PRODUCTS OF ZERO-ONE MATRICES

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1. Introduction. Let P be a finite set with p objects o_j, $j = 1, 2, \ldots, p$, and let $\{S_i\}$, $i = 1, 2, \ldots, n$, be a family of n subsets of P. The incidence matrix $A = (a_{ij})$ for the family $\{S_i\}$ is defined by the rules: $a_{ij} = 1$ if $o_j \in S_i$ and $a_{ij} = 0$ if $o_j \notin S_i$. Then, if $AA^T = B = (b_{ij})$ (where A^T denotes the transpose of A), it is easy to see that $b_{ij} = |S_i \cap S_j|$, $i = 1, \ldots, n$, $j = 1, \ldots, n$, so that the elements of B are integers with $b_{ii} \geqslant b_{ij} \geqslant 0$.

Conversely, if an $n \times n$ symmetric matrix B with non-negative integral elements is given, one may ask whether there exists a zero-one matrix, A, such that $B = AA^T$. In combinatorial terms, this is tantamount to asking whether the intersection pattern presented by B is realizable for some family of subsets of a suitable finite set. Let us say that B is realizable if this is so. Evidently, a necessary condition for realizability is that $b_{ii} \geqslant b_{ij} \geqslant 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, n$, but, as we shall show, this condition is not sufficient if $n > 2$.

If B is realizable, one may seek to determine the smallest value of p for which there exists an $n \times p$ zero-one matrix A with $AA^T = B$. (Clearly, if $q > p$, there exists an $n \times q$ matrix $\bar{A}$ with $\bar{A}\bar{A}^T = B$; $\bar{A}$ may be obtained by adjoining $q - p$ columns of zeros to A.) We call this minimum value of p the content of B and represent it by $C(B)$. Combinatorially, the content of B is simply the number of objects in the smallest possible set that contains a family of subsets with the intersection pattern presented by B.

The principal aims of this paper are the determination of necessary and sufficient conditions for realizability of a given matrix and the acquisition of formulas for the content of a given realizable matrix. These problems are completely solved for $n \leqslant 4$, the case $n = 4$ presenting by far the greatest difficulty. We also obtain partial results for (k, λ) matrices, i.e. matrices of the form $(k - \lambda)I_n + \lambda J_n$, where k and λ are non-negative integers with $k \geqslant \lambda$, I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix all of whose elements are 1. As an application, we prove anew Qvist's theorem **(8)** stating that a finite projective plane of odd order N cannot contain an $(N + 2)$-arc, i.e. a set of $N + 2$ points no three of which are collinear.

Section 2 contains some elementary remarks about realizability and content. In § 3 we transform the realizability problem into a problem in linear diophantine analysis. An independent set of necessary conditions is obtained.

Received April 12, 1966.

https://doi.org/10.4153/CJM-1968-029-7 Published online by Cambridge University Press

In § 4, the core of the paper, the problem of the calculation of the content of a matrix is transformed into a problem in integral linear programming. Consideration of the dual problem leads one immediately to an examination of the $n \times n$ matrices all of whose principal submatrices have element sums $\leqslant 1$. These matrices form a convex subset of the n^2-dimensional vector space over the reals; the extreme points, or vertices, of this set are then a natural object of study. Their determination for $n \leqslant 4$ opens the way for the solution of the realizability and content problems for $n \leqslant 4$ in § 5. The length of this section is due to the fact that we are dealing with an integral, rather than real, linear programming problem. In § 6, we find extreme matrices for $n > 4$ with the same type of symmetry as the (k, λ) matrices. The content of some (k, λ) matrices is computed and the afore-mentioned application is made. In § 7 a connection is made with the Hasse–Minkowski theory of congruence of matrices (or quadratic forms), while in § 8 analogues of the concepts of realizability and content are discussed for non-symmetric matrices. In particular, it is shown that every non-negative integral matrix is the product of two zero–one matrices.

In **(6)**, Hall discusses intersection patterns. He assumes that one knows only whether the sets $S_i \cap S_j$ are empty or non-empty so that the available information can be conveyed by a zero–one matrix. He treats problems of content for this situation. Goodman **(5)** assumes that the actual objects in $S_i \cap S_j$ are given for $i \neq j$ and answers the realizability and content questions that arise. In a sense, then, the problems treated in the present paper are intermediate between those discussed by Hall and by Goodman. Long ago Boole **(1)** was concerned with intersection patterns and related combinatorial problems. The concepts of realizability and content are implicit in his work. However, he made no use of matrices. He was aware of the importance of the quantities x_τ (cf. § 3) which play an essential role in the present investigation.

2. Elementary properties.

THEOREM 2.1. *If B_1 and B_2 are realizable matrices of order n, then $B_1 + B_2$ is realizable and $C(B_1 + B_2) \leqslant C(B_1) + C(B_2)$.*

Proof. Let $B_1 = A_1 A_1^T$ and $B_2 = A_2 A_2^T$, where A_1 has n rows and p_1 columns and A_2 has n rows and p_2 columns. Let A be the matrix with n rows and $p_1 + p_2$ columns obtained by placing A_2 to the right of A_1. Then $B_1 + B_2 = AA^T$ so that $B_1 + B_2$ is realizable. Moreover, if A_1 and A_2 are chosen so that $p_1 = C(B_1)$ and $p_2 = C(B_2)$, our construction implies that $C(B_1 + B_2) \leqslant C(B_1) + C(B_2)$.

COROLLARY 2.1. *If B is realizable and m is a non-negative integer, then mB is realizable and $C(mB) \leqslant mC(B)$.*

The converse of the first part of Corollary (2.1) is false. The matrix

$$B_0 = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

is not realizable. This may be seen by a straightforward combinatorial argument in which one considers the various (and not very numerous) possibilities for the sets S_j, $j = 1, 2, 3, 4, 5$. On the other hand, $2B_0$ is realizable, as one sees by taking $S_1 = \{o_1, o_2, o_3, o_4\}$, $S_2 = \{o_1, o_2, o_5, o_6\}$, $S_3 = \{o_3, o_4, o_5, o_6\}$, $S_4 = \{o_1, o_3, o_5, o_7\}$, $S_5 = \{o_2, o_4, o_6, o_8\}$. Each of these sets has four objects and any two have two objects in common except for S_4 and S_5 which are disjoint.

It will follow from Theorems 5.1–5.4 that the realizability of mB implies that of B when $n \leqslant 4$. In § 7, we give an example of a matrix B for which $C(2B) < C(B)$.

Evidently $I_n = I_n I_n^T$ is realizable. Moreover $C(I_n) = n$, since in this case the sets S_j, $j = 1, 2, \ldots, n$, must each have one object and no two have any object in common. J_n is realizable since $J_n = J_{n1} J_{n1}^T$ where J_{n1} is an $n \times 1$ matrix consisting entirely of ones. This representation shows that $C(J_n) = 1$. Theorem 2.1 now implies

COROLLARY 2.2. *If B is an $n \times n$ (k, λ) matrix, B is realizable and*

$$C(B) \leqslant nk - (n - 1)\lambda.$$

It is obvious that rows and columns of zeros may be removed from a matrix without altering its realizability or content. Also, if two rows (and hence two columns), are identical, one of the rows may be removed without altering either realizability or content. If $\bar{B}$ is a principal submatrix of the realizable matrix B, then clearly $\bar{B}$ is realizable and $C(\bar{B}) \leqslant C(B)$. Every proper principal submatrix of B_0 is realizable even though B_0 is not.

From the equation $B = AA^T$ we deduce that $C(B) \geqslant \operatorname{rank} B$ and that if $C(B) = n$, then $\det B$ is an integral square. Another obvious lower bound for $C(B)$ is $\mu(B)$, where $\mu(B)$ denotes the largest element of B. Furthermore, if B has d distinct rows, then, since the set P has 2^p distinct subsets, we have $2^p \geqslant d$ so that $C(B) \geqslant \log_2 d$.

An upper bound for $C(B)$ is given by

$$\operatorname{tr} B = \sum_{i=1}^{n} b_{ii}.$$

For the set S_i has b_{ii} elements; hence P need not have more than $\sum_{i=1}^{n} b_{ij}$ elements. Clearly $C(B) = \operatorname{tr} B$ if and only if B is diagonal.

3. Necessary conditions for realizability. Let us suppose that B, and hence the intersection pattern presented by B, is realizable, so that

$$b_{ij} = |S_i \cap S_j|, \qquad i, j = 1, 2, \ldots, n,$$

for some family $\{S_i\}$ of subsets of a finite set P. Let $\omega_n = \{1, 2, 3, \ldots, n\}$ and let $\tau \subset \omega_n$. Let x_τ denote the number of elements of P belonging to precisely those sets S_i for which $i \in \tau$. Thus

$$x_\tau = |\bigcap_{i \in \tau} S_i \cap \bigcap_{j \notin \tau} \bar{S}_j|$$

where $\bar{S}_j$ is the complement of S_j in P. It is easy to see that

$$\sum_{\{i,j\} \subseteq \tau} x_\tau = b_{ij}, \qquad i, j = 1, 2, \ldots, n. \tag{3.1}$$

Since B is symmetric, the system (3.1) consists of $\binom{n+1}{2}$ independent equations in $2^n - 1$ unknowns (we can assume τ non-empty). It is immediate that B is realizable only if the system has a solution in non-negative integers. Moreover, if non-negative integers x_τ are given satisfying (3.1), sets S_i with $|S_i \cap S_j| = b_{ij}$ may readily be constructed from the definitions of x_τ. Hence the solvability of (3.1) in non-negative integers is also a sufficient condition for the realizability of B.

Suppose that σ and ρ are a pair of subsets of ω_n. Define the function $f_{\sigma\rho}(B)$ by

$$f_{\sigma\rho}(B) = \sum_{\substack{i \leqslant j \\ \{i,j\} \subset \sigma}} b_{ij} + \sum_{\substack{i < j \\ \{i,j\} \subset \rho}} b_{ij} - \sum_{\substack{i \in \sigma \\ j \in \rho}} b_{ij}. \tag{3.2}$$

We prove:

THEOREM 3.1. *If B is realizable, then $f_{\sigma\rho}(B) \geqslant 0$ for all pairs (σ, ρ) of subsets of ω_n.*

Proof. The theorem will be established if we show that when equations (3.1) are inserted into (3.2) the coefficient of x_τ will be non-negative, for all subsets, τ, of ω_n. Let $\alpha_{\tau\sigma}$ be equal to the number of unordered pairs of (possibly equal) elements in both τ and σ. Let $\beta_{\tau\rho}$ be the number of unordered pairs of unequal elements in both τ and ρ. Let $\gamma_{\tau\sigma\rho}$ be the number of ordered pairs of elements (i, j) where i is in τ and σ and j is in τ and ρ. Then it follows from (3.1) and (3.2) that the coefficient of x_τ, when (3.1) is substituted in (3.2), will be $\alpha_{\tau\sigma} + \beta_{\tau\rho} - \gamma_{\tau\sigma\rho}$.

