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BOURGAIN ALGEBRAS OF DOUGLAS ALGEBRAS

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Introduction. Let A be a Banach algebra and let B be a linear subspace of A. Recall that A has the Dunford Pettis property if whenever $f_n \longrightarrow 0$ weakly in A and $\varphi_n \longrightarrow 0$ weakly in A^* then $\varphi_n(f_n) \longrightarrow 0$. Bourgain showed that H^∞ has the Dunford Pettis property using the theory of ultraproducts. The Dunford Pettis property is related to the notion of Bourgain algebra, denoted B_b, introduced by [6] Cima and Timoney. The algebra B_b is the set of f in A such that if $f_n \longrightarrow 0$ weakly in B then $\mathrm{dist}(ff_n, B) \longrightarrow 0$. Bourgain showed [2] that a closed subspace X of $C(L)$, where L is a compact Hausdorff space, has the Dunford Pettis property if $X_b = C(L)$. Cima and Timoney proved that B_b is a closed subalgebra of A and that if B is an algebra then $B \subset B_b$. In this paper we study the Bourgain algebra associated with various algebras of functions on the unit circle T.

Let D be the open unit disc and let H^∞ denote the space of bounded analytic functions on D. By identifying H^∞ with boundary functions, we may consider H^∞ as a closed subalgebra of L^∞, the algebra of bounded measurable functions on the unit circle T. A closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. We denote the space of continuous functions on T by C. Then [14] $H^\infty + C$ is a Douglas algebra. In this paper, we describe the Bourgain algebras in the case where $A = L^\infty$ and B is a Douglas algebra. The case $H^\infty + C$ was studied by Cima, Janson and Yale [5]. Using Fefferman's duality theorem, they showed that $H^\infty_b = H^\infty + C$. We present another proof of Cima, Janson and Yale's result.

Using these methods, we are also able to extend the result to Douglas algebras. We also prove some very nice properties of these algebras, and it is our hope that the simplicity of our proofs will allow them to extend to other algebras.

We now recall the information necessary to the proof.

A sequence $\{z_n\}$ in D is called interpolating for H^∞ if whenever $\{w_n\}$ is a bounded sequence of complex numbers then there exists a function f in H^∞ such that $f(z_n) = w_n$ for all n. A Blaschke product b, where

$$b(z) = \prod \frac{-\overline{z_n}(z - z_n)}{|z_n|(1 - \bar{z}_n z)}$$

is called *interpolating* if the zero sequence $\{z_n\}$ is interpolating. A function u in H^∞ is called inner if u has modulus one a.e. on the unit circle. An interpolating Blaschke product

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is a typical inner function. By the Chang Marshall theorem [4,12] every Douglas algebra B coincides with the closed subalgebra generated by H^∞ and the complex conjugate of interpolating Blaschke products b with $\bar{b} \in B$. We denote the maximal ideal space of B by $M(B)$. Making the usual identifications, we have $M(L^\infty) \subset M(B) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary for every Douglas algebra B.

We shall use Hoffman's results about Gleason parts. For a point φ in $M(H^\infty)$ recall that the pseudohyperbolic distance from φ to another point ψ in $M(H^\infty)$ is given by

$$\rho(\psi,\varphi) = \sup\bigl\{|\psi(f)| : \varphi(f) = 0, \|f\| \leqq 1, f \in H^\infty\bigr\}.$$

For z, w in D, Schwarz's lemma shows that

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|.$$

If we define $\sim$ by $\psi \sim \varphi$ if and only if $\rho(\psi,\varphi) < 1$, then $\sim$ defines an equivalence relation on $M(H^\infty)$ and the equivalence classes are called the Gleason parts. The Gleason part associated with a point φ will be denoted $P(\varphi)$. Hoffman [11] showed that a Gleason part is either a singleton, in which case we call the part trivial, or there exists a map L_φ of the disc onto the part $P(\varphi)$ which is one to one and $\hat{f} \circ L_\varphi$ is an H^∞ function whenever $\hat{f}$ is the Gelfand transform of some $f \in H^\infty$. In the latter case we call the part nontrivial. Since we are dealing only with uniform algebras, in what follows we shall identify $\hat{f}$ with f.

