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Relative annihilators in semilattices

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An a-distributive (respectively a-implicative) semilattice S is a lower semilattice (with greatest lower bound denoted by juxtaposition) in which the annihilator $\langle x, a \rangle$, that is $\{y \in S : xy \leq a\}$, is an ideal (respectively a principal ideal) for the fixed element a and any x of S. These semilattices appear as natural links between general and distributive semilattices on the one hand, and between pseudo-complemented and implicative semilattices on the other hand. Prime and dense elements, as well as maximal and prime filters, are essential. Mandelker's result, a lattice L is distributive if and only if $\langle x, y \rangle$ is an ideal for any $x, y \in L$, is extended to semilattices.

0. Introduction

Implicative lattices have been extensively studied by algebraists, topologists and logicians, unfortunately under various names among which we quote relatively pseudo-complemented lattices and brouwerian lattices. An implicative lattice L is a set on which is defined, besides the two lattice-operations, a third binary operation, $*$, whose meaning is the following: for any $a, b \in L$, $ax \leq b \leftrightarrow x \leq a * b$. Since the latter involves only one of the lattice-operations, it is quite reasonable to consider implicative semilattices as in [2].

A semilattice S with least element 0 is said to be pseudo-

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complemented if, for any $a \in S$, there exists an element a^* satisfying $ax = 0$ if and only if $x \leq a^*$. Clearly any implicative semilattice bounded below is pseudo-complemented. Nevertheless, most pseudo-complemented semilattices are not implicative and furthermore an implicative semilattice need not have a zero. Whereas every implicative semilattice is distributive, this property is no longer true in a pseudo-complemented semilattice. Thus most statements are not transferable from implicative semilattices to pseudo-complemented ones and *vice versa*. Apart from the elementary arithmetical properties, the link between both theories is weak. In order to fill up the gap, we introduce a restricted form of implicativity, the implicativity with respect to a fixed element a, not necessarily zero: in an a-implicative semilattice, $x * a$ is defined for all x. These semilattices have nice properties, which generalize those of pseudo-complemented semilattices and which are valid trivially in any implicative semilattice.

Extensive use is made of the notion of annihilator of an element a relative to an element b, in symbols $\langle a, b \rangle$, introduced by Mandelker in [1]. Let us recall that the annihilator $\langle a, b \rangle$ of a relative to b is the family of all elements x such that $ax \leq b$. Using Zorn's Lemma, we can enlarge to semilattices the nice characterization of distributive lattices given by Mandelker: a lattice L (here: a semilattice) is distributive if and only if, for any $a, b \in L$, $\langle a, b \rangle$ is an ideal.

In Section 1 we introduce all notions and notations we need.

Section 2 is devoted to the study of the a-dense elements of a general semilattice S, that is, elements x for which $\langle x, a \rangle$ is contained in $(a]$, the principal ideal generated by a. It is the natural extension of the notion of dense element in a semilattice bounded below. The properties of these elements are closely linked to those of the filters maximal with respect to the property of not containing a, in short: a-maximal filters.

In Section 3 we impose upon the semilattice S a slight condition: S has a fixed element a such that $\langle x, a \rangle$ is an ideal for any $x \in S$. It is a partial distributivity that we name a-distributivity. The special case $a = 0$ has already been studied by us in [5] and [6]. A characterization of a-distributivity in terms of a-maximal filters is provided and

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the subset $\mathcal{D}(S)$ of all elements $a \in S$ for which S is a-distributive is investigated. Prime elements are here of utmost importance. As already mentioned, $\mathcal{D}(S) = S$ if and only if S is distributive.

The last section deals with a-implicative semilattices, that is, semilattices S in which, for any $x \in S$, $\langle x, a \rangle$ is a principal ideal denoted by $(x*a]$. It is proved that a semilattice S is a-distributive if and only if its ideal lattice is $(a]$-implicative. The rules of computation in an a-implicative semilattice are given and special attention is paid to the multiplicative closure operator φ : $x \to (x*a) * a$, especially to the location of its invariant elements. Finally the congruence Θ_a defined by $(x, y) \in \Theta_a$ if and only if $x * a = y * a$, is briefly studied.

The reader will have no difficulty in deducing properties of pseudo-complemented and implicative semilattices from our statements. In order to keep clarity and to avoid any lengthening, we do not mention these particular results: they are obvious corollaries.