Let $|\tau \cap \sigma| = u$, $|\tau \cap \rho| = v$. Then $\alpha_{\tau\sigma} = \frac{1}{2}(u^2 + u)$, $\beta_{\tau\rho} = \frac{1}{2}(v^2 - v)$, $\gamma_{\tau\sigma\rho} = uv$ so that

$$\alpha_{\tau\sigma} + \beta_{\tau\rho} - \gamma_{\tau\sigma\rho} = \tfrac{1}{2}[(u - v + 1)(u - v)].$$

The quadratic function $z(z + 1)$ is non-negative for integral z. Hence, setting $z = u - v$, we find that $\alpha_{\tau\sigma} + \beta_{\tau\rho} - \gamma_{\tau\sigma\rho} \geqslant 0$.

We shall see in § 5 that the converse of Theorem 3.1 is true if $n \leqslant 4$. However, if $n > 4$, the converse is false. The 5×5 matrix B_0 mentioned in § 2 is not realizable but must satisfy all inequalities of the form $f_{\sigma\rho}(B_0) \geqslant 0$ since these are linear homogeneous inequalities satisfied by the elements of the realizable matrix $2B_0$. We conjecture that if $f_{\sigma\rho}(B) \geqslant 0$ for all pairs of subsets (σ, ρ) of ω_n, then the system (3.1) has a non-negative (but not necessarily integral) solution, but we have been unable to prove this.

For further reference we list some of the inequalities $f_{\sigma\rho}(B) \geqslant 0$ for $2 \leqslant n \leqslant 4$:

(3.3) $\sigma = \emptyset,\ \rho = \{1, 2\},\ f_{\sigma\rho}(B) = b_{12} \geqslant 0;$

(3.4) $\sigma = \{1\},\ \rho = \{2\},\ f_{\sigma\rho}(B) = b_{11} - b_{12} \geqslant 0;$

(3.5) $\sigma = \{1\},\ \rho = \{2, 3\},\ f_{\sigma\rho}(B) = b_{11} + b_{23} - b_{12} - b_{13} \geqslant 0;$

(3.6) $\sigma = \{1\},\ \rho = \{2, 3, 4\},$
$$f_{\sigma\rho}(B) = b_{11} + b_{23} + b_{24} + b_{34} - b_{12} - b_{13} - b_{14} \geqslant 0;$$

(3.7) $\sigma = \{1, 2\},\ \rho = \{3, 4\},$
$$f_{\sigma\rho}(B) = b_{11} + b_{12} + b_{22} + b_{34} - b_{13} - b_{14} - b_{23} - b_{24} \geqslant 0.$$

It is easy to show that if $\sigma \cap \rho = \nu$ and $\sigma' = \sigma - \nu$, $\rho' = \rho - \nu$, then $f_{\sigma\rho}(B) = f_{\sigma'\rho'}(B)$ so that we can assume that $\sigma \cap \rho \neq \emptyset$. In this case the general form of $f_{\sigma\rho}(B)$ is determined by $|\sigma|$ and $|\rho|$. Let us say that the inequalities $f_{\sigma\rho}(B) \geqslant 0$ and $f_{\sigma^*\rho^*}(B) \geqslant 0$ are of the same *type* if $\sigma \cap \rho = \emptyset$, $\sigma^* \cap \rho^* = \emptyset$, $|\sigma| = |\sigma^*|$, and $|\rho| = |\rho^*|$. Then if $n = 4$, there are 6 inequalities of type (3.3), 12 of type (3.4), 12 of type (3.5), 4 of type (3.6), and 6 of type (3.7), 40 in all.

4. Extreme matrices. In the notation of § 3, we have

$$|P| \geqslant \textstyle\sum_{\tau \subseteq \omega_n} x_\tau. \tag{4.1}$$

The reason we do not have equality is that some objects in P may not belong to any of the sets S_i. (Equality could be restored if we were to allow τ to be empty.) If $|P| = C(B)$, then clearly equality holds in (4.1). Hence the problem of determining the content of the realizable matrix B may be regarded as a problem in integral linear programming; that is, we are to find integers x_τ such that

$$x_\tau \geqslant 0, \tag{4.2}$$

$$\textstyle\sum_{\{i,j\} \subseteq \tau} x_\tau = b_{ij}, \qquad i = 1, 2, \ldots, n, \qquad j = 1, 2, \ldots, n, \tag{4.3}$$

$$\textstyle\sum_{\tau \subseteq \omega_n} x_\tau \text{ is a minimum.} \tag{4.4}$$

The minimum of $\sum_\tau x_\tau$, referred to as the value of the programming problem, is in this case equal to $C(B)$.

We shall make some general remarks about real linear programming. The canonical linear programming problem may be put in the following form.

We are to find a (real) column vector x such that

$$x \geqslant 0, \tag{4.5}$$

$$Dx = b, \tag{4.6}$$

$$cx \text{ is a minimum.} \tag{4.7}$$

Here x and b are column vectors, c is a row vector and D a matrix, the dimensions of x, b, c, and D being compatible for multiplication. (According to the usual convention, $z \geqslant w$ means that each component of z exceeds the corresponding component of w.) The problem dual to (4.5)–(4.7) is that of finding a row vector y (not necessarily non-negative) such that

$$yD \leqslant c, \tag{4.8}$$

$$yb \text{ is a maximum.} \tag{4.9}$$

If x is any vector satisfying (4.5) and (4.6) and y is any vector satisfying (4.8), then

$$yb = y(Dx) = (yD)x \leqslant cx. \tag{4.10}$$

Thus the maximum of yb does not exceed the minimum of cx. According to the fundamental theorem of linear programming, if both (4.5)–(4.6) and (4.8) have solutions (are "feasible"), the problem (4.5)–(4.7) and its dual (4.8)–(4.9) have solutions and $\max yb = \min cx$; that is, the two programming problems have the same value.

In order to formulate the problem dual to (4.2)–(4.4) we first order the set of non-empty subsets of ω_n in an arbitrary manner. The column vector x with $2^n - 1$ elements will have x_τ lying above x_σ if τ precedes σ in this ordering. In the same way, we order the n^2 pairs (i, j), $i = 1, 2, \ldots, n$, $j = 1, \ldots, n$, arbitrarily and construct the column vector b from the matrix B. The matrix D will have n^2 rows and $2^n - 1$ columns. Let the columns of D be indexed by the subsets of ω_n, ordered as above, and let the rows of D be indexed by the pairs (i, j) ordered as above. Then the element $d_{\tau, ij}$ of D is 1 if $\{i, j\} \subseteq \tau$ and is zero otherwise. (In this case, because of the symmetry of B, the system (4.6) of n^2 equations in $2^n - 1$ unknowns contains only $\binom{n+1}{2}$ independent equations.) We introduce a row vector y with n^2 elements y_{ij} indexed by the pairs (i, j) ordered as above and associate with y an $n \times n$ symmetric matrix $Y = (y_{ij})$. In terms of the matrix Y it is now easy to describe the constraint (4.8) of the dual problem.

The row vector c consists of $2^n - 1$ ones. Inequality (4.8) is $\sum_\tau d_{\tau, ij} y_{ij} \leqslant 1$, which, because of the definition of $d_{\tau, ij}$, is

$$\textstyle\sum_{\{i,j\} \subseteq \tau} y_{ij} \leqslant 1, \qquad \text{for all } \tau \subseteq \omega_n. \tag{4.11}$$

Let Y_τ be the principal submatrix of Y consisting of those elements y_{ij} such that $\{i, j\} \subseteq \tau$. Then (4.11) says that the sum of the elements in Y_τ does not exceed 1. We shall call a symmetric matrix Y all of whose principal submatrices have this property admissible. Let $Y \cdot B = yb$. $Y \cdot B$ may be regarded as the "scalar product" of the matrices Y and B. Then the problem dual to (4.2)–(4.4) is that of maximizing $Y \cdot B$ as Y runs over the set $\mathfrak{Y}_n$ of admissible matrices of order n. It is clear from (4.10) that if Y is admissible and B is realizable

$$C(B) \geqslant Y \cdot B, \tag{4.12}$$

The inequalities (4.2)–(4.3) have solutions if B is realizable. Since admissible matrices obviously exist, the dual problem is also feasible. Hence, by the fundamental theorem, the minimum value of $\sum_\tau x_\tau$ subject to (4.2) and (4.3) (where we do *not* demand that the numbers x_τ be integral) is equal to the maximum value of $Y \cdot B$ for $Y \in \mathfrak{Y}_n$.

$\mathfrak{Y}_n$ is a convex subset of the $\binom{n+1}{2}$-dimensional vector space of real symmetric $n \times n$ matrices. The maximum value of $Y \cdot B$ will be attained at one of the vertices or extreme points of $\mathfrak{Y}_n$. At these extreme points, which we call extreme matrices, some set of $\binom{n+1}{2}$ of the inequalities (4.11) become independent equalities.

Let $\mathfrak{E}_n$ be the set of extreme points of $\mathfrak{Y}_n$. Then the value of the real linear programming problem (4.2)–(4.4) is the maximum of $E \cdot B$ for $E \in \mathfrak{E}_n$. The value of the integral linear programming problem (4.2)–(4.4) may, of course, exceed the value of the real linear programming problem so that $C(B) \geqslant \max E \cdot B$.

Extreme matrices may be found by considering all possible sets of $\binom{n+1}{2}$ equalities in (4.10) and solving the resulting systems of linear equations. Naturally, this process does not always produce an admissible matrix. By taking advantage of the symmetry of the problem, one may reduce the number of systems to be solved. We say that two extreme matrices are of the same *class* if each may be obtained from the other by permuting rows and columns. We list below representatives of each class for $n \leqslant 4$. With each matrix of a given class we have indicated the number of matrices in the class. We order the classes in $\mathfrak{E}_n$ arbitrarily and denote by $\mathfrak{E}_{nm}$ the mth class in $\mathfrak{E}_n$:

$$n = 1 \quad \mathfrak{E}_{11}: [1], 1 \tag{4.13}$$

$$n = 2 \quad \mathfrak{E}_{21}: \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}, 1 \tag{4.14}$$

$$n = 3 \quad \mathfrak{E}_{31}: \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}, 1 \quad \mathfrak{E}_{32}: \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 3 \tag{4.15}$$

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(4.16) $n = 4$

$$\mathfrak{E}_{41}: \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}, 1 \qquad \mathfrak{E}_{42}: \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 4$$

$$\mathfrak{E}_{43}: \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 6 \qquad \mathfrak{E}_{44}: \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix}, 4$$

$$\mathfrak{E}_{45}: \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}, 1$$

Note that there is 1 extreme matrix of order 1, 1 of order 2, and that there are 4 of order 3 and 16 of order 4. Further calculation will doubtless yield more insight into the structure of $\mathfrak{E}_n$ when $n > 4$. An obvious conjecture is that $|\mathfrak{E}_n| = 4^{n-2}$ if $n > 1$.