In a recent survey article Yale [16] summarizes the results in this area. We offer some specific answers to the questions stated in Yale's article (for Douglas algebras). We will also show that $(H^\infty)_b = H^\infty + C$. The proof we give here is similar to, but simpler than, that given by Cima, Janson and Yale. These results will be given in the next section.

Bourgain algebras of Douglas algebras. For an inner function u we let $Z(u) = \{\psi \in M(H^\infty) : \psi(u) = 0\}$. The following lemma appeared in a proof in [5]. For completeness, we state it again here.

LEMMA 1. *Suppose that $\{f_n\}$ is a sequence of H^∞ functions such that $\sum_{n=1}^\infty |f_n(z)| < M$ for all $z \in D$. Then $f_n \longrightarrow 0$ weakly in H^∞.*

PROOF. As in [5] if $\varphi \in (H^\infty)^*$ and $\alpha_n = \exp\bigl(-i \arg \varphi(f_n)\bigr)$ then for any positive integer N we have

$$\sum_1^N |\varphi(f_n)| = \sum_1^N \alpha_n \varphi(f_n) = \varphi\Bigl(\sum_1^N \alpha_n f_n\Bigr) \leqq \|\varphi\| \Bigl\|\sum_1^N \alpha_n f_n\Bigr\| \leqq \|\varphi\| M. \qquad ■$$

Before we proceed with the proof of the next theorem, we briefly review some facts about interpolating sequences. All of the material we use here appears (essentially) in [10, p. 194–206].

If $\{\psi_1, \ldots, \psi_n\}$ are finitely many points in the maximal ideal space of a Douglas algebra B, then it is not difficult to see that the map $T\colon B \longrightarrow C^n$ given by

$T(f) = \big(f(\psi_1), f(\psi_2), \ldots, f(\psi_n)\big)$ is a surjective linear map between Banach spaces. Let T_1 denote the map from $B/\ker T$ onto C^n induced by T. By the Open Mapping Theorem, T_1 is bounded below. Thus there is a constant M such that $M\|f|\{\psi_j\}_1^n\|_\infty = M\|T_1(f + \operatorname{Ker} T)\| \geqq \|f + \operatorname{Ker} T\|$. As a consequence, we have the following fact which we shall use in the proof of Lemma 2: If f is a function in B such that $\|f|\{\psi_j\}_1^n\| < \varepsilon$, then there exists a function g in B such that $g(\psi_j) = f(\psi_j)$ for $j = 1, \ldots, n$ and $\|g\| < N\varepsilon$ (where N is a constant independent of f). If $\{\psi_n\}$ is an infinite interpolating sequence, we can replace C^n by ℓ^∞ and the remarks above still hold.

The interpolating sequence $S = \{\psi_n\}$ (see [10]) is discrete in its relative topology as a subset of $M(B)$, its closure $\bar{S}$ in $M(B)$ is homeomorphic to the Čech compactification of S and is totally disconnected.

THEOREM 2. *Let B be a Douglas algebra and let c be an interpolating Blaschke product. Then $\bar{c} \in B_b$ if and only if $Z(c) \cap M(B)$ is a finite set.*

PROOF. Suppose that $Z(c) \cap M(B) = \{\psi_1, \ldots, \psi_m\}$. Choose $\{f_n\}_{n=1}^\infty$ in B with $f_n \longrightarrow 0$ weakly. Then $f_n(\psi_j) \longrightarrow 0$ for $j = 1, 2, \ldots, m$. Now use the remarks above to obtain a sequence $\{g_n\}$ in B such that $g_n(\psi_j) = f_n(\psi_j)$ for $j = 1, \ldots, m$ and $\|g_n\| \longrightarrow 0$. By [1,9] $\bar{c}(f_n - g_n) \in B$, $n = 1, 2, \ldots$. Therefore

$$\begin{aligned} \operatorname{dist}(\bar{c}f_n, B) &= \operatorname{dist}\big(\bar{c}(f_n - g_n) + \bar{c}g_n, B\big) \\ &= \operatorname{dist}(\bar{c}g_n, B) \\ &\leqq \|g_n\| \longrightarrow 0. \end{aligned}$$

Thus $\bar{c} \in B_b$.