1. Preliminaries

Throughout this paper the word semilattice will mean *lower semilattice*, that is a commutative idempotent semigroup or, equivalently, a partially ordered set in which any two elements a and b have a greatest lower bound, denoted by $a \cdot b$ or simply ab, the partial ordering being defined by $a \leq b \Leftrightarrow ab = a$. The greatest lower bound of the family $\{a_i : i \in I\}$ will be denoted by $\prod\{a_i : i \in I\}$ or $\prod_{i \in I} a_i$ or even $\prod_i a_i$ when no confusion is allowed. The least and greatest elements of a semilattice S, when they exist, will be denoted by 0 and 1 respectively. When S is a lattice, the second binary operation will be denoted by $+$.

The symbols $\cap$, $\cup$, $-$, $\subseteq$ and $\subset$ will be used in their usual set-theoretical meaning: intersection, union, difference, inclusion and strict inclusion.

A *filter* of a semilattice S is a non-empty subset F of S such that $xy \in F$ if and only if $x \in F$ and $y \in F$. The *principal filter*

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generated by an element a of S, that is, the set $\{x : x \in S, x \geq a\}$, will be denoted by $[a)$. A filter F of S is *prime* if, whenever two filters F_1 and F_2 are such that $\emptyset \neq F_1 \cap F_2 \subseteq F$, then F_1 or F_2 belongs to F.

An *ideal* I of a semilattice S is a non-empty subset of S satisfying

(i) $y \leq x$ and $x \in I$ imply $y \in I$;

(ii) for any $x, y \in I$, there exists $z \in I$ such that $z \geq x$ and $z \geq y$.

We shall use the symbol $(a]$ to denote the *principal ideal* generated by a. In an up-directed semilattice S, a filter F is prime if and only if $S - F$ is an ideal of S.

When ordered by inclusion, the set $F(S)$ of all filters of an up-directed semilattice S is a lattice in which, for any $F_1, F_2 \in F(S)$, $F_1 \cdot F_2 = F_1 \cap F_2$ and $F_1 + F_2$ is the filter generated by $F_1 \cup F_2$. Similar considerations are valid also for the set $T(S)$ of all ideals of S.

A semilattice S is called *modular* if, for any x, y, z in S such that $y \geq z \geq xy$, there exists $x_1 \geq x$ satisfying $z = x_1 y$. The following condition is stronger: a semilattice S is *distributive* if $z \geq xy$ $(x, y, z \in S)$ implies the existence in S of x_1, y_1 such that $x_1 \geq x$, $y_1 \geq y$ and $x_1 y_1 = z$. We proved in [6] that Stone's Separation Theorem can be extended as follows: an up-directed semilattice is distributive if and only if for any filter F and any ideal I such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint from I.

An element a of a semilattice S is *irreducible* if $a = bc$ implies $a = b$ or $a = c$. An element a of S is *prime* if $(a]$ is a prime ideal, that is, $a \geq bc$ implies $a \geq b$ or $a \geq c$. Obviously any prime element is irreducible and the two notions coincide if S is distributive.

Finally we shall make use of the notion of multiplicative closure operator. A mapping $\varphi : S \to S$ of a partially ordered set S into itself is called a *closure operator* if for any $x, y \in S$,

(i) $x \leq x\varphi$;

(ii) $x \leq y$ implies $x\varphi \leq y\varphi$;

(iii) $(x\varphi)\varphi = x\varphi$.

If the partially ordered set S is endowed with a binary operation, called multiplication and denoted by juxtaposition, the closure operator φ is termed *multiplicative* if, for any $x, y \in S$,

(iv) $(xy)\varphi = (x\varphi)(y\varphi)$.

2. a-dense elements and a-maximal filters

The following definition makes sense for any element a of a semilattice S.

DEFINITION 2.1. An element x of the semilattice S is *a-dense* (a, fixed element of S) if $\langle x, a \rangle \subseteq (a]$, that is, for every $y \in S$, $xy \leq a$ implies $y \leq a$. When $a = 0$, the expression 0-dense element is shortened to dense element.

The set of a-dense elements of S will be denoted by D_a; when S is bounded below, D_0 (shortly D) is the dense set of S in the usual sense. The following facts are easily verified:

(1) D_a is either the empty set or a filter;

(2) the following three conditions are equivalent: $a = 1$, $D_a \ni a$ and $D_a = S$;

(3) if x is a-dense and $xy = a$, then $y = a$.