It is easy to see that the $n \times n$ matrix with ones on the principal diagonal and $-\frac{1}{2}$ elsewhere is admissible and, in fact, extreme. Application of (4.12) gives, for arbitrary realizable matrices B,

$$C(B) \geqslant \sum_{i=1}^{n} b_{ii} - \sum_{i<j} b_{ij}. \tag{4.17}$$

THEOREM 4.1. *If B is an integral symmetric matrix with non-negative elements b_{ij} such that*

$$b_{ii} \geqslant \sum_{j \neq i} b_{ij}, \qquad \text{for } i = 1, 2, \ldots, n,$$

then B is realizable and

$$C(B) = \sum_{i=1}^{n} b_{ii} - \sum_{i<j} b_{ij}.$$

Proof. An integral solution of the system (3.1) is

$$\begin{aligned} x_{\{i\}} &= b_{ii} - \textstyle\sum_{j \neq i} b_{ij}, && i = 1, 2, \ldots, n, \\ x_{\{i,j\}} &= b_{ij}, && 1 \leqslant i < j \leqslant n, \\ x_{\tau} &= 0, && \text{otherwise.} \end{aligned}$$

The hypothesis implies that this is a non-negative solution, so that B is realizable. Moreover,

$$\sum_{\tau} x_{\tau} = \sum_{i=1}^{n} b_{ii} - 2\sum_{i<j} b_{ij} + \sum_{i<j} b_{ij} = \sum_{i=1}^{n} b_{ii} - \sum_{i<j} b_{ij}$$

so that

$$C(B) \leqslant \sum_{i=1}^{n} b_{ii} - \sum_{i<j} b_{ij}.$$

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But (4.17) implies that

$$C(B) = \sum_{i=1}^{n} b_{ii} - \sum_{i<j} b_{ij}.$$

COROLLARY 4.1. *If b_{ij}, $1 \leqslant i < j \leqslant n$, are given non-negative integers, the intersection pattern $b_{ij} = |S_i \cap S_j|$ is realizable.*

Proof. We may choose integers b_{ii} such that $b_{ii} \geqslant \sum_{j \neq i} b_{ij}$ for $i = 1, 2, \ldots, n$. The corollary follows immediately from Theorem 4.1.

5. Realizability and content for $n \leqslant 4$.

THEOREM 5.1. *Let $B = [b_{11}]$. Then B is realizable if and only if b_{11} is a non-negative integer. $C(B) = b_{11}$.*

Proof. The necessity of the realizability condition is obvious. The remainder of the theorem follows from Theorem 4.1.

THEOREM 5.2. *Let B be an integral symmetric matrix of order 2. Then B is realizable if and only if the three inequalities of types (3.3) and (3.4) are satisfied. If B is realizable, $C(B) = b_{11} + b_{22} - b_{12}$.*

Proof. Necessity follows from Theorem 3.1. The remainder of the theorem follows from Theorem 4.1.

Now let B be a matrix of order 3. Let $M(B) = \max E \cdot B$ for $E \in \mathfrak{E}_3$, so that $M(B)$ is the largest of four numbers. We have

THEOREM 5.3. *Let B be an integral symmetric matrix of order 3. Then B is realizable if and only if the 12 inequalities of types (3.3), (3.4), and (3.5) are satisfied. If B is realizable, $C(B) = M(B)$.*

Proof. The necessity of the realizability conditions follows from Theorem 3.1. We shall establish their sufficiency and show that $C(B) = M(B)$ simultaneously.

Suppose first that $M(B) = E \cdot B$ where $E \in \mathfrak{E}_{31}$. Then the condition that $E \cdot B \geqslant E' \cdot B$ where $E' \in \mathfrak{E}_{32}$ gives

$$\begin{aligned} b_{11} &\geqslant b_{12} + b_{13}, \\ b_{22} &\geqslant b_{12} + b_{23}, \\ b_{33} &\geqslant b_{13} + b_{23}. \end{aligned} \tag{5.1}$$

It follows from Theorem 4.1 that B is realizable and that

$$C(B) = b_{11} + b_{22} + b_{33} - b_{12} - b_{13} - b_{23} = M(B).$$

Now let $M(B) = E \cdot B$, where $E \in \mathfrak{E}_{32}$. Because of the symmetry in the hypotheses and conclusions of Theorem (5.3) we may assume that

$$E = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.2}$$

Then the condition $E \cdot B \geqslant E' \cdot B$ where $E' \in \mathfrak{E}_{31}$ gives

$$b_{13} + b_{23} \geqslant b_{33} \tag{5.3}$$

while the condition that $E \cdot B \geqslant E'' \cdot B$ where $E'' \in \mathfrak{E}_{32}$, $E'' \neq E$, gives

$$\begin{aligned} b_{11} + b_{23} &\geqslant b_{12} + b_{33}, \\ b_{22} + b_{13} &\geqslant b_{12} + b_{33}. \end{aligned} \tag{5.4}$$

The system (3.1) consists, when $n = 3$, of three equations similar to

$$x_1 + x_{12} + x_{13} + x_{123} = b_{11}$$

and of three equations similar to

$$x_{12} + x_{123} = b_{12}.$$

These six equations in seven unknowns have the particular integral solution

$$\begin{aligned} x_1 &= b_{11} + b_{23} - b_{12} - b_{33}, \\ x_2 &= b_{22} + b_{13} - b_{12} - b_{33}, \\ x_3 &= 0, \\ x_{12} &= b_{12} + b_{33} - b_{13} - b_{23}, \\ x_{13} &= b_{33} - b_{23}, \\ x_{23} &= b_{33} - b_{13}, \\ x_{123} &= b_{13} + b_{23} - b_{33}. \end{aligned} \tag{5.5}$$

Now $x_1 \geqslant 0$ and $x_2 \geqslant 0$ because of (5.4), $x_{12} \geqslant 0$ because of (3.5) (after a permutation of subscripts), $x_{13} \geqslant 0$ and $x_{23} \geqslant 0$ because of (3.4) (after a permutation of subscripts), and $x_{123} \geqslant 0$ because of (5.3). (Henceforth, when referring to inequalities of types (3.3)–(3.7) we shall omit the remark "after a permutation of subscripts" in cases where it clearly applies.) It follows that B is realizable. Moreover, we have, from (5.5), that

$$\textstyle\sum_\tau x_\tau = b_{11} + b_{22} - b_{12} = E \cdot B.$$

Hence $C(B) \leqslant E \cdot B$. But $C(B) \geqslant E \cdot B$ from (4.12). Hence $C(B) = E \cdot B = M(B)$ when $E \in \mathfrak{E}_{32}$. Thus Theorem (5.3) is proved.

We introduce the symbol $\langle s \rangle$ to denote the smallest integer greater than or equal to the real number s. Suppose now that B is an integral matrix of order 4. Let $M(B) = \max \langle E \cdot B \rangle$, for $E \in \mathfrak{E}_4$, so that $M(B)$ is the largest of 16 numbers. (Evidently the symbol $\langle\ \rangle$ is required only when $E \in \mathfrak{E}_{45}$.)

THEOREM 5.4. *Let B be an integral symmetric matrix of order* 4. *Then B is realizable if and only if the* 40 *inequalities of types* (3.3), (3.4), (3.5), (3.6), *and* (3.7) *are satisfied. If B is realizable, $C(B) = M(B)$ unless $B = I_4 + J_4$. In this case $M(I_4 + J_4) = 4$ but $C(I_4 + J_4) = 5$.*

Proof. The method of proof is essentially that of Theorem 5.3. However, the details are more cumbersome and a somewhat different type of argument is required to handle matters when $E \in \mathfrak{E}_{45}$.

As before, the necessity of the realizability conditions follows from Theorem 3.1. Also, if B is realizable, then $C(B) \geqslant M(B)$ by (4.12). Hence we need only establish the sufficiency of the realizability conditions and the inequality $C(B) \leqslant M(B)$ for $B \neq I_4 + J_4$.

When $n = 4$ the system of equations (3.1) consists of four equations of the form

$$x_1 + x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} + x_{1234} = b_{11} \tag{5.6}$$

and of six equations of the form

$$x_{12} + x_{123} + x_{124} + x_{1234} = b_{12}. \tag{5.7}$$

We shall refer to the entire system of 10 equations in 15 unknowns as system A. Our task will be to show that if the inequalities (3.3)–(3.7) are satisfied, system A has a non-negative integral solution with $\sum_\tau x_\tau = M(B)$.

If $M(B) = E \cdot B$ where $E \in \mathfrak{E}_{41}$, the argument is the same as for the corresponding case when $n = 3$.