Next suppose that $Z(c) \cap M(B)$ is an infinite set. Let $\{z_n\}$ denote the zero sequence of c in D. Since $Z(c)$ is homeomorphic to the Čech compactification of $\{z_n\}$ [10, p. 205], and because every infinite set in a Hausdorff space contains an infinite discrete subset, we can take a sequence $\{\lambda_j\} \in Z(c) \cap M(B)$ and a sequence $\{V_j\}$ of disjoint open and closed subsets of $Z(c)$ such that $\lambda_j \in V_j \cap M(B)$. Since λ_j belongs to the closure of $\{z_n\}$ [10, p. 206], we see that λ_j is contained in the closure of $V_j \cap \{z_n\}$ for each j. As in [5] we can find P. Beurling functions [7, Theorem VII.2.1] f_k in H^∞ and a positive number M such that

$$f_k(z_j) = \delta_{jk} \text{ and } \sum |f_k(z)| < M \text{ on } D.$$

For each j, let $F_j = \sum_{k \in I} f_k$ where $I = \{k : z_k \in V_j\}$.

Now since $\sum |f_k| < M$ on D, we see that $F_j \in H^\infty$ for all j. Furthermore, since the sets V_j are disjoint and $\sum |f_j| < M$ on D we have $\sum |F_j| < M$ and $F_j = 1$ on $V_j \cap \{z\}$. Thus $F_j(\lambda_j) = 1$ and, by Lemma 1, $F_j \longrightarrow 0$ weakly in H^∞ (hence in B). But

$$\begin{aligned} \operatorname{dist}(\bar{c}F_j, B) &= \operatorname{dist}(F_j, cB) \\ &\geqq \|F_j|Z(c) \cap M(B)\| \\ &\geqq |F_j(\lambda_j)| \\ &= 1. \end{aligned}$$

Hence $\bar{c} \notin B_b$. ∎

COROLLARY 3. *Let A and B be Douglas algebras with $A \subset B$. Then $A_b \subset B_b$.*

PROOF. We shall use the Chang Marshall theorem.

Let c be an interpolating Blaschke product with $\bar{c} \in A_b$. By Theorem 2, $Z(c) \cap M(A)$ is finite. Since $A \subset B$, $M(B) \subset M(A)$. Thus $Z(c) \cap M(B)$ is finite. By Theorem 2, $\bar{c} \in B_b$. By the Chang Marshall Theorem $A_b \subset B_b$. ∎

We now give our proof of the result (i)⇔(iv) in [5].

THEOREM 4. $H_b^\infty = H^\infty + C$.

PROOF. We first show that $\bar{z} \in H_b^\infty$. Since we know from Cima and Timoney's results that H_b^∞ is an algebra containing H^∞, this will prove that $H^\infty + C \subset H_b^\infty$. Let $\{f_n\}$ be a sequence in H^∞ such that $f_n \longrightarrow 0$ weakly in H^∞. Since $\bar{z}\big(f_n - f_n(0)\big) \in H^\infty$, we have

$$\begin{aligned}\operatorname{dist}(\bar{z}f_n, H^\infty) &= \operatorname{dist}\big(\bar{z}(f_n - f_n(0)) + \bar{z}f_n(0), H^\infty\big)\\ &= \operatorname{dist}\big(\bar{z}f_n(0), H^\infty\big)\\ &\leqq |f_n(0)|\end{aligned}$$

Since $f_n \longrightarrow 0$ weakly, $f_n(0) \longrightarrow 0$. Thus $\bar{z} \in H_b^\infty$ and we have established the containment of $H^\infty + C$ in H_b^∞.