DEFINITION 2.2. A filter F of the semilattice S will be called *a-maximal* (a, fixed element of S distinct from 1, if the latter exists) if F is maximal with respect to the property of not containing a. Since $a \neq 1$, there exists $b \nleq a$ and $[b) \not\ni a$. An easy application of Zorn's Lemma enables us to infer the existence of an a-maximal filter. Here also 0-maximal will be shortened to maximal.

THEOREM 2.3. *A filter F of the semilattice S is a-maximal if*

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and only if F *does not contain* a *and, for every* $x \notin F$, $\langle x, a \rangle$ *meets* F .

Proof 1°. IF: If F does not contain a but is not a-maximal, there exists a filter G such that $G \supset F$ and $G \not\ni a$. Let us choose $x \in G - F$. By hypothesis there is $y \in F$ such that $xy \leq a$. But $xy \in G$ and G would contain a .

2°. ONLY IF: Let F be an a-maximal filter. If $x \notin F$, $F + [x) \supset F$ and $F + [x) \ni a$. Hence there exists $y \in F$ such that $xy \leq a$ and $\langle x, a \rangle \cap F \neq \emptyset$.

The following statement shows that the concepts of a-dense element and a-maximal filter are closely linked.

THEOREM 2.4. *In a semilattice* S , *the subset* D_a *is the intersection of the* a*-maximal filters.*

Proof. Let $\{F_i : i \in I\}$ be the family of a-maximal filters of S . We first show that, for every $i \in I$, $F_i \supseteq D_a$. Therefore let us suppose there is $x \in D_a$ but $x \notin F_i$ for some $i \in I$. By Theorem 2.3 one can find $y \in F_i$ such that $xy \leq a$. Since $x \in D_a$, this implies $y \leq a$ and $F_i \ni a$, a contradiction.

Now let b be an element of $\cap\{F_i : i \in I\}$ which does not belong to D_a . Then there exists c such that $bc \leq a$ and $c \nleq a$. The family of filters containing c but not a is not empty; by Zorn's Lemma there exists a filter F maximal with respect to this property. Clearly $F = F_i$ for some $i \in I$, $b \in F$, $bc \in F$, which is impossible since $F \not\ni a \geq bc$.

THEOREM 2.5. *In a semilattice* S , *the following are equivalent:*

(1) S *has exactly one* a*-maximal filter;*

(2) D_a *is an* a*-maximal filter.*

Moreover they imply

(3) D_a *is a prime filter.*

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Proof. By the preceding theorem the equivalence of (1) and (2) is clear since the family of a-maximal filters is totally unordered. Let us now consider a semilattice S in which D_a is an a-maximal filter. Then, for every $x \notin D_a$, there exists $y \in D_a$ satisfying $xy \leq a$. Since $y \in D_a$, this implies $x \leq a$. As conversely $x \leq a$ implies $x \notin D_a$, D_a is the set-complement of the principal ideal $(a]$ and D_a is a prime filter.

REMARK 2.6. Generally conditions (2) and (3) are not equivalent. An example is provided by a generalized boolean lattice (that is, a distributive relatively complemented lattice bounded below but not above) to which an element 1 is added. In such a lattice, $D = [1)$ is a prime filter but is far from being a maximal filter. Nevertheless we will show in Section 4 that the previous conditions become equivalent in an a-implicative semilattice.

When the element a is prime, the subset D_a takes an advantageous form.

THEOREM 2.7. *In a semilattice S, an element $a \neq 1$ is prime if and only if $D_a = S - (a]$.*

Proof 1°. IF: Let us suppose $D_a = S - (a]$ for some $a \neq 1$. We then have $D_a \neq \emptyset$. If $yz \leq a$ and $y \not\leq a$ (that is, $y \in D_a$), then $z \leq a$ by definition of D_a and a is prime.

2°. ONLY IF: Let $a \neq 1$ be a prime element of S. Clearly if $x \leq a$, then $x \notin D_a$; on the contrary, if $x \not\leq a$, then $xy \leq a$ implies $y \leq a$ and $x \in D_a$.

3. a-distributive semilattices

DEFINITION 3.1. A semilattice S is *a-distributive* (a, fixed element of S) if, for any $x \in S$, $\langle x, a \rangle$ is an ideal.

This definition applies to lattices as well and can be reformulated as follows: a lattice L is a-distributive if, for any $x \in L$, $xy \leq a$

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and $xz \leq a$ imply $x(y+z) \leq a$. Moreover $x(y+z) = a$ follows from $xy = a$ and $xz = a$. Clearly a lattice is a-distributive if and only if it is so as a semilattice.