To treat the case $M(B) = E \cdot B$ where $E \in \mathfrak{E}_{42}$ we need the following lemmas:

LEMMA 5.1. *The system of m inequalities and one equation*

$$\alpha_i \leqslant z_i \leqslant \beta_i, \qquad i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} z_i = \gamma,$$

where α_i, β_i, and γ are integers, has an integral solution if and only if

$$\alpha_i \leqslant \beta_i, \qquad i = 1, 2, \ldots, m,$$

and

$$\sum_{i=1}^{m} \alpha_i \leqslant \gamma \leqslant \sum_{i=1}^{m} \beta_i.$$

Proof. The necessity is obvious. We establish the sufficiency by induction. The lemma is clear for $m = 1$. There is no loss in generality in assuming that $\beta_m - \alpha_m$ is minimal. We assert, that there exists a number θ in $[\alpha_m, \beta_m]$ such that $\gamma - \theta$ is in

$$\left[\sum_{i=1}^{m-1} \alpha_i, \sum_{i=1}^{m-1} \beta_i \right].$$

For otherwise, for all ξ in $[\alpha_m, \beta_m]$ either

$$\gamma - \xi < \sum_{i=1}^{m-1} \alpha_i$$

or

$$\gamma - \xi > \sum_{i=1}^{m-1} \beta_i.$$

In particular, we must have

$$\gamma - \alpha_m > \sum_{i=1}^{m-1} \beta_i,$$

since

$$\gamma - \alpha_m < \sum_{i=1}^{m-1} \alpha_i$$

implies

$$\gamma < \sum_{i=1}^{m} \alpha_i,$$

contradicting the hypothesis. Similarly

$$\gamma - \beta_m < \sum_{i=1}^{m-1} \alpha_i.$$

Thus

$$\gamma > \alpha_m + \sum_{i=1}^{m-1} \beta_i \quad \text{and} \quad \gamma < \beta_m + \sum_{i=1}^{m-1} \alpha_i$$

whence

$$\alpha_m + \sum_{i=1}^{m-1} \beta_i < \beta_m + \sum_{i=1}^{m-1} \alpha_i$$

so that

$$\sum_{i=1}^{m-1} (\beta_i - \alpha_i) < \beta_m - \alpha_m,$$

contradicting the minimality of $\beta_m - \alpha_m$. Set $z_m = [\theta]$. Then, since α_m, β_m, γ, $\sum \alpha_i$, and $\sum \beta_i$ are integers, we have $\alpha_m \leqslant z_m \leqslant \beta_m$ and

$$\sum_{i=1}^{m-1} \alpha_i \leqslant \gamma - z_m \leqslant \sum_{i=1}^{m-1} \beta_i.$$

The system

$$\alpha_i \leqslant z_i \leqslant \beta_i, \qquad i = 1, 2, \ldots, m-1,$$
$$\sum_{i=1}^{m-1} z_i = \gamma - z_m$$

has, by the inductive hypothesis, an integral solution and this solution leads immediately to an integral solution of the original system.

LEMMA 5.2. *The system*

$$\alpha_{ij} \leqslant z_i, \qquad i = 1, \ldots, m, \; j = 1, \ldots, r_i,$$
$$z_i \leqslant \beta_{ik}, \qquad i = 1, \ldots, m, \; k = 1, \ldots, s_i,$$
$$\sum z_i = \gamma$$

where α_{ij}, β_{ik}, and γ are integers has an integral solution if and only if

$$\alpha_{ij} \leqslant \beta_{ik}, \qquad i = 1, \ldots, m, \; j = 1, \ldots, r_i, \; k = 1, \ldots, s_i,$$

and

$$\sum \alpha_{i,j_i} \leqslant \gamma, \qquad \sum \beta_{i,k_i} \geqslant \gamma$$

for all choices of the integers $j_1, j_2, \ldots, j_r, k_1, k_2, \ldots, k_m$ *such that* $1 \leqslant j_i \leqslant r_i$, $1 \leqslant k_i \leqslant s_i$.

Proof. Setting $\alpha_i = \max_j \alpha_{ij}$, $\beta_i = \min_k \beta_{ik}$, we see that Lemma 5.2 follows immediately from Lemma 5.1.

Now suppose that $M(B) = E \cdot B$, where $E \in \mathfrak{E}_{42}$. With no loss in generality, we may assume that

$$E = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that

$$E \cdot B = b_{11} + b_{22} + b_{33} - b_{12} - b_{13} - b_{23}. \tag{5.8}$$

We seek solutions of system A with $x_4 = x_{123} = x_{1234} = 0$. Then system A is equivalent to the system

$$\begin{aligned} x_1 &= b_{11} - b_{12} - b_{13} - x_{14}, \\ x_2 &= b_{22} - b_{23} - b_{12} - x_{24}, \\ x_3 &= b_{33} - b_{13} - b_{23} - x_{34}, \\ x_{12} &= b_{44} + b_{12} - b_{14} - b_{24} - x_{34}, \\ x_{13} &= b_{44} + b_{13} - b_{14} - b_{34} - x_{24}, \\ x_{23} &= b_{44} + b_{23} - b_{24} - b_{34} - x_{14}, \\ x_{124} &= b_{14} + b_{24} - b_{44} + x_{34}, \\ x_{134} &= b_{14} + b_{34} - b_{44} + x_{24}, \\ x_{234} &= b_{24} + b_{34} - b_{44} + x_{14}, \\ x_{14} + x_{24} + x_{34} &= 2b_{44} - b_{14} - b_{24} - b_{34}. \end{aligned} \tag{5.9}$$

In order to apply Lemma 5.2, we put

$$\begin{aligned} \alpha_{11} &= 0, & \beta_{11} &= b_{11} - b_{12} - b_{13}, \\ \alpha_{12} &= b_{44} - b_{24} - b_{34}, & \beta_{12} &= b_{44} + b_{23} - b_{24} - b_{34}, \\ \alpha_{21} &= 0, & \beta_{21} &= b_{22} - b_{12} - b_{23}, \\ \alpha_{22} &= b_{44} - b_{14} - b_{34}, & \beta_{22} &= b_{44} + b_{13} - b_{14} - b_{34}, \\ \alpha_{31} &= 0, & \beta_{31} &= b_{33} - b_{13} - b_{23}, \\ \alpha_{32} &= b_{44} - b_{14} - b_{24}, & \beta_{32} &= b_{44} + b_{12} - b_{14} - b_{24}, \end{aligned}$$
$$\gamma = 2b_{44} - b_{14} - b_{24} - b_{34}.$$

If we put $z_1 = x_{14}$, $z_2 = x_{24}$, $z_3 = x_{34}$, we see that the system (5.9) has a solution in non-negative integers if the system

$$\begin{gathered} \alpha_{ij} \leqslant z_i, \quad i = 1, 2, 3, j = 1, 2, \\ z_i \leqslant \beta_{ik}, \quad i = 1, 2, 3, k = 1, 2, \\ z_1 + z_2 + z_3 = \gamma \end{gathered}$$

has a solution in integers. According to Lemma 5.2, this will be the case if

(a) $\alpha_{i1} \leqslant \beta_{i1}$, $i = 1, 2, 3$, or
(a′) $b_{11} \geqslant b_{12} + b_{23}$, $b_{22} \geqslant b_{12} + b_{23}$, $b_{33} \geqslant b_{13} + b_{23}$;
(b) $\alpha_{i1} \leqslant \beta_{i2}$, $i = 1, 2, 3$, or

(b′) $b_{44} + b_{23} \geqslant b_{24} + b_{34},\ b_{44} + b_{13} \geqslant b_{14} + b_{34},\ b_{44} + b_{12} \geqslant b_{14} + b_{24}$;
(c) $\alpha_{i2} \leqslant \beta_{i1},\ i = 1, 2, 3$, or
(c′) $b_{11} + b_{24} + b_{34} \geqslant b_{44} + b_{12} + b_{13}$,
$b_{22} + b_{14} + b_{34} \geqslant b_{44} + b_{12} + b_{23}$,
$b_{33} + b_{14} + b_{24} \geqslant b_{44} + b_{13} + b_{23}$;
(d) $\alpha_{i2} \leqslant \beta_{i2},\ i = 1, 2, 3$, or
(d′) $b_{23} \geqslant 0,\ b_{13} \geqslant 0,\ b_{12} \geqslant 0$;
(e) $\alpha_{11} + \alpha_{21} + \alpha_{31} \leqslant \gamma$, or
(e′) $2b_{44} \geqslant b_{14} + b_{24} + b_{34}$;
(f) $\alpha_{11} + \alpha_{21} + \alpha_{32} \leqslant \gamma,\ \alpha_{11} + \alpha_{22} + \alpha_{31} \leqslant \gamma,\ \alpha_{12} + \alpha_{21} + \alpha_{31} \leqslant \gamma$, or
(f′) $b_{44} \geqslant b_{34},\ b_{44} \geqslant b_{24},\ b_{44} \geqslant b_{14}$;
(g) $\alpha_{11} + \alpha_{22} + \alpha_{32} \leqslant \gamma,\ \alpha_{12} + \alpha_{21} + \alpha_{32} \leqslant \gamma,\ \alpha_{12} + \alpha_{22} + \alpha_{31} \leqslant \gamma$, or
(g′) $b_{14} \geqslant 0,\ b_{24} \geqslant 0,\ b_{34} \geqslant 0$;
(h) $\alpha_{12} + \alpha_{22} + \alpha_{32} \leqslant \gamma$, or
(h′) $b_{14} + b_{24} + b_{34} \geqslant b_{44}$;
(i) $\beta_{11} + \beta_{21} + \beta_{31} \geqslant \gamma$, or
(i′) $b_{11} + b_{22} + b_{33} + b_{14} + b_{24} + b_{34} \geqslant 2(b_{44} + b_{12} + b_{13} + b_{23})$;
(j) $\beta_{11} + \beta_{21} + \beta_{32} \geqslant \gamma,\ \beta_{11} + \beta_{22} + \beta_{31} \geqslant \gamma,\ \beta_{12} + \beta_{21} + \beta_{31} \geqslant \gamma$, or
(j′) $b_{11} + b_{22} + b_{34} \geqslant b_{44} + b_{12} + b_{13} + b_{23}$,
$b_{11} + b_{33} + b_{24} \geqslant b_{44} + b_{12} + b_{13} + b_{23}$,
$b_{22} + b_{33} + b_{14} \geqslant b_{44} + b_{12} + b_{13} + b_{23}$;
(k) $\beta_{11} + \beta_{22} + \beta_{32} \geqslant \gamma,\ \beta_{12} + \beta_{21} + \beta_{32} \geqslant \gamma,\ \beta_{12} + \beta_{22} + \beta_{31} \geqslant \gamma$, or
(k′) $b_{11} \geqslant b_{14},\ b_{22} \geqslant b_{24},\ b_{33} \geqslant b_{34}$;

and finally

(l) $\beta_{12} + \beta_{22} + \beta_{32} \geqslant \gamma$, or
(l′) $b_{44} + b_{12} + b_{13} + b_{23} \geqslant b_{14} + b_{24} + b_{34}$.

Now inequalities (b′), (d′), (f′), (g′), (k′), and (l′) follow from the realizability conditions (3.3)–(3.7). The remaining inequalities may be deduced from the fact that $M(B) = E \cdot B$ where E is given by (5.8). Thus the condition $E \cdot B \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{42}$, $E' \neq E$, gives the inequalities (c′). Similarly, the condition $E \cdot B \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{41}$, gives (h′); the condition $E \cdot B \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{43}$, gives (a′) and (j′); the condition $E \cdot B \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{44}$, gives (e′); and the condition $E \cdot B \geqslant E' \cdot \mathrm{B}$, where $E' \in \mathfrak{E}_{45}$, gives (i′). Thus, if $M(B) = E \cdot B$, system A has a solution in non-negative integers with $x_4 = x_{123} = x_{1234} = 0$. Thus B is realizable and it follows from (5.8) and (5.9) that $\sum_\tau x_\tau = E \cdot B = M(B)$.

