $k$ for which $x$ belongs to $T^k(E)$ but not to $T^{k+1}(E)$. Hence, by (ii), $Tx$ belongs to $T^{k+1}(E)$ but not to $T^{k+2}(E)$. Therefore $Tx$ cannot be zero. This completes the proof of Theorem (4.1).

With slight modifications, the above proof remains true in the case that $T$ is a closed operator. Instead of using the fact that: if 0 is in the boundary of the spectrum of a bounded operator $T$, then $T$ is neither bounded below nor surjective [**22**, pp. 22, 278–279; **29**, pp. 493–494], one uses the analogous result for closed operators [**23**, pp. 258 and 233].

THEOREM (4.2). *Suppose $T$ and $A$ are commuting bounded operators on a Banach space $E$. If $T$ is quasi-nilpotent, compact, or a Riesz operator, respectively, then its restriction to $A(E)$ has the same property. If $T$ has any one of the above properties and if $A$ has infinite rank, then $TA(E)$ has infinite co-dimension in $A(E)$.*

*Proof.* It is easy to see that the spectrum of the restriction of $T$ to $A(E)$ is a subset of the spectrum of $T$. Thus if $T$ is quasi-nilpotent, so is its restriction. If $T$ is compact, an easy direct calculation, using formula (3.1), shows that the restriction of $T$ to $A(E)$ is compact.

Suppose $T$ is a Riesz operator, and let $B$ be the algebra of bounded operators on $E$ which commute with $A$. Each member of $B$ has the same spectrum in $B$ as in the algebra of all bounded operators on $E$. If $V$ belongs to $B$, let $V_r$ be the restriction of $V$ to $A(E)$. The map $V \to V_r$ is a continuous algebra homomorphism from $B$ into the algebra of bounded operators on $A(E)$. Suppose that $\lambda$ is a non-zero element of the spectrum of $T$ and that $P$ is the projection associated with $\lambda$ as an element of the spectrum of $T$. Then, either $\lambda$ is in the resolvent of $T_r$, or $\lambda$ is a pole of $T_r$ and $P_r$ is the projection associated with $\lambda$ as an element of the spectrum of $T_r$. Thus the spectrum of $T_r$ is a subset of the spectrum of $T$ and all non-zero elements of the spectrum of $T_r$ are poles of finite rank. Hence $T_r$ is a Riesz operator.

Finally, 0 belongs to the boundary of the spectrum of each quasi-nilpotent, compact, or Riesz operator; so the final statement in Theorem (4.2) now follows from Theorem (4.1)(B).

We now prove the dual of Theorem (4.2) for null-spaces.

COROLLARY (4.3). *Suppose $T$ and $A$ are commuting bounded linear operators on a Banach space $E$. If $T$ is a Riesz operator, if $A$ has infinite rank, and if $TA(E)$ is a closed subspace of $E$, then $N(A)$ has infinite co-dimension in $N(AT)$.*

*Proof.* Let $T_r$ be the restriction of $T$ to $A(E)$. $T_r$ has closed range, so, by [**7**, Theorem IV.1.2, p. 95], the range of $T_r^*$ is the orthogonal complement of $N(T_r)$. But $T_r$, and hence $T_r^*$, are Riesz operators, by Theorem (4.2). Thus the dimension of $N(T_r)$, which equals the co-dimension of the range of $T_r^*$, is, by Theorem (4.2) again, infinite. But $N(T_r) = N(T) \cap A(E)$ is easily shown to be linearly isomorphic to $N(AT)/N(T)$. This completes the proof.

We now extend Theorem (4.2) to the case that $A$ is just a closed operator.

THEOREM (4.4). *Suppose that $T$ is a bounded operator on a Banach space $E$ and that $A$ is a closed operator which commutes with $T$ (i.e., $TA \subseteq AT$) and*

https://doi.org/10.4153/CJM-1974-137-7 Published online by Cambridge University Press

*has infinite-dimensional range. If $T$ is a quasi-nilpotent, compact, or Riesz operator, then the restrictions of $T$ to the domain and range of $A$ are the same kind of operator (i.e., quasi-nilpotent, compact, or Riesz) as $T$.*

*Proof.* We first consider the domain of $A$. The case where $T$ is compact is an easy calculation using formula (3.2).

For $T$ a quasi-nilpotent or a Riesz operator, let $B$ be the algebra of bounded operators on $E$ which commute with $A$. For $V$ in $B$, let $V_r$ be the restriction of $V$ to the domain of $A$. The map $V \to V_r$ is a continuous algebra homomorphism from $B$ into the algebra of bounded operators on the domain of $A$. Hence, if $T$ is quasi-nilpotent, so is $T_r$. If $T$ is a Riesz operator, its spectrum as an operator does not disconnect the plane, so that its operator spectrum is the same as its spectrum in $B$ [**22**, Theorem (1.6.13), p. 34]. The proof that $T_r$ is a Riesz operator is now the same as the analogous proof in Theorem (4.2).