The following theorem provides us with a characterization of a-distributivity in terms of filters. Its proof is similar to that of [6], Theorem 2. Nevertheless, for the sake of completeness, we give it in every detail.

THEOREM 3.2. *A semilattice S is a-distributive if and only if it is up-directed and any a-maximal filter is prime.*

Proof 1°. IF: We have to prove that, for any $x \in S$, $I = \langle x, a \rangle$ is an ideal. For any $y, z \in I$, the set of all upper bounds of $\{y, z\}$ is a filter F since S is up-directed. The set $G = \{t \in S : t \geq xf, f \in F\}$ is also a filter. If $G \not\ni a$, then G is contained in an a-maximal filter M . By hypothesis M is prime and, since $[y) \cap [z) = F \subseteq M$, either $[y) \subseteq M$ or $[z) \subseteq M$. But $y \in M$ (or $z \in M$) would imply (owing to $x \in M$) $xy \in M$ and $a \in M$, a contradiction. Hence $G \ni a$, that is, there exists $f \in F$ such that $xf \leq a$, and consequently y, z have in I the upper bound f .

2°. ONLY IF: First, any a-distributive semilattice is up-directed since $\langle a, a \rangle = S$ is an ideal, whence any two elements of S have an upper bound. Then let F be an a-maximal filter of S which is not prime. There exist two filters G and H such that $G \cap H \subseteq F$, $G \nsubseteq F$ and $H \nsubseteq F$. So we can find $x \in G-F$ and $y \in H-F$. Since F is a-maximal, by Theorem 2.3 there exist z and t in F belonging respectively to $\langle x, a \rangle$ and $\langle y, a \rangle$. Since x and y both belong to $\langle zt, a \rangle$, which is an ideal, an element $u \geq x, y$ can be found in the same annihilator, whence $ztu \leq a$. Since z, t and u all belong to F , F contains a , a new contradiction.

Mandelker has shown ([1], Theorem 1) that a lattice L is distributive if and only if it is a-distributive for all $a \in L$. Using Zorn's Lemma we can extend this property to a semilattice.

THEOREM 3.3. *A semilattice S is distributive if and only if it is a-distributive for all $a \in S$.*

Proof 1°. IF: We have to prove that, for any a, b, c such that

$c \geq ab$, there exist elements a_1, b_1 satisfying $a_1 \geq a$, $b_1 \geq b$ and $a_1 b_1 = c$. Let us denote by F_1 (respectively F_2) the non-empty set of upper bounds of $\{a, c\}$ (respectively $\{b, c\}$). F_1 and F_2 are filters, as well as $F_1 + F_2 = \{z \in S : z \geq xy,\ x \in F_1,\ y \in F_2\}$. If $F_1 + F_2$ does not contain c , there is a c-maximal filter G containing $F_1 + F_2$. Since S is c-distributive, G is prime by virtue of Theorem 3.2. As $\emptyset \neq F_1 = [a) \cap [c) \subseteq G$, the element a has to belong to G . Similarly we can show that G contains b , whence $ab \in G$ and $c \in G$, which is impossible. In conclusion, $F_1 + F_2$ does contain c , that is, there exist elements d, e such that $d \geq a, c$ and $e \geq b, c$ together with $de \leq c$. As we also have $de \geq c$ (owing to $d \geq c$ and $e \geq c$), we get $c = de$ and d, e are respectively the desired elements a_1, b_1 .

2°. ONLY IF: Let $y, z \in \langle x, a \rangle$. We have to show that there is an element t satisfying $t \geq y, z$ and $t \in \langle x, a \rangle$. Since a distributive semilattice is up-directed, $F = [y) \cap [z)$ is a filter, as well as $G = F + [x)$. If $G \not\ni a$, there is a prime filter P containing G but not a . Since P is prime and contains F , either $y \in P$ or $z \in P$. We have either $xy \in P$ or $xz \in P$, which implies $a \in P$, a contradiction. Thus we may conclude that $G \ni a$ and the existence of the desired element t is established.

It is most natural to investigate the subset of elements a of a semilattice S for which S is a-distributive. This subset, denoted by $\mathcal{D}(S)$, enjoys the following properties.

THEOREM 3.4. *In a semilattice* S , $\mathcal{D}(S)$ *is a subsemilattice of* S *including all prime elements of* S . *If* S *is complete, then so is* $\mathcal{D}(S)$.