To treat the case in which $M(B) = E \cdot B$, where $E \in \mathfrak{E}_{43}$, we require some additional lemmas.

LEMMA 5.3. *The inequalities*

$$\alpha_i \leqslant z_i \leqslant \beta_i, \qquad i = 1, 2, \ldots, m,$$
$$\nu \leqslant \sum z_i \leqslant \mu$$

where α_i, β_i, ν, μ are integers have an integral solution if and only if

$$\alpha_i \leqslant \beta_i, \qquad i = 1, 2, \ldots, m,$$
$$\nu \leqslant \mu,$$
$$\nu \leqslant \sum \beta_i, \qquad \sum \alpha_i \leqslant \mu.$$

Proof. The necessity is obvious. The hypotheses imply the existence of an integer γ such that $\sum \alpha_i \leqslant \gamma \leqslant \sum \beta_i$ and $\nu \leqslant \gamma \leqslant \mu$. The lemma follows from Lemma 5.1.

LEMMA 5.4. *The inequalities*

$$\begin{aligned} &\alpha_{ij} \leqslant z_i \leqslant \beta_{ij}, \qquad i = 1, \ldots, m, \qquad j = 1, \ldots, r,\\ &\nu_k \leqslant \sum z_i, \qquad k = 1, 2, \ldots, p,\\ &\sum z_i \leqslant \mu_l, \qquad l = 1, 2, \ldots, q, \end{aligned}$$

where α_{ij}, β_{ij}, ν_k, μ_l are integers, have an integral solution if and only if

$$\begin{aligned} &\alpha_{ij} \leqslant \beta_{ik}, \qquad i = 1, \ldots, m,\ j, k = 1, \ldots, r,\\ &\nu_k \leqslant \mu_l, \qquad k = 1, \ldots, p,\ l = 1, \ldots, q,\\ &\sum \alpha_{i,j(i)} \leqslant \mu_l, \quad \nu_k \leqslant \sum \beta_{i,j(i)}, \qquad k = 1, \ldots, p,\ l = 1, \ldots, q, \end{aligned}$$

where the integers $j(i)$ are chosen arbitrarily between 1 *and* r.

Proof. Lemma 5.4 follows from Lemma 5.3 in the same way that Lemma 5.2 follows from Lemma 5.1.

Now suppose that $M(B) = E \cdot B$ where $E \in \mathfrak{E}_{43}$. There is no loss in generality in assuming that

$$E = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that

$$M(B) = b_{11} + b_{22} - b_{12}. \tag{5.10}$$

We seek solutions of system A for which $x_3 = x_4 = x_{34} = 0$. Then system A reduces to

$$\begin{aligned} x_{12} &= b_{11} + b_{22} - b_{33} - b_{44} - b_{12} + b_{34} - (x_1 + x_2),\\ x_{13} &= b_{11} - b_{12} - b_{44} + b_{24} - x_1,\\ x_{14} &= b_{11} - b_{12} - b_{33} + b_{23} - x_1,\\ x_{23} &= b_{22} - b_{12} - b_{44} + b_{14} - x_2,\\ x_{24} &= b_{22} - b_{12} - b_{33} + b_{13} - x_2,\\ x_{123} &= b_{33} + 2b_{44} - b_{11} - b_{22} + 2b_{12} - b_{14} - b_{24} - b_{34} + (x_1 + x_2),\\ x_{124} &= 2b_{33} + b_{44} - b_{11} - b_{22} + 2b_{12} - b_{14} - b_{24} - b_{34} + (x_1 + x_2),\\ x_{134} &= b_{33} + b_{44} - b_{11} + b_{12} - b_{23} - b_{24} + x_1\\ x_{234} &= b_{33} + b_{44} - b_{22} + b_{12} - b_{13} - b_{14} + x_2,\\ x_{1234} &= b_{11} + b_{22} - 2b_{33} - 2b_{44} - 2b_{12} + b_{13} + b_{14} + b_{23} + b_{24}\\ &\qquad + b_{34} - (x_1 + x_2). \end{aligned} \tag{5.11}$$

In order to apply Lemma 5.3, we put

$$\begin{aligned}
\alpha_{11} &= b_{11} + b_{23} + b_{24} - b_{33} - b_{44} - b_{12}, \qquad \alpha_{12} = 0,\\
\alpha_{21} &= b_{22} + b_{13} + b_{14} - b_{33} - b_{44} - b_{12}, \qquad \alpha_{22} = 0,\\
\beta_{11} &= b_{11} - b_{12} - b_{44} + b_{24}, \qquad \beta_{12} = b_{11} - b_{12} - b_{33} + b_{23},\\
\beta_{21} &= b_{22} - b_{12} - b_{44} + b_{14}, \qquad \beta_{22} = b_{22} - b_{12} - b_{33} + b_{13},\\
\mu_1 &= b_{11} + b_{22} - b_{33} - b_{44} - b_{12} + b_{34},\\
\mu_2 &= b_{11} + b_{22} - 2b_{33} - 2b_{44} - 2b_{12} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34},\\
\nu_1 &= b_{11} + b_{22} + b_{14} + b_{24} + b_{34} - b_{33} - 2b_{44} - 2b_{12},\\
\nu_2 &= b_{11} + b_{22} + b_{13} + b_{23} + b_{34} - 2b_{33} - b_{44} - 2b_{12}.
\end{aligned}$$

If we put $z_1 = x_1$, $z_2 = x_2$, we see that the system (5.11) has a solution in non-negative integers if the system

$$\begin{aligned}
\alpha_{ij} \leqslant z_1, \quad z_i \leqslant \beta_{ik}, &\qquad i, j, k = 1, 2,\\
\nu_j \leqslant z_1 + z_2 \leqslant \mu_k, &\qquad j, k = 1, 2,
\end{aligned}$$

has an integral solution. According to Lemma 5.4, this will be the case if

(a) $\alpha_{11} \leqslant \beta_{11}$, $\alpha_{11} \leqslant \beta_{12}$ or
(a′) $b_{33} \geqslant b_{23}$, $b_{44} \geqslant b_{24}$;
(b) $\alpha_{21} \leqslant \beta_{21}$, $\alpha_{21} \leqslant \beta_{22}$ or
(b′) $b_{33} \geqslant b_{13}$, $b_{44} \geqslant b_{14}$;
(c) $\alpha_{12} \leqslant \beta_{11}$, $\alpha_{12} \leqslant \beta_{12}$, $\alpha_{22} \leqslant \beta_{21}$, $\alpha_{22} \leqslant \beta_{22}$ or
(c′) $b_{11} + b_{24} \geqslant b_{44} + b_{12}$, $b_{11} + b_{23} \geqslant b_{33} + b_{12}$,
$b_{22} + b_{14} \geqslant b_{44} + b_{12}$, $b_{22} + b_{13} \geqslant b_{33} + b_{12}$;
(d) $\nu_1 \leqslant \mu_1$, $\nu_2 \leqslant \mu_1$ or
(d′) $b_{44} + b_{12} \geqslant b_{14} + b_{24}$, $b_{33} + b_{12} \geqslant b_{13} + b_{23}$;
(e) $\nu_1 \leqslant \mu_2$, $\nu_2 \leqslant \mu_2$ or
(e′) $b_{13} + b_{23} \geqslant b_{33}$, $b_{14} + b_{24} \geqslant b_{44}$;
(f) $\nu_1 \leqslant \beta_{11} + \beta_{21}$, $\nu_2 \leqslant \beta_{12} + \beta_{22}$ or
(f′) $b_{44} \geqslant b_{34}$, $b_{33} \geqslant b_{34}$;
(g) $\nu_1 \leqslant \beta_{11} + \beta_{22}$, $\nu_2 \leqslant \beta_{12} + \beta_{21}$ or
(g′) $b_{44} + b_{13} \geqslant b_{14} + b_{34}$, $b_{33} + b_{14} \geqslant b_{13} + b_{34}$;
(h) $\nu_1 \leqslant \beta_{12} + \beta_{22}$, $\nu_2 \leqslant \beta_{11} + \beta_{21}$ or
(h′) $2b_{44} + b_{13} + b_{23} \geqslant b_{33} + b_{14} + b_{23} + b_{34}$,
$2b_{33} + b_{14} + b_{24} \geqslant b_{44} + b_{13} + b_{23} + b_{34}$;
(i) $\nu_1 \leqslant \beta_{12} + \beta_{21}$, $\nu_2 \leqslant \beta_{11} + \beta_{22}$ or
(i′) $b_{44} + b_{23} \geqslant b_{24} + b_{34}$, $b_{33} + b_{24} \geqslant b_{23} + b_{34}$;
(j) $\alpha_{11} + \alpha_{21} \leqslant \mu_1$ or
(j′) $b_{33} + b_{34} + b_{44} + b_{21} \geqslant b_{13} + b_{14} + b_{23} + b_{24}$;
(k) $\alpha_{11} + \alpha_{22} \leqslant \mu_1$, $\alpha_{21} + \alpha_{12} \leqslant \mu_1$ or
(k′) $b_{22} + b_{34} \geqslant b_{23} + b_{24}$, $b_{11} + b_{34} \geqslant b_{13} + b_{14}$;
(l) $\alpha_{12} + \alpha_{22} \leqslant \mu_1$ or
(l′) $b_{11} + b_{22} + b_{34} \geqslant b_{33} + b_{44} + b_{12}$;
(m) $\alpha_{11} + \alpha_{21} \leqslant \mu_2$ or
(m′) $b_{34} \geqslant 0$;

(n) $\alpha_{11} + \alpha_{22} \leqslant \mu_2$, $\alpha_{12} + \alpha_{21} \leqslant \mu_2$ or
(n′) $b_{22} + b_{13} + b_{14} + b_{34} \geqslant b_{33} + b_{44} + b_{12}$,
$b_{11} + b_{23} + b_{24} + b_{34} \geqslant b_{33} + b_{44} + b_{12}$;
(o) $\alpha_{12} + \alpha_{22} \leqslant \mu_2$ or
(o′) $b_{11} + b_{22} + b_{13} + b_{14} + b_{23} + b_{24} + b_{34} \geqslant 2(b_{33} + b_{44} + b_{12})$.

Now a′, b′, d′, f′, g′, i′, j′, k′, and m′ are consequences of (3.3)–(3.7) while the remaining conditions follow from the fact that

$$M(B) = E\cdot B = b_{11} + b_{22} + b_{12}.$$

Thus the condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{42}$, gives (e′) and (n′); the condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{43}$, $E' \neq E$, gives (c′) and (l′); the condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{44}$, gives (h′); and the condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{45}$, gives (o′). Thus system A has a non-negative integral solution with $x_3 = x_4 = x_{34} = 0$ and B is realizable. It follows from (5.10) and (5.11) that $\sum_\tau x_\tau = M(B)$.