If we consider $A$ as a bounded operator from its domain to $E$, then $TA = AT_r$. The statements about the restrictions of $T$ to the range of $A$ will therefore follow from the following lemma.

LEMMA (4.5). *Suppose that $S$ and $T$ are bounded operators on Banach spaces $F$ and $E$, respectively, and that $A$ is a bounded operator from $F$ to $E$ for which $TA = AS$. If $S$ is quasi-nilpotent, compact, or Riesz, then the restriction of $T$ to the range of $A$ has the same property as $S$.*

*Proof.* Again, the case where $S$ is compact is an easy calculation, using formula (3.1). For $S$ quasi-nilpotent or Riesz, let $B$ be the algebra of bounded operators on $F$ which map the null-space of $A$ into itself. As in Theorem (4.4), if $S$ is a quasi-nilpotent or Riesz operator, then $S$ has the same spectrum in $B$ that it has as an operator on $E$. For each $V$ in $B$, define the operator $V'$ on $A(E)$ by $V'(Ax) = AVx$. It is easy to see that the map $V \to V'$ is a continuous algebra homomorphism from $B$ into the algebra of bounded operators on $A(E)$. Moreover $S'$ is just the restriction of $T$ to $A(E)$. The rest of the proof follows exactly as in Theorem (4.2).

Suppose $T$ is a Riesz operator on $E$ and $M$ is a closed subspace of $E$ for which $T(M) \subseteq M$. If we let $A$ be the injection of $M$ into $E$, we see, as a special case of Theorem (4.4), that $T$ restricted to $M$ is Riesz. Similarly, if we let $A$ be the natural projection of $E$ onto $E/M$; we see, as a special case of Lemma (4.5), that the operator induced by $T$ on $E/M$ is Riesz. These two special cases are due to T. T. West [**26**, p. 746].

We conclude this section with an example that shows that our results about the restrictions of $T$ to ranges of operators commuting with $T$ do not generalize to ranges of arbitrary operators. The example also shows that Theorem (2.1) is in a sense optimal. Let $E = F = l_1$ and let $\{e_n\}$ be the natural basis of $l_1$. Define the weighted shift $T$ on $E$ by $Te_n = e_{n+1}/(n+1)$. $T$ is compact and quasi-nilpotent. Define $A$ from $F$ to $E$ by $Ae_n = e_n/n!$. Then $T(Ae_n) = Ae_{n+1}$, so $T$ restricted to $A(F)$ is just the simple unilateral shift on $l_1$. Hence 0 is an

https://doi.org/10.4153/CJM-1974-137-7 Published online by Cambridge University Press

interior point of the spectrum of the restriction of $T$ to $A(F)$, and $TA(F)$ has co-dimension exactly equal to 1 in $A(F)$. Essentially the same construction can be carried out on any infinite-dimensional Banach space $E$; by using an arbitrary biorthogonal sequence $\{b_n, \beta_n\}$ in place of $\{e_n\}$ in $E$ and adjusting the weights on $T$ and $A$ so that $T$ is still compact and quasi-nilpotent and so that $TAe_n = Ae_{n+1}$. Thus we have the following theorem.

THEOREM (4.6). *If $E$ is an infinite-dimensional Banach space, then there is a compact quasi-nilpotent operator $T$ on $E$ and an operator range $M$ in $E$ such that $T(M)$ has co-dimension* 1 *in $M$, and such that* 0 *is an interior point of the spectrum of the restriction of $T$ to $M$.*

**5. Boundary points of the spectrum.** In this section we apply the methods of this paper, in particular Theorem (4.1), to the study of the boundary of the spectrum of an arbitrary bounded operator $T$. For simplicity, we normalize to the case where 0 is the boundary point of the spectrum. The major result in this section is Theorem (5.2), which is a restatement of Theorem 1(C). This result, and its dual for null-spaces, is reformulated in Theorem (5.4) as a characterization of the poles of a bounded operator.

We first consider the case where 0 is an isolated point of the spectrum of $T$.

THEOREM (5.1). *Suppose that $T$ and $A$ are commuting bounded linear operators on a Banach space $E$, that* 0 *is an isolated point in the spectrum of $T$, and that $P$ is the projection associated with* 0. *If $AP$ has infinite rank, then*

(A) *$TA(E)$ has infinite co-dimension in $A(E)$.*

*(B) If $TA(E)$ is closed, then $N(A)$ has infinite co-dimension in $N(AT)$.*

*Proof.* Let $T_r$ and $A_r$ be the restrictions of $T$ and $A$, respectively, to $P(E)$. Then $T_r$ is quasi-nilpotent, $A_r$ has infinite rank, and $T_r$ and $A_r$ commute. Hence, by Theorem (4.2), the range of $T_rA_r$ has infinite co-dimension in the range of $A_r$. Since $P(E)$ reduces $A$ and $TA$, this proves (A). If $TA$ has closed range, then so does $T_rA_r$, so (B) follows from Corollary (4.3).