Proof. If S is complete and, for every $i \in I$, $a_i \in \mathcal{D}(S)$, then the subset $J = \left\langle x, \prod_i a_i \right\rangle$ is an ideal for every $x \in S$ and $\prod_i a_i \in \mathcal{D}(S)$. Indeed, $y \in \left\langle x, \prod_i a_i \right\rangle$ implies $y \in \langle x, a_i \rangle$ for every $i \in I$. Hence if $y, z \in J$, we can find elements t_i satisfying

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$t_i \geq y, z$ and $t_i \in \langle x, a_i \rangle$ for every $i \in I$. Consequently, $\prod_i t_i \in J$ with $\prod_i t_i \geq y, z$. If S is not complete, the previous argument holds for I finite.

Finally, any prime element p of S belongs to $\mathcal{D}(S)$ since $\langle x, p \rangle$ equals S or $(p]$ according as $x \leq p$ or $x \nleq p$, hence $\langle x, p \rangle$ is an ideal in both cases.

COROLLARY 3.5. *Any complete semilattice in which each element has a representation as a meet of prime elements, is distributive.*

Proof. The statement is a direct consequence of Theorems 3.4 and 3.3.

REMARK 3.6. We take this opportunity to point out that, even in a complete lattice L, $\mathcal{D}(L)$ does not include all irreducible elements of L. For instance, in the five-element non-modular lattice $\{0, a, b, c, 1\}$ with $0 < a < b < 1$, $0 < c < 1$, a and c incomparable, b and c incomparable, the element a does not belong to $\mathcal{D}(L)$ although it is irreducible. Actually, $\langle b, a \rangle = \{0, a, c\}$ is not an ideal.

THEOREM 3.7. *In a semilattice* S, *if* $a \in \mathcal{D}(S)$, *then* $D_a \cap [a)$ *is the dense set of* $[a)$.

Proof. If $a = 1$, the statement is trivial. Hence we may assume $a \neq 1$ in the sequel.

Let us first prove that if x is a dense element of $[a)$, then $x \in D_a$. Let us consider an element y such that $xy \leq a$. From $y \in \langle x, a \rangle$ and $a \in \langle x, a \rangle$ we deduce the existence of an element $z \in \langle x, a \rangle$ with $z \geq a, y$. Since $x > a$, $z \geq a$ and $xz \leq a$, we necessarily have $xz = a$ and, owing to the density of x in $[a)$, $z = a$. We conclude that $y \leq a$ and $x \in D_a$.

Now let us denote by y an element of $D_a \cap [a)$ distinct from 1. For any z of $[a)$ such that $yz = a$, we have $z = a$, proving that y is dense in $[a)$.

4. a-implicative semilattices

DEFINITION 4.1. A semilattice S is *a-implicative* (a, fixed

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element of S) if, for any $x \in S$, $\langle x, a \rangle$ is a principal ideal, denoted by $(x * a]$. In other words, for any $x \in S$, we have $xy \leq a$ if and only if $y \leq x * a$.

From the point of view of universal algebra, an a-implicative algebra can be regarded as an algebra $S = \langle S; \cdot, {}^{*a}, a \rangle$ with a binary operation $\cdot$, a unary operation $x \to x^{*a} = x * a$, and the distinguished element a .

An a-implicative semilattice is of course a-distributive and always has a greatest element 1 since $x \in \langle a, a \rangle$ is satisfied for any $x \in S$, hence $a * a = 1$; conversely, a semilattice S which is bounded above is always 1-implicative and $x * 1 = 1$ for any $x \in S$.

In case S has a least element 0 and S is 0-implicative, it is customary to say that S is *pseudo-complemented* and to denote $x * 0$ by x^* .

A semilattice is *implicative* if it is a-implicative for every $a \in S$.

If S is a-implicative, then $[a)$ is pseudo-complemented; if we denote by x^+ the pseudo-complement of x $(x \geq a)$ in $[a)$, we have $x^+ = x * a$. The converse of this statement is not true as is easily verified.

In a semilattice satisfying the ascending chain condition, all ideals are principal. As a consequence, if such a lattice is a-distributive, it is also a-implicative. The following theorem throws light on the link between a-distributivity and a-implicativity.