Suppose next that $M(B) = E\cdot B$ where $E \in \mathfrak{E}_{44}$. There is no loss in generality in assuming that

$$E = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix}$$

so that

$$M(B) = b_{11} + b_{22} + b_{33} - 2b_{44} - b_{12} - b_{13} - b_{23} + b_{14} + b_{24} + b_{34}. \tag{5.12}$$

We seek an integral solution of system A for which $x_4 = x_{14} = x_{24} = x_{34} = x_{123} = 0$. This solution is readily found to be

$$\begin{aligned} x_1 &= b_{11} - b_{12} - b_{13} + b_{14} + b_{24} + b_{34} - 2b_{44}, \\ x_2 &= b_{22} - b_{12} - b_{23} + b_{14} + b_{24} + b_{34} - 2b_{44}, \\ x_3 &= b_{33} - b_{13} - b_{23} + b_{14} + b_{24} + b_{34} - 2b_{44}, \\ x_{12} &= b_{12} + b_{44} - b_{14} - b_{24}, \\ x_{13} &= b_{13} + b_{44} - b_{14} - b_{34}, \\ x_{23} &= b_{23} + b_{44} - b_{24} - b_{34}, \\ x_{124} &= b_{44} - b_{34}, \qquad x_{134} = b_{44} - b_{24}, \qquad x_{234} = b_{44} - b_{14}, \\ x_{1234} &= b_{14} + b_{24} + b_{34} - 2b_{44}. \end{aligned} \tag{5.13}$$

Here $x_{12} \geqslant 0$, $x_{13} \geqslant 0$, $x_{23} \geqslant 0$ by (3.5) and $x_{124} \geqslant 0$, $x_{134} \geqslant 0$, and $x_{234} \geqslant 0$ by (3.4). The condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{42}$, gives $x_{1234} \geqslant 0$ and the condition that $E\cdot B \geqslant E'\cdot B$, where $E' \in \mathfrak{E}_{43}$, gives $x_1 \geqslant 0$, $x_2 \geqslant 0$, $x_3 \geqslant 0$. Hence system A has a non-negative integral solution so that B is realizable. From (5.12) and (5.13) we calculate that $\sum_\tau x_\tau = M(B)$.

Finally, let us suppose that $M(B) = \langle E \cdot B \rangle$, where $E \in \mathfrak{E}_{45}$, so that

$$E = \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

and

$$(5.14) \quad M(B) = \langle N(B) \rangle = \langle E \cdot B \rangle = \langle \tfrac{2}{3}(b_{11} + b_{22} + b_{33} + b_{44}) \\ - \tfrac{1}{3}(b_{12} + b_{13} + b_{23} + b_{14} + b_{24} + b_{34}) \rangle.$$

Since this is the last case to be considered, we may assume that $M(B) > E' \cdot B$, i.e. strict inequality holds, where E' is any extreme matrix not in $\mathfrak{E}_{45}$. There are three possibilities:

(i) $M(B) = N(B)$,
(ii) $M(B) = N(B) + \frac{1}{3}$,
(iii) $M(B) = N(B) + \frac{2}{3}$.

In case (i) we seek a solution of system A with $x_1 = x_2 = x_3 = x_4 = x_{1234} = 0$. Such a solution, which for future reference we label x_{12}^*, x_{13}^*, etc., is

$$(5.15) \quad \begin{aligned} x_{12} &= x_{12}^* = N(B) - b_{33} - b_{44} + b_{34}, \\ x_{13} &= x_{13}^* = N(B) - b_{22} - b_{44} + b_{24}, \\ x_{14} &= x_{14}^* = N(B) - b_{22} - b_{33} + b_{23}, \\ x_{23} &= x_{23}^* = N(B) - b_{11} - b_{44} + b_{14}, \\ x_{24} &= x_{24}^* = N(B) - b_{11} - b_{33} + b_{13}. \\ x_{34} &= x_{34}^* = N(B) - b_{11} - b_{22} + b_{12}, \\ x_{123} &= x_{123}^* = N(B) - b_{11} - b_{22} - b_{33} + b_{12} + b_{13} + b_{23}, \\ x_{124} &= x_{124}^* = N(B) - b_{11} - b_{22} - b_{44} + b_{12} + b_{14} + b_{24}, \\ x_{134} &= x_{134}^* = N(B) - b_{11} - b_{33} - b_{44} + b_{13} + b_{14} + b_{34}, \\ x_{234} &= x_{234}^* = N(B) - b_{22} - b_{33} - b_{44} + b_{23} + b_{24} + b_{34}, \end{aligned}$$

This solution is integral since $N(B)$ is integral in case (i). Moreover, x_{123}, x_{124}, x_{134}, x_{234} are $\geqslant 0$ since $N(B) \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{42}$, and x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, and x_{34} are $\geqslant 0$ since $N(B) \geqslant E' \cdot B$, where $E' \in \mathfrak{E}_{43}$. Hence B is realizable, and from (5.14) and (5.15) one calculates that

$$\textstyle\sum_\tau x_\tau = N(B) = M(B).$$

Suppose next that (ii) holds, so that $N(B) + \frac{1}{3}$ is an integer. There is no loss in generality in assuming that $b_{11} \geqslant 1$. We seek a solution of system A with $x_1 = 1$, $x_2 = x_3 = x_4 = x_{1234} = 0$. System A now has the integral solution

$$(5.16) \quad \begin{aligned} x_{12} &= x_{12}^* - \tfrac{2}{3}, & x_{13} &= x_{13}^* - \tfrac{2}{3}, & x_{14} &= x_{14}^* - \tfrac{2}{3}, \\ x_{23} &= x_{23}^* + \tfrac{1}{3}, & x_{24} &= x_{24}^* + \tfrac{1}{3}, & x_{34} &= x_{34}^* + \tfrac{1}{3}, \\ x_{123} &= x_{123}^* + \tfrac{1}{3}, & x_{124} &= x_{124}^* + \tfrac{1}{3}, & x_{134} &= x_{134}^* + \tfrac{1}{3}, \\ & & x_{234} &= x_{234}^* - \tfrac{2}{3}. \end{aligned}$$

It follows as before that x_{23}, x_{24}, x_{123}, x_{124}, and x_{134} are non-negative. If $E' \in \mathfrak{E}_{42}$, then $M(B) = N(B) + \frac{1}{3} > E' \cdot B$ so that $N(B) + \frac{1}{3} \geqslant E \cdot B + 1$ or $N(B) \geqslant E' \cdot B + \frac{2}{3}$ from which it follows easily that $x_{234} \geqslant 0$. In a similar way, we may show that $x_{12} \geqslant 0$, $x_{13} \geqslant 0$, and $x_{14} \geqslant 0$. Thus B is realizable and from (5.14), (5.15), and (5.16) (remembering that $x_1 = 1$) we calculate that $\sum_\tau x_\tau = M(B)$.

If (iii) holds, then $N(B) + \frac{2}{3}$ is an integer. We shall investigate solutions of system A of the types

(a) $x_1 = 1, \quad x_2 = 1, \quad x_3 = x_4 = x_{1234} = 0,$
(b) $x_1 = 1, \quad x_2 = x_3 = x_4 = 0, \quad x_{1234} = 1.$

A solution of type (a) will have two of the x_i equal to 1, the other two zero, and $x_{1234} = 0$ while a solution of type (b) will have one of the x_i equal to 1, the others 0, and $x_{1234} = 1$.

An integral solution of type (a) with $x_1 = 1$ and $x_2 = 1$ is

$$(5.17)\quad \begin{aligned} x_{12} &= x_{12}^* - \tfrac{4}{3}, & x_{34} &= x_{34}^* + \tfrac{2}{3}, \\ x_{13} &= x_{13}^* - \tfrac{1}{3}, & x_{14} &= x_{14}^* - \tfrac{1}{3}, \quad x_{23} = x_{23}^* - \tfrac{1}{3}, \quad x_{24} = x_{24}^* - \tfrac{1}{3}, \\ x_{123} &= x_{123}^* + \tfrac{2}{3}, & x_{124} &= x_{124}^* + \tfrac{2}{3}, \\ x_{134} &= x_{134}^* - \tfrac{1}{3}, & x_{234} &= x_{234}^* - \tfrac{1}{3}. \end{aligned}$$

Arguing as before, we see that (5.17) gives a non-negative integral solution of system A with $\sum_\tau x_\tau = M(B)$ unless $x_{12}^* = \frac{1}{3}$. There will be a non-negative integral solution of type (a) to system A unless simultaneously we have

$$(5.18)\qquad x_{12}^* = x_{13}^* = x_{14}^* = x_{23}^* = x_{24}^* = x_{34}^* = \tfrac{1}{3}.$$

A solution of type (b) with $x_1 = 1$ and $x_{1234} = 1$ is

$$(5.19)\quad \begin{aligned} x_{12} &= x_{12}^* - \tfrac{1}{3}, & x_{13} &= x_{13}^* - \tfrac{1}{3}, & x_{14} &= x_{14}^* - \tfrac{1}{3}, \\ x_{23} &= x_{23}^* + \tfrac{2}{3}, & x_{24} &= x_{24}^* + \tfrac{2}{3}, & x_{34} &= x_{34}^* + \tfrac{2}{3}, \\ x_{123} &= x_{123}^* - \tfrac{1}{3}, & x_{124} &= x_{124}^* - \tfrac{1}{3}, & x_{134} &= x_{134}^* - \tfrac{1}{3}, \\ & & x_{234} &= x_{234}^* - \tfrac{4}{3}. \end{aligned}$$

Then, as before, (5.19) will yield a non-negative integral solution of system A unless $x_{234}^* = \frac{1}{3}$. There will be a solution of type (b) to system A unless we have simultaneously

$$(5.20)\qquad x_{123}^* = x_{124}^* = x_{134}^* = x_{234}^* = \tfrac{1}{3}.$$

Theorem 5.4 is proved except for the case in which (5.18) and (5.20) both hold. If (5.18) and (5.20) are inserted in (5.15), we have a system of linear equations in the elements of B. This system has the unique solution $b_{ii} = 2$, $b_{ij} = 1$, $i \neq j$, so that $b = I_4 + J_4$, the exceptional case of the theorem.

$I_4 + J_4$ is realizable by Corollary 2.2 and $M(I_4 + J_4) = 4$. But $C(I_4 + J_4)$ cannot be $\leqslant 4$ because the determinant of $I_4 + J_4$ is 5, which is not a square. Thus $C(I_4 + J_4) \geqslant 5$. But, by Corollary 2.2, $C(I_4 + J_5) \leqslant 5$; hence $C(I_4 + J_4) = 5$. This completes the proof of Theorem 5.4.