The above proof remains valid if $T$ is only a closed operator with 0 in its spectrum, since the projection $P$ can be defined in this case, and the restriction of $T$ to $P(E)$ is still a bounded quasi-nilpotent operator [**23**, Theorem 5.7-B, p. 299].

THEOREM (5.2). *Suppose that $T$ is a bounded linear operator on a Banach space $E$ and that* 0 *is a boundary point of the spectrum of $T$. If* 0 *is not a pole of $T$, then, for all non-negative integers $n$, $T^{n+1}(E)$ has infinite co-dimension in $T^n(E)$.*

*Proof.* Because of Theorem (5.1)(A), we can assume that 0 is a limit of boundary points of the spectrum of $T$. By Theorem (4.1)(A), it will be enough to show that 0 is in the spectra of the restrictions of $T$ to each $T^n(E)$. This will follow, by an induction on $n$, from the following lemma.

LEMMA (5.3). *If $T$ is a bounded operator on a Banach space $E$ and if $\lambda$ is a non-zero boundary point of the spectrum of $T$, then $\lambda$ is in the spectrum of the restriction of $T$ to $T(E)$.*

*Proof.* Let $B$ be the algebra of bounded operators on $E$ which commute with $T$. $T$ has the same spectrum in $B$ as it does as an operator on $E$, so $\lambda - T$ is a topological divisor of zero in $B$ [**22**, Theorem (1.5.9), p. 22]. Hence there is a sequence of bounded operators $\{V_n\}$, each of norm 1 and each commuting with $T$, for which $\lim||(\lambda - T)V_n|| = 0$. Since $\lim||TV_n|| = |\lambda| \neq 0$, we can also assume that there is a positive number $d$ for which all $||TV_n|| > d$.

Define the norm $||\cdot||'$ on $T(E)$ by $||y||' = \inf\{||x|| : Tx = y\}$. For each $V$ in $B$, let $||V||'$ be the operator norm of the restriction of $V$ to $T(E)$; and notice that $||V||' \leqq ||T|| \, ||V||$. Hence $\lim||(\lambda - T)V_n||' = 0$, and we must only show that $\{||V_n||'\}$ is bounded away from 0.

Fix $n$, and choose $||x|| < 1$ for which $||V_nTx|| > d$. Then $||Tx||' < 1$, so

$$||V_n||' > ||V_nTx||' = ||TV_nx||' = \inf\{||x|| : Ty = TV_nx\}.$$

But, if $||Ty|| = ||TV_nx|| > d$, then $||V_n||' > ||y|| > d/||T||$. Since this lower bound on $||V_n||'$ is independent of $n$, this completes the proofs of Lemma (5.3) and Theorem (5.2).

It is possible to modify the above proof of Lemma (5.3) in the case that $T$ is only a closed operator, and thus prove Theorem (5.2) for closed operators. One can combine [**23**, Theorem 5.1-D, p. 258] with an easy adaptation of [**22**, Theorem (1.4.7), p. 12] to show that bounded operators $V_n$ commuting with $T$ can be found for which $\lim||V_n(\lambda - T)|| = 0$, and for which the restriction of each $V_n$ to the closure of the domain of $T$ has norm 1. Thus there is a $d > 0$, independent of $n$, such that for each sufficiently large $n$ we can find an $x$ in the domain of $T$ with $||x|| < 1$ and $||V_nx|| > d$. The rest of the proof of Lemma (5.3) requires only minor modifications.

In Theorem (5.4), below, we reformulate Theorem (5.2), and its dual for null-spaces, as a characterization of those points in the spectrum of a bounded operator $T$ which are poles of $T$.

THEOREM (5.4). *Suppose that $T$ is a bounded linear operator on a Banach space $E$ and that 0 is a boundary point of the spectrum of $T$. Then the following are equivalent:*

(A) *0 is an isolated point of the spectrum of $T$ and is a pole of $T$.*

(B) *There is a non-negative integer $n$ for which $T^{n+1}(E)$ has finite co-dimension in $T^n(E)$.*

(C) *There is a non-negative integer $n$ for which $N(T^n)$ has finite co-dimension $N(T^{n+1})$, and there is an $m > n$ for which $T^m(E)$ is closed.*

*Proof.* The equivalence of (A) and (B) is just Theorem (5.2). Also, it is clear that (A) implies (C).

https://doi.org/10.4153/CJM-1974-137-7 Published online by Cambridge University Press

Suppose $T$ satisfies (C). Since the dimension of $N(T^m)/N(T^{m-1})$ is no greater than the dimension of $N(T^{n+1})/N(T^n)$, we can assume that $m = n + 1$. Hence $T^{n+1}(E)$ is closed, and by restricting to $T^n(E)$ if necessary, we can assume $T^n(E)$ is also closed. Then $T^*$ satisfies (B), by [**7**, Theorem IV.1.2, p. 95], so 0 is a pole of $T^*$, and hence also of $T$. This completes the proof.