THEOREM 4.2. *A semilattice* S *is* a*-distributive if and only if* $T(S)$ *is* a *-implicative.*

Proof 1°. IF: We have to show that, for any $x \in S$, the subset $I = \langle x, a \rangle$ is an ideal. Let y_1 and y_2 be two elements of I : $(x] \cap (y_1] \subseteq (a]$ and $(x] \cap (y_2] \subseteq (a]$. Since $T(S)$ is $(a]$-implicative, $(y_1] \subseteq (x] * (a] = J$ and, similarly, $(y_2] \subseteq J$. As J is an ideal, it contains an element b such that $b \geq y_1, y_2$. From $(b] \subseteq J$ follows $(x] \cap (b] \subseteq (a]$, hence $xb \leq a$.

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2°. ONLY IF: Let I be an element of $T(S)$. For every $x \in I$, $J_x = \langle x, a \rangle$ is an ideal containing a, whence $J = \cap\{J_x : x \in I\}$ is also an ideal. We are going to prove that $I * (a] = J$. First, $I \cap J \subseteq (a]$. If not, there exist $i \in I$ and $j \in J$ such that $ij \nleq a$, which is impossible given the way J has been constructed. Secondly, if $K \nsubseteq J$, there is $k \in K$ such that $xk \nleq a$ for some $x \in I$, hence $I \cap K \nsubseteq (a]$.

The following rules of computation have been proved for implicative semilattices ([2], [3], [4]). Clearly they are still valid in an a-implicative semilattice.

THEOREM 4.3. *If x, y, z are any elements of an a-implicative semilattice, then the following hold:*

(1) $x * a \geq a$;

(2) $x \leq a$, $x * a = 1$ *and* $x * a \geq x$ *are equivalent;*

(3) $1 * a = a$;

(4) $x * a = x$ *if and only if* $x = a = 1$;

(5) $(x*a) * a \geq x, a$;

(6) *if* $x \leq y$, *then* $x * a \geq y * a$ *and* $(x*a) * a \leq (y*a) * a$;

(7) $((x*a)*a) * a = x * a$ *and* $(((x*a)*a)*a) * a = (x*a) * a$;

(8) $(x*a)((x*a)*a) = a$.

As far as we know, the following important formula is novel.

THEOREM 4.4. *If x and y are any elements of an a-implicative semilattice, then*

(9) $((xy)*a) * a = ((x*a)*a)((y*a)*a)$.

Proof. Let us denote by t the right-side member of (9). From $x \leq (x*a) * a$ and $y \leq (y*a) * a$, we deduce $xy \leq t$, hence $(xy) * a \geq t * a$.

We now intend to prove that $(xy) * a \leq t * a$, that is, any z satisfying $xyz \leq a$ also satisfies $tz \leq a$. From $tzxy \leq a$ follows $tzx \leq y * a$. Since tzx $(\leq t) \leq (y*a) * a$, we have $tzx \leq (y*a)((y*a)*a) = a$, hence $tz \leq x * a$. Since $tz \leq (x*a) * a$, we finally obtain $tz \leq a$ and $(xy) * a = t * a$. Then

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$((xy)*a) * a = (t*a) * a \geq t$. The opposite inclusion being trivial, the proof ends.

COROLLARY 4.5. *In an a-implicative semilattice, the mapping* $\varphi : x \to (x*a) * a$ *is a multiplicative closure.*

Proof. Formulae (5), (6) and (7) make φ a closure operator. The multiplicative character of this closure is given by (9).

COROLLARY 4.6. *In an a-implicative algebra* $S = \langle S; \cdot, ^{*a}, a \rangle$ *, the mapping* $\varphi : x \to (x*a) * a$ *is an idempotent endomorphism which preserves all congruences.*

Proof. Theorem 4.4 shows that $(xy)\varphi = (x\varphi)(y\varphi)$. Moreover, $(x^{*a})\varphi = ((x*a)*a) * a = (x\varphi)^{*a}$, $a\varphi = a$ and $x\varphi^2 = x$. Finally, for any $\Theta \in C(S)$, the congruence lattice of S , $(x, y) \in \Theta$ implies $(x\varphi, y\varphi) \in \Theta$.

DEFINITION 4.7. An element x of an a-implicative semilattice S will be called *a-closed* if $(x*a) * a = x$. The subset of a-closed elements of S is denoted in the sequel by C_a . Clearly an element x of an a-implicative semilattice S is a-closed if and only if there is $y \in S$ satisfying $y * a = x$.

DEFINITION 4.8. A subset A of a partially ordered set S will be called *semiconvex* if $a, b \in A$ $(a \leq b)$ implies $x, y \in A$ for any x, y satisfying $xy = a$ and $x + y = b$. This definition generalizes the one we introduced in [4] for lattices. Obviously a convex subset of a partially ordered set is semiconvex.