One may inquire whether a proper subset of the 40 realizability conditions (3.3)–(3.7) would be sufficient. We show that this is not the case by exhibiting symmetric integral matrices $B_{\sigma\rho}$ satisfying $f_{\sigma\rho}(B_{\sigma\rho}) < 0$ but $f_{\sigma'\rho'}(B_{\sigma\rho}) \geqslant 0$ if either $\sigma \neq \sigma'$ or $\rho \neq \rho'$. Here (σ, ρ), (σ', ρ') are taken from the 40 pairs of subsets of ω_4 used in (3.3)–(3.7). Clearly it is enough to produce one matrix for each of the five types of inequality:

$$\sigma = \emptyset,\ \rho = \{1, 2\},\ B_{\sigma\rho} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma = \{1\},\ \rho = \{2\},\ B_{\sigma\rho} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\sigma = \{1\},\ \rho = \{2, 3\},\ B_{\sigma\rho} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\sigma = \{1\},\ \rho = \{2, 3, 4\},\ B_{\sigma\rho} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma = \{1, 2\},\ \rho = \{3, 4\},\ B_{\sigma\rho} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 3 & 2 & 2 \\ 2 & 2 & 4 & 1 \\ 2 & 2 & 1 & 4 \end{bmatrix}$$

Similarly one may show that the realizability conditions in Theorems 5.2 and 5.3 are independent. Also, matrices B with $M(B) = E \cdot B$, where E is any given extreme matrix of order not exceeding 4, may be exhibited. Hence the portions of Theorems 5.2, 5.3, and 5.4 referring to $C(B)$ cannot be simplified.

6. (k, λ) Matrices.

THEOREM 6.1. *If $n \geqslant 3$, the extreme matrices in $\mathfrak{Y}_n$ of the form*

$$\alpha I_n + (\beta - \alpha) J_n$$

are the matrices

$$E_{n,r} = \{(2r - 1)I_n - J_n\}/r(r - 1) \tag{6.1}$$

where r is an integer and $2 \leqslant r \leqslant n - 1$.

Proof. Let $F = F(\alpha, \beta, n) = \alpha I_n + (\beta - \alpha)J_n$. If $g(t)$ is the sum of the elements in a principal submatrix of F of order t, then

$$g(t) = (\beta - \alpha)t^2 + \alpha t. \tag{6.2}$$

Suppose that F is admissible and that there exists an integer r such that $2 \leqslant r \leqslant n - 1$, $g(r - 1) < 1$, $g(r) = 1$, $g(r + 1) < 1$. We can represent F in the form $aE_{n,r} + bE_{n,r+1}$ since the determination of a and b requires the solution of a consistent system of two linear equations in two unknowns. A straightforward calculation shows that $g(r - 1) < 1$ implies that $b > 0$, $g(r) = 1$ implies that $a + b = 1$, and $g(r + 1) < 1$ implies that $a > 0$. Consequently, F is not extreme. If $g(1) = 1$, $g(2) < 1$, then $\beta = 1$, $\alpha < 3/2$, and, for $t \geqslant 2$, $g(t) < t^2 + 3/2(t - t^2) \leqslant 1$. Thus F has only n principal submatrices for which the element sum is 1 so that F is not extreme. Similarly, $g(n) = 1$, $g(n - 1) < 1$ is impossible if F is extreme. It follows that F is extreme only if there exists an integer r with $2 \leqslant r \leqslant n$ such that $g(r) = g(r - 1) = 1$. This implies, by a simple computation, that $F = E_{n,r}$. However, $E_{n,n}$ has only $r + 1$ principal submatrices with element sum 1 and so is not extreme if $n \geqslant 3$, since then

$$\binom{n+1}{2} > n + 1.$$

It is easy to see that $E_{n,r}$ is admissible. The element sum of a principal submatrix of $E_{n,r}$ of order t is given by

$$h(t) = \{(2r - 1)t - t^2\}/r(r - 1),$$

a quadratic function of t having its maximum at $t = r - \frac{1}{2}$. Thus the value of $h(t)$ cannot exceed its values at r and $r - 1$, which are both 1. (Note that it is essential that r be an integer for $E_{n,r}$ to be admissible.)

To prove that $E_{n,r}$ is extreme, we must show that the condition that all principal submatrices of orders $r - 1$ and r have element sum 1 determines $E_{n,r}$ uniquely. Let $Y = (y_{ij})$ be a symmetric matrix of order $n > r$ satisfying this condition. Then the elements of Y satisfy $\binom{n}{r-1}$ equations of the type

$$\sum_{i=1}^{r-1} y_{ii} + 2 \sum_{1 \leqslant i < j \leqslant r-1} y_{ij} = 1 \tag{6.3}$$

and $\binom{n}{r}$ equations of the type

$$\sum_{i=1}^{r} y_{ii} + 2 \sum_{1 \leqslant i < j \leqslant r} y_{ij} = 1. \tag{6.4}$$

Subtracting (6.3) from (6.4) we have

$$\sum_{i=1}^{r-1} y_{i,r} = -\tfrac{1}{2} y_{rr}.$$

By symmetry, the sum of any set of $r-1$ off-diagonal elements in the rth column of Y is $-\frac{1}{2}y_{rr}$. This implies that all these off-diagonal elements are equal, so that we have, for $i \neq r$,

$$y_{i,r} = -y_{r,r}/\{2(r-1)\}.$$

The same argument may be carried through for any column so that

$$y_{ij} = -y_{jj}/\{2(r-1)\}, \qquad \text{for } i \neq j.$$

But since Y is symmetric we may proceed in the same way with the rows of Y and deduce that

$$y_{ij} = -y_{ii}/\{2(r-1)\} \qquad \text{for } i \neq j.$$

Hence $y_{ii} = y_{jj} = \gamma$ and $y_{ij} = -\gamma/\{2(r-1)\}$ if $i \neq j$. Insertion in (6.3) gives $\gamma = 2/r$ so that $y_{ii} = 2/r$, $y_{ij} = -1/\{r(r-1)\}$ for $i \neq j$, and $Y = E_{n,r}$.

Let B be a (k, λ) matrix, $B = \lambda I_n + (k-\lambda)J_n$. If $k > \lambda$, then B is non-singular, since, as is well-known,

$$\det B = (k-\lambda)^{n-1}(k+(n-1)\lambda). \tag{6.5}$$

Let $C(B) = C(n, k, \lambda)$. An upper bound for $C(n, k, \lambda)$ is given by Corollary 2.2. We now establish a theorem giving a lower bound for $C(n, k, \lambda)$.

Let δ_{ij} be the Kronecker symbol. Consider the real linear programming problem:

$$\begin{gathered} x_\tau \geqslant 0, \\ \sum\nolimits_{\{i,j\}\subseteq\tau} x_\tau = \lambda + (k-\lambda)\delta_{ij}, \qquad 1 \leqslant i \leqslant j \leqslant n, \\ \sum\nolimits_{\tau\subseteq\omega_n} x_\tau = \text{minimum}. \end{gathered} \tag{6.6}$$

Let us call the value of the program $C^*(n, k, \lambda)$. Clearly

$$C(n, k, \lambda) \geqslant \langle C^*(n, k, \lambda)\rangle \tag{6.7}$$

since $C(n, k, \lambda)$ is the value of the integral program (6.6).

THEOREM 6.2.

$$C^*(n, k, \lambda) = \frac{n}{s+1}\left(2k - \frac{(n-1)\lambda}{s}\right),$$

where

$$s = \left[\frac{(n-1)\lambda}{s} + 1\right]. \tag{6.8}$$

Proof. Since $k \geqslant \lambda$, $1 \leqslant s \leqslant n$. Now,

$$C^*(n, k, \lambda) \geqslant E_{n,s+1} \,.\, B = \frac{n}{s+1}\left(2k - \frac{(n-1)\lambda}{s}\right).$$

https://doi.org/10.4153/CJM-1968-029-7 Published online by Cambridge University Press

On the other hand, a solution of (6.6) is

$$
(6.9)\qquad
\begin{aligned}
&x_\tau = 0, \qquad |\tau| \neq s,\ s+1,\\
&x_\tau = u = (ks - \lambda(n-1))\Big/\binom{n-1}{s-1}, \qquad |\tau| = s,\\
&x_\tau = v = (\lambda(n-1) - k(s-1))\Big/\binom{n-1}{s}, \qquad |\tau| = s+1.
\end{aligned}
$$

For, from (6.8), we have immediately that $x_\tau \geqslant 0$, for all $\tau \subseteq \omega_n$. Now, if $i \neq j$,

$$\sum_{\{i,j\}\subseteq\tau} x_\tau = \binom{n-2}{s-2}u + \binom{n-2}{s-1}v = \lambda$$

and

$$\sum_{i\in\tau} x_\tau = \binom{n-1}{s-1}u + \binom{n-1}{s}v = k,$$

so that (6.9) yields a solution of (6.6). Moreover, one can also deduce from (6.9) that

$$\sum x_\tau = \binom{n}{s}u + \binom{n}{s+1}v = \frac{n}{s+1}\left(2k - \frac{(n-1)\lambda}{s}\right).$$

Hence

$$C^*(n, k, \lambda) \leqslant \frac{n}{s+1}\left(2k - \frac{(n-1)\lambda}{s}\right).$$

Theorem (6.2) follows.

Note that the solution (6.9) is integral if $s = 1$ or if $s = n - 1$. Now $s = 1$ implies that $k \geqslant (n-1)\lambda$ and $s = n - 1$ implies that

$$k \leqslant \frac{n-1}{n-2}\lambda.$$

In these cases, $C(n, k, \lambda) = C^*(n, k, \lambda)$.

From Theorem (6.2) we obtain

COROLLARY 6.1. *If* $k \geqslant (n-1)\lambda$,

$$C(n, k, \lambda) = nk - \binom{n}{2}\lambda,$$

and if

$$k \leqslant \frac{n-1}{n-2}\lambda, \qquad C(n, k, \lambda) = 2k - \lambda.$$

The first portion of Corollary 6.1 could also have been deduced from Theorem 4.1.

In many combinatorial problems, k is a divisor of $(n-1)\lambda$. In particular, this occurs with the configuration known as a symmetric block design. A symmetric block design with parameters $n > k \geqslant \lambda$ is an incidence system consisting of n objects, n sets of these objects (called blocks) such that any block has k objects, any object belongs to k blocks, any two blocks have λ objects in common, and any two objects belong to λ blocks in common. (These properties are not independent.) A finite projective plane of order N is a symmetric block design with parameters $N^2 + N + 1$, $N + 1$, and 1.