To show equality, we show that H_b^∞ does not contain the conjugate of any interpolating Blaschke product. By the Chang Marshall theorem we know that any Douglas algebra containing $\bar{z}$ and no conjugates of (infinite) interpolating Blaschke products must be $H^\infty + C$.

Let c be an interpolating Blaschke product and let $\{z_n\}$ denote the zero sequence of c. As in [5] we can find P. Beurling functions [7, Theorem VII.2.1] f_n in H^∞ and a positive integer M such that

$$f_n(z_m) = \delta_{nm} \text{ and } \sum |f_n(z)| < M \text{ for all } z \text{ in } D.$$

By Lemma 1, $f_n \longrightarrow 0$ weakly but

$$\begin{aligned}\operatorname{dist}(\bar{c}f_n, H^\infty) &= \|\bar{c}f_n + H^\infty\|\\ &= \|f_n + cH^\infty\|\\ &\geqq |f_n(z_n)|\\ &\geqq 1.\end{aligned}$$

Thus $\bar{c} \notin H_b^\infty$, completing the proof. ∎

Theorem 2 gives a characterization of the interpolating Blaschke products invertible in B_b for an arbitrary Douglas algebra B. Using some results about division in Douglas algebras, we can study inner functions which are invertible in B_b. We do so in Corollary 5 below.

For a function f in L^∞, let $H^\infty[f]$ denote the closed subalgebra of L^∞ generated by H^∞ and f.

COROLLARY 5. *Let B be a Douglas algebra and u an inner function. If $Z(u) \cap M(B)$ is infinite, then $\bar{u} \notin B_b$.*

PROOF. There are two cases to consider: either u vanishes at a trivial point in $M(B)$ or else it does not.

CASE 1. Suppose that u vanishes at a trivial point ψ in $M(B)$. Then [17] there exists an interpolating Blaschke product v such that $H^\infty[\bar{u}] = H^\infty[\bar{v}]$. Since $M(H^\infty[\bar{c}]) = \{\varphi \in M(H^\infty) : |\varphi(c)| = 1\}$ for any inner function c, we see that $|\psi(v)| < 1$. If we can show that $Z(v) \cap M(B)$ is infinite, then $\bar{v} \notin B_b$ by Theorem 2. Since $H^\infty \subset B \subset B_b$ and $\bar{v} \in H^\infty[\bar{u}]$ we would be able to conclude that $\bar{u} \notin B_b$. So we will assume that $Z(v) \cap M(B)$ is finite.

Suppose that $Z(v) \cap M(B) = \{\psi_1, \dots, \psi_n\}$. Since v is interpolating, all the zeros of v are in nontrivial parts and hence ψ is not in the same part as any ψ_j. Thus we can find $h_1, \dots, h_n$ in H^∞ such that $\|h_j\| \leqq 1$, $\psi_j(h_j) = 0$ and $|\psi(h_j)|$ is as close to 1 as we like. Thus, if $h = h_1 \cdots h_n$, then we have a function h in H^∞ of norm less than or equal to one which vanishes on the zeros of v in $M(B)$. Furthermore, since we chose v satisfying $|\psi(v)| < 1$ we may assume that $|\psi(v)| < |\psi(h)|$. By [1,9] we know that there exists $g \in B$ such that $h = gv$. Since $|v| = 1$ on $M(L^\infty)$ and $M(L^\infty)$ is the Shilov boundary for B, we see that $\|g\| \leqq 1$. Since $\psi \in M(B)$ we have $|\psi(h)| \leqq |\psi(v)| < |\psi(h)|$, a contradiction. So $Z(v) \cap M(B)$ is infinite, and we are done with Case 1.