LEMMA 4.9. *In a modular semilattice* S *, the subset* $F_\varphi = \{x \in S : x\varphi = x\}$ *of all* φ*-invariant elements of a multiplicative closure* φ *is semiconvex.*

Proof. Let us assume $a, b \in F_\varphi$, $xy = a$ and $x + y = b$. We have to show that $x\varphi = x$ and $y\varphi = y$. First $a = a\varphi = (x\varphi)(y\varphi)$ since φ is multiplicative. Moreover, since $x \leq x\varphi \leq b\varphi = b$ and $y \leq y\varphi \leq b\varphi = b$, clearly $x\varphi + y\varphi = b$. As S is modular and $y\varphi \geq y \geq x(y\varphi) = a$, there is $z \in S$ satisfying $z \geq x$ and $y = z(y\varphi)$. Consequently we have $z \geq y$, $z \geq x+y = b \geq y\varphi$ and finally $y = y\varphi$. The inversion of the parts played by x and y gives $x = x\varphi$. //

THEOREM 4.10. *In an a-implicative algebra* $S = \langle S; \cdot, {}^{*a}, a \rangle$, C_a *is a subalgebra whose least and greatest elements are* a *and* 1 *respectively. If, moreover,* S *is modular, then* C_a *is semiconvex.*

Proof. Since the endomorphism $\varphi : x \to (x*a) * a$ maps S onto C_a , the latter is a subalgebra of S . Since $(a*a) * a = a$ and $(x*a) * a \geq a$ for any $x \in S$, the least element of C_a is a . Clearly its greatest element is 1 . The last part is a direct consequence of Corollary 4.5 and Lemma 4.9.

Our next project is to describe a-dense elements and a-maximal filters in an a-implicative semilattice. It is an easy exercise to verify that an element x of an a-implicative semilattice is a-dense if and only if one of the following equivalent conditions is satisfied:

(i) $\langle x, a \rangle = (a]$;

(ii) $x * a = a$;

(iii) $(x*a) * a = 1$.

THEOREM 4.11. *In an a-implicative semilattice* S *, an element* x *is a-dense if there exists* $y \in S$ *such that* x *is an upper bound of* $\{y, y*a\}$.

Proof. From $x \geq y$ and $x \geq y * a$, we deduce respectively $x * a \leq y * a$ and $x * a \leq (y*a) * a$, hence $x * a \leq (y*a)\big((y*a)*a\big) = a$. Since $x * a \geq a$ always holds, we get $x * a = a$ and x is a-dense.

REMARK 4.12. The previous condition can be restated as follows: for every $x \in S$, $[x) \cap [x*a) \subseteq D_a$.

THEOREM 4.13. *In an a-implicative semilattice* S *, a filter* F *is a-maximal if and only if, for any* $x \in S$ *,* F *contains exactly one of the elements* x *and* $x * a$.

Proof 1°. IF: Since F contains $1 = a * a$, it does not contain a . Let $y \notin F$; by hypothesis $y * a \in F$. By the very definition of $y * a$ we have $y(y*a) \leq a$. Theorem 2.3 enables us to infer that F is a-maximal.

2°. ONLY IF: Let F be a-maximal. F cannot contain both x and $x * a$ since otherwise it would contain $a \geq x(x*a)$. Now let us suppose $x \notin F$ and $x * a \notin F$. The filter $F + [x)$ contains a , hence there exists $y \in F$ such that $yx \leq a$. But this means $y \leq x * a$ and $F \ni x * a$, a contradiction. In conclusion, F contains exactly one of the elements x and $x * a$.

THEOREM 4.14. *In an a-implicative semilattice* S *, a prime filter* P *which contains* D_a *but not* a *is a-maximal.*

Proof. If $x \in P$, then $x * a \notin P$ since $x(x*a) \leq a$. If $x \notin P$, $x * a \in P$ because $\emptyset \neq [x) \cap [x*a) \subseteq D_a \subseteq P$ jointly with $[x) \nsubseteq P$ implies $[x*a) \subseteq P$. To conclude it suffices to set forth Theorem 4.13.

COROLLARY 4.15. *In an a-implicative semilattice,* D_a *is a prime filter if and only if it is an a-maximal filter.*

Proof. Owing to Theorem 2.5, we only need show that D_a is a-maximal if it is prime. But this is straightforward by Theorem 4.14 since $D_a \not\ni a$.