It is not difficult to use the characterization of poles in Theorem (5.4) to characterize Riesz operators, meromorphic operators, operators of finite type, and similar classes of operators, just as was done by Lay [**20**, pp. 209–213], from his characterizations of poles. Also, Theorem (5.2) and Corollary (5.4), when taken together, can be used to obtain sufficient conditions for a point to be in the interior of the spectrum (compare [**22**, pp. 278–279] and [**20**, pp. 206–209]). We leave the details of these applications to the reader.

## References


**1.** N. Aronszajn and E. Gagliardo, *Interpolation spaces and interpolation methods*, Ann. Mat. Pura Appl. *68* (1965), 51–117.

**2.** S. R. Caradus, *Operators of Riesz type*, Pacific J. Math. *18* (1966), 61–71.

**3.** J. Dixmier, *Étude sur les varietes et les opérateurs de Julia*, Bull. Soc. Math. France *77* (1949), 11–101.

**4.** N. Dunford and J. T. Schwartz, *Linear operators*, Part I (Interscience, New York, 1958).

**5.** P. A. Fillmore and J. P. Williams, *On operator ranges*, Advances in Math. *7* (1971), 254–281.

**6.** C. Foias, *Invariant para-closed subspaces*, Indiana Univ. Math. J. *21* (1972), 887–906.

**7.** S. Goldberg, *Unbounded linear operators* (McGraw-Hill, New York, 1966).

**8.** S. Grabiner, *Ranges of operator iterates, preliminary report*, Notices Amer. Math. Soc. *18* (1971), 416.

**9.** ——— *Ranges of quasi-nilpotent operators*, Illinois J. Math. *15* (1971), 150–152.

**10.** ——— *A formal power series operational calculus for quasi-nilpotent operators*, Duke Math. J. *38* (1971), 641–658.

**11.** ——— *Ranges of operator iterates*, Notices Amer. Math. Soc. *20* (1973), A-154.

**12.** P. Hilton, *Lectures on homological algebra* (American Mathematical Society, Providence, 1971).

**13.** B. E. Johnson, *Continuity of linear operators commuting with continuous linear operators*, Trans. Amer. Math. Soc. *128* (1967), 88–102.

**14.** ——— *Continuity of operators commuting with quasi-nilpotent operators*, Indiana Univ. Math. J. *20* (1971), 913–915.

**15.** B. E. Johnson and A. M. Sinclair, *Continuity of linear operators commuting with continuous linear operators, II*, Trans. Amer. Math. Soc. *146* (1969), 533–540.

**16.** T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. *6* (1958), 261–322.

**17.** ——— *Perturbation theory of linear operators* (Springer-Verlag, New York, 1966).

**18.** G. Köthe, *Die Bildräume abgeschlossener Operatoren*, J. Reine Angew. Math. *232* (1968), 110–111.

**19.** S. N. Krachkovskii and A. S. Dikanskii, *Fredholm operators and their generalizations*, Progr. Math. *10* (1971), 37–72.

**20.** D. C. Lay, *Spectral analysis using ascent, descent, nullity and defect*, Math. Ann. *184* (1970), 197–214.

**21.** G. W. Mackey, *On the domains of closed linear transformations in Hilbert space*, Bull. Amer. Math. Soc. *52* (1946), 1009.

**22.** C. E. Rickart, *Banach algebras* (Van Nostrand, Princeton, 1960).

**23.** A. E. Taylor, *Introduction to functional analysis* (Wiley, New York, 1958).
**24.** ——— *Theorems on ascent, descent, nullity and defect of linear operators*, Math. Ann. *163* (1966), 18–49.
**25.** T. T. West, *Riesz operators in Banach spaces*, Proc. London Math. Soc. *16* (1966), 131–140.
**26.** ——— *The decomposition of Riesz operators*, Proc. London Math. Soc. *16* (1966), 737–752.
**27.** A. Wilansky, *Functional analysis* (Blaisdell, New York, 1964).
**28.** ——— *Topics in functional analysis* (Springer-Verlag, Berlin-Heidelberg-New York, 1967).
**29.** B. Yood, *Transformations between Banach spaces in the uniform topology*, Ann. of Math. *50* (1949), 486–503.

*Claremont Graduate School,*
*Claremont, California*