CASE 2. If u does not vanish at any trivial point in $M(B)$, then [8] there exist n interpolating Blaschke products $b_1, \dots, b_n$ and an L^∞ function g invertible in B such that $u = b_1 \cdots b_n g$. Since $Z(u) \cap M(B)$ is infinite, for some j, $Z(b_j) \cap M(B)$ is infinite. By Theorem 2, $\bar{b}_j \notin B_b$. Since $\bar{b}_j = \bar{u} b_1 \cdots b_{j-1} b_{j+1} \cdots b_n g \in \bar{u}B$, $\bar{u} \notin B_b$, completing the proof of Case 2. ∎

The next result shows that $B_b \neq C\big(M(L^\infty)\big)$ for any Douglas algebra $B \neq L^\infty$. (Recall that the negation of this result implies that an algebra has the Dunford Pettis property.)

COROLLARY 6. *Let B be a Douglas algebra with $B \neq L^\infty$. Then $B_b \neq L^\infty$.*

PROOF. By the Chang Marshall theorem there is an interpolating Blaschke product c such that $\bar{c} \notin B$. By [13] there is an inner function u such that

$$\{\varphi \in M(H^\infty + C) : |\varphi(c)| < 1\} \subset \{\varphi \in M(H^\infty + C) : |\varphi(u)| = 0\}.$$

Because $\bar{c} \notin B$, the unimodular function c is not invertible in B. Thus there is a point φ in $M(B)$ such that $\varphi(c) = 0$. Not that since c is an interpolating Blaschke product, $P(\varphi)$ is nontrivial [11]. Now $c \circ L_\varphi$ is an analytic function bounded by one on the unit disc and vanishing at 0. Thus c is less than one on $P(\varphi)$, and hence u vanishes on $P(\varphi)$. Since $\varphi \in M(B)$, $P(\varphi) \subset M(B)$. Thus $Z(u) \cap M(B)$ is infinite and Corollary 5 implies that $u \notin B_b$. Hence $B_b \neq L^\infty$. ∎

COROLLARY 7. $(B_b)_b = B_b$.

PROOF. We know that $B_b \subset (B_b)_b$. Now let c be an interpolating product invertible in $(B_b)_b$. By Theorem 2, $Z(c) \cap M(B_b)$ is a finite set. We claim that $E = Z(c) \cap$

$[M(B) - M(B_b)]$ is also finite. If E is infinite, then either (a) E has a cluster point in $M(B) - M(B_b)$ or (b) $\bar{E} - E \subset M(B_b)$. Suppose that (a) occurs. Since $Z(c)$ is homeomorphic to the Čech compactification of $Z(c) \cap D$, there exists an open and closed subset U of $Z(c)$ such that $U \cap M(B_b) = \emptyset$ and $U \cap M(B)$ is an infinite set. Let c_1 be the subproduct of c with zeroes in $U \cap D$. Since U is closed in $Z(c)$, we have $Z(c_1) = U$. Since $Z(c_1) \cap M(B_b) = \emptyset$, $\bar{c}_1 \in B_b$. On the other hand, since $Z(c_1) \cap M(B) = U \cap M(B)$ is infinite, by Theorem 2 this is impossible.

Suppose that (b) occurs. Then there is a sequence $\{\alpha_n\}$ in E and a sequence $\{V_n\}$ of disjoint open and closed subsets of $Z(c)$ [10, p. 205] such that $\alpha_n \in V_n$. Therefore for every bounded sequence of complex numbers $\{w_n\}$ there is a function h in H^∞ such that

$$h = w_n \text{ on } D \cap V_n.$$

Since α_n is contained in the closure of $D \cap V_n$, $h(\alpha_n) = w_n$. Thus $\{\alpha_n\}$ is an interpolating sequence and hence [10 p. 205] the closure of $\{\alpha_n\}$ is homeomorphic to the Čech compactification of the integers. Thus $\overline{\{\alpha_n\}} - \{\alpha_n\} \subset M(B_b) \cap Z(c)$ is infinite and by Theorem 2, this implies $\bar{c} \notin (B_b)_b$. This contradiction implies that $Z(c) \cap M(B)$ is finite and by Theorem 2, we have $\bar{c} \in B_b$. ∎

By the Chang Marshall theorem, every Douglas algebra B is generated by H^∞ and the complex conjugates of the inner functions invertible in B. If countably many conjugates of inner functions together with H^∞ generate the algebra, then we say that the algebra is countably generated.