Let us denote by $A(S)$ the subset of elements a of S for which S is a-implicative.

THEOREM 4.16. *For any semilattice* S *,* $A(S)$ *enjoys the following properties:*

(1) $A(S)$ *is empty if and only if* S *has no greatest element;*

(2) $A(S) \subseteq \mathcal{D}(S)$ *;*

(3) $A(S)$ *is a sublattice of* S *; if* S *is complete, so is* $A(S)$ *;*

(4) $A(S)$ *includes all prime elements of* S *if* S *has a greatest element.*

Proof. (1) and (2) are obvious. (3) is a direct consequence of the formula: $x * \left(\prod_{i \in I} a_i\right) = \prod_{i \in I} (x*a_i)$, true for any index set I if S is complete. In fact, if, for every $i \in I$, $a_i \in A(S)$, then we have

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$$xy \le \prod_{i\in I} a_i \Leftrightarrow xy \le a_i \text{ for every } i \in I$$
$$\Leftrightarrow y \le x * a_i \text{ for every } i \in I$$
$$\Leftrightarrow y \le \prod_{i\in I} (x*a_i) .$$

Finally, for any prime element p of S we have $x * p = 1$ if $x \le p$ and $x * p = p$ if $x \nleq p$.

To end with we focus our attention on a congruence of the a-implicative algebra.

THEOREM 4.17. *In an a-implicative algebra $S = \langle S; \cdot, {}^{*a}, a\rangle$, the relation Θ_a defined by*

$$(x, y) \in \Theta_a \text{ if and only if } x * a = y * a$$

is a fully invariant congruence of S. Moreover, if S is a lattice, Θ_a is also a fully invariant congruence of $S' = \langle S; +, \cdot, {}^{*a}, a\rangle$.

Proof. Theorem 4.4 shows that, for every $z \in S$, $(x, y) \in \Theta_a$ implies $(xz, yz) \in \Theta_a$. Clearly $(x, y) \in \Theta_a$ implies $(x*a, y*a) \in \Theta_a$.

If S is a lattice, $(x, y) \in \Theta_a$ implies $(x+z, y+z) \in \Theta_a$ since $(x+z) * a = (x*a)(z*a)$.

That Θ_a is fully invariant is also obvious: for every $\xi \in E(S)$, the endomorphism monoid of S, $(x, y) \in \Theta_a$ implies $x\xi * a = (x*a)\xi = (y*a)\xi = y\xi * a$, hence $(x\xi, y\xi) \in \Theta_a$.

Now we enumerate some elementary properties of Θ_a. The easy proof is left to the reader.

(1) For every $x \in S$, $(x, (x*a)*a) \in \Theta_a$.

(2) For every $x \in S$, $[x]\Theta_a$ contains exactly one element of C_a ; this element is $(x*a) * a$ and constitutes the maximum element of the class.

(3) $[a]\Theta_a = (a]$ is the kernel of Θ_a and the least element of S/Θ_a . $[1]\Theta_a = D_a$ is the antikernel of Θ_a and the greatest element of S/Θ_a .

(4) S/Θ_a is a boolean algebra. In fact, if we adopt the notation of 4.1, $\Theta_a|[a) = \sim$, where $\sim$ is the congruence on $[a)$ defined by $(x, y) \in \sim$ if and only if $x^+ = y^+$.

(5) If there exists $d \in D_a$ such that $xd = yd$, then $(x, y) \in \Theta_a$.

References

[1] Mark Mandelker, "Relative annihilators in lattices", *Duke Math. J.* 37 (1970), 377-386.

[2] William C. Nemitz, "Implicative semi-lattices", *Trans. Amer. Math. Soc.* 117 (1965), 128-142.

[3] Helena Rasiowa and Roman Sikorski, *The mathematics of metamathematics* (Monografie Matematyczne, Tom 41. Państwowe Wydawnictwo Naukowe, Warszawa, 1963).

[4] Jules Varlet, "Contribution à l'étude des treillis pseudo-complémentés et des treillis de Stone", *Mém. Soc. Roy. Sci. Liège Coll. in* - 8° (5) 8 No. 4 (1963), 1-71.

[5] J.C. Varlet, "A generalization of the notion of pseudo-complementedness", *Bull. Soc. Roy. Sci. Liège* 37 (1968), 149-158.

[6] J.C. Varlet, "Distributive semilattices and boolean lattices", *Bull. Soc. Roy. Sci. Liège* 41 (1972), 5-10.

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