COROLLARY 8. *If B is a countably generated Douglas algebra, $B \neq H^\infty$, then $B_b = B$.*

PROOF. Let $B = H^\infty[\bar{I}_n; n = 1, 2 \dots]$, where each I_n is an inner function. Suppose that c is an interpolating Blaschke product with $\bar{c} \notin B$. Since we know that $B \subset B_b$, by the Chang Marshall theorem it is enough to show that $\bar{c} \notin B_b$. Consider the function

$$F = \sum_1^\infty \left(\frac{1}{2^n}\right)|I_n| \text{ on } M(H^\infty).$$

Since $M(B) = \{x \in M(H^\infty) : |I_n(x)| = 1 \text{ for every } n\}$, we see that $F = 1$ on $M(B)$ and $F < 1$ on $M(H^\infty) - M(B)$. Since $\bar{c} \notin B$, $Z(c) \cap M(B) \neq \emptyset$. Let $\{z_n\}$ denote the zero sequence of c in D. Since $Z(c) - D = \mathrm{cl}\{z_n\} - \{z_n\}$ [10, p. 205] and $Z(c) \cap M(B) \neq \emptyset$, there is an interpolating subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with $F(z_{n_j}) \longrightarrow 1$.

Let c_1 be the interpolating Blaschke product with zeroes $\{z_{n_j}\}$. Then $c = c_1 c_2$. Since $Z(c_1) - D = \mathrm{cl}\{z_{n_j}\} - \{z_{n_j}\}$, $F = 1$ on $Z(c_1) - D$ so that $Z(c_1) - D \subset M(B)$. Since $Z(c_1) - D$ is always infinite for an interpolating Blaschke product c_1, by Theorem 2,

$\bar{c}_1 \notin B_b$. Since B_b is an algebra containing H^∞ and $\bar{c}_1 = \bar{c}c_2$, we have $\bar{c} \notin B_b$. Since c was arbitrary $B_b = B$. ■

The following example shows that B_b need not equal B for Douglas algebras (other than $H^\infty + C$). Recall that a Blaschke product is thin if it is interpolating with zero sequence $\{z_n\}$ and

$$\lim \prod_{\substack{n=1 \\ n \neq m}}^{\infty} \rho(z_n, z_m) = 1$$

A QC level set E is a subset of $M(L^\infty)$ such that (the Gelfand transform of) any function f in $H^\infty + C$ whose conjugate is also in $H^\infty + C$ is constant on E. Sundberg and Wolff [15] showed that a thin Blaschke product has at most one zero in $M(H^\infty|E) = \{\varphi \in M(H^\infty) : \operatorname{supp} \mu_\varphi \subset E\}$. It is well known that the algebra

$$B = H^\infty_E = \{g \in L^\infty : g|E \in H^\infty|E\}$$

has

$$M(B) = M(L^\infty) \cup M(H^\infty|E) = M(L^\infty) \cup \{\varphi \in M(H^\infty) : \operatorname{supp} \mu_\varphi \subset E\}$$

and is a closed subalgebra of L^∞. If we choose a thin Blaschke product c and a set E such that $M(H^\infty|E)$ contains a zero of c, then c has precisely one zero in $M(B)$. Thus $\bar{c} \notin B$, but by Lemma 1, $\bar{c} \in B_b$. Hence B_b properly contains B.

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