COROLLARY 6.2. *If k divides $(n - 1)\lambda$, then*

$$C^*(n, k, \lambda) = \frac{nk^2}{k + (n - 1)\lambda}.$$

Proof. In this case

$$s = \frac{(n-1)\lambda}{k} + 1.$$

The corollary follows at once.

It is interesting to compare the lower bound $C^*(n, k, \lambda)$ for $C(n, k, \lambda)$ with the lower bound n (the rank of a (k, λ) matrix of order n if $k > \lambda$). From Corollary 6.2 we see at once that when k divides $(n - 1)\lambda$, $C^*(n, k, \lambda) \gtreqless n$ according as $k^2 - k \gtreqless (n - 1)\lambda$. Notice that in the case of equality, $k^2 - k = (n - 1)\lambda$. This is always true for symmetric block designs.

THEOREM 6.3. *A symmetric block design with parameters $n > k > \lambda$ exists if and only if $C(n, k, \lambda) = n$.*

Proof. If a design exists, there is a 0–1 matrix A of order n such that $AA^T = (k - \lambda)I_n + \lambda J_n$. Hence $C(n, k, \lambda) \leqslant n$. But $C(n, k, \lambda) \geqslant n$ since a (k, λ) matrix with $k > \lambda$ is non-singular. Thus $C(n, k, \lambda) = n$.

On the other hand, if $C(n, k, \lambda) = n$, there is a 0–1 matrix A of order n such that $AA^T = (k - \lambda)I_n + \lambda J_n$. Each row of A contains exactly k ones. Hence $AJ_n = kJ_n$. It follows from a theorem of Ryser **(9)** that $J_n A = kJ_n$ and $A^TA = (k - \lambda)I_n + \lambda J_n$. These imply that a design exists with parameters n, k, λ.

COROLLARY 6.3. *If $k^2 - k \neq (n - 1)\lambda$ or if n is odd and $k - \lambda$ is not a square, then $C(n, k, \lambda) > n$.*

The corollary follows from Theorem 6.3 and the remarks preceding it and from equation (6.5).

COROLLARY 6.4. *A projective plane of order N exists if and only if*

$$C(N^2 + N + 1, N + 1, 1) = N^2 + N + 1.$$

If a plane of order N does not exist, then

$$C(N^2 + N + 1, N + 1, 1) > N^2 + N + 1.$$

When n is large in comparison with k and λ, $C^*(n, k, \lambda)$ can give a very poor estimate of $C(n, k, \lambda)$. In fact, the crude upper bound of Corollary 2.2 actually gives $C(n, k, \lambda)$ in this situation. More precisely, we have

THEOREM 6.4. *If*

$$n > \binom{k}{\lambda}(k - \lambda) + 1 \qquad (\lambda > 0), \tag{6.10}$$

then $C(n, k, \lambda) = nk - (n - 1)\lambda$.

Proof. Let the matrix $(k - \lambda)I_n + \lambda J_n$ be the intersection matrix for a certain family of subsets $\{S_i\}$, $i = 1, 2, \ldots, n$, of a set, P. Let S_1 contain objects $o_1, o_2, \ldots, o_k$. Some set of λ of these objects, say $\{o_1, o_2, \ldots, o_\lambda\} = L$, must be contained in at least

$$t = \left\langle (n - 1) \Big/ \binom{k}{\lambda} \right\rangle$$

of the remaining sets. Let these sets be $S_2, S_3, \ldots, S_{t+1}$. Consider now any of the sets S_q where $q > t + 1$. Suppose that S_q contains at most a proper (possibly empty) subset M of L. Let $|M| = \mu$, where $0 \leqslant \mu < \lambda$. Then S_q must contain $\lambda - \mu$ objects from each of the sets $S_1 - L, S_2 - L, \ldots, S_{t+1} - L$ and these objects are distinct, since $S_i \cap S_j = L$ for $1 \leqslant i < j \leqslant t + 1$. Thus S_q contains at least $\mu + (t + 1)(\lambda - \mu)$ objects. It follows from (6.10) and the definition of t that

$$(t + 1) > k - \lambda + 1 = \max_{0 \leqslant \mu \leqslant \lambda - 1} \frac{k - \mu}{\lambda - \mu}$$

so that $\mu + (t + 1)(\lambda - \mu) > k$, contradicting $|S_q| = k$.

It follows that all sets S_i, $i = 1, \ldots, n$, contain L and have, therefore, no further objects in common. Thus

$$|P| \geqslant \left| \bigcup_{i=1}^{n} S_i \right| = \lambda + n(k - \lambda) = nk - (n - 1)\lambda.$$

Hence $C(n, k, \lambda) \geqslant nk - (n - 1)\lambda$. The theorem now follows from Corollary 2.2.

Note that when $k = N + 1$ and $\lambda = 1$,

$$\binom{k}{\lambda}(k - \lambda) + 1 = N^2 + N + 1,$$

indicating that finite projective planes play a critical role in the theory of content. We can see from Theorems 6.3 and 6.4 that $C(n, k, \lambda)$ may behave quite irregularly (though of course monotonically) as a function of n. Thus, since a plane of order 11 exists, $C(133, 12, 1) = 133$ but, according to Theorem 6.4, $C(134, 12, 1) = 1475$. It would be of considerable interest to know $C(N^2 + N + 1, N + 1, 1)$ when a plane of order N fails to exist.

We can now give an example of matrix B for which $C(2B) < C(B)$. Let $B = I_7 + J_7$. It follows from Theorem 6.4 that $C(B) = 8$. On the other hand $C(2B) = 7$. To see this, observe that here $k = 4$, $\lambda = 2$, $n = 7$. Consider a plane Π of order 2 with seven points and seven lines L_i, $i = 1, 2, \ldots, 7$. Let $S_i = \Pi - L_i$. Then $|S_i| = 7 - 3 = 4$ and $|S_i \cap S_j| = 2$ if $i \neq j$. Thus $C(2B) \leqslant 7$. But since the rank of $2B$ is 7 we have $C(2B) = 7$.

We turn to the case in which n is "small" in comparison with k and λ. First we have, as an immediate consequence of Theorems 5.1–5.4,

THEOREM 6.5. *$C(1, k, \lambda) = k$, $C(2, k, \lambda) = 2k - \lambda$. If $k \geqslant 2\lambda$, $C(3, k, \lambda) = 3k - 3\lambda$; if $2\lambda \geqslant k \geqslant \lambda$, $C(3, k, \lambda) = 2k - \lambda$. If $k \geqslant 3\lambda$, $C(4, k, \lambda) = 4k - 6\lambda$; if $3\lambda \geqslant k \geqslant 3\lambda/2$ and $(k, \lambda) \neq (2, 1)$, $C(4, k, \lambda) = 8/3k - 2\lambda$ while $C(4, 2, 1) = 5$; if $3\lambda/2 \geqslant k \geqslant \lambda$, then $C(4, k, \lambda) = 2k - \lambda$.*

An alternative formulation is

COROLLARY 6.5. *If $n \leqslant 3$, $C(n, k, \lambda) = C^*(n, k, \lambda)$. If $n = 4$, $C(n, k, \lambda) = C^*(n, k, \lambda)$ save when $k = 2$, $\lambda = 1$.*

Much of Theorem 6.5 can be deduced from Corollary 6.1.

Corollary 6.5 suggests

THEOREM 6.6. *For each n there exists a positive integer $D(n)$ such that $0 \leqslant C(n, k, \lambda) - C^*(n, k, \lambda) < D(n)$.*

Proof. We seek solutions of the constraints in (6.6) with

$$\begin{aligned} &x_\tau = t, && |\tau| = 1,\\ &x_\tau = u, && |\tau| = s, \text{ where } s \text{ is given by (6.8)},\\ &x_\tau = v, && |\tau| = s + 1,\\ &x_\tau = w, && \tau = \omega_n,\\ &x_\tau = 0, && \text{otherwise}. \end{aligned} \tag{6.11}$$

The equations in (6.6) become

$$\begin{aligned} t + \binom{n-1}{s-1}u + \binom{n-1}{s}v \quad w &= k,\\ \binom{n-2}{s-2}u + \binom{n-2}{s-1}v + w &= \lambda. \end{aligned} \tag{6.12}$$

Solving (6.8) for u and v in terms of t and w, we have

$$\begin{aligned} u &= ((k - w - t)s - (\lambda - w)(n-1)) \Big/ \binom{n-1}{s-1},\\ v &= ((\lambda - w)(n-1) - (k - w - t)(s-1)) \Big/ \binom{n-1}{s}. \end{aligned} \tag{6.13}$$

Let $m = s\binom{n-1}{s}$. If t and w are chosen so that

$$t \equiv k - \lambda \pmod{m}, \qquad w \equiv \lambda \pmod{m}, \tag{6.14}$$

then u and v will be integers. We now show that we can determine t and w so that u and v will also be non-negative.

If we set $ks - \lambda(n-1) = a$, then $1 \leqslant a \leqslant k$ by (6.8). The conditions $u \geqslant 0$, $v \geqslant 0$ may be written

$$\begin{aligned} w(n-1-s) - ts + a &\geqslant 0, \\ t(s-1) - w(n-s) + k - a &\geqslant 0. \end{aligned} \tag{6.15}$$

It is evident that the estimate in Theorem 6.6 need be established only for sufficiently large values of k. For since $\lambda \leqslant k$, only a finite number of matrices are excluded if we assume $k > K(n)$. Therefore we can assume that at least one of the numbers a/s, $(k-a)/(n-s)$ exceeds m. For otherwise we should have an upper bound for k.

Suppose $a/s \geqslant m$ and $(k-a)/(n-s) \geqslant m$. Let Q_1 be the square of side m in the (t, w) plane with vertices $(0, 0)$, $(0, m)$, $(m, 0)$, and (m, m). If we notice that m and s are bounded when n is fixed, we see that if $k > K(n)$, then (6.15) is satisfied at the vertices of Q_1 and hence throughout Q_1 since the solution set of (6.15) is convex. Thus we can find a point (t_0, w_0) in Q_1 with integral coordinates at which the values of u and v given by (6.13) will be non-negative integers. Note that $|t_0| \leqslant m$, $|w_0| \leqslant m$.

Now suppose that $a/s \leqslant m$ and $(k-a)/(n-s) \geqslant m$. In the (t, w) plane let Q_2 be the square of side m with vertices

$$\left(m, \frac{ms-a}{n-1-s}\right), \left(0, \frac{ms-a}{n-1-s}\right), \left(0, \frac{m(n-1)-a}{n-1-s}\right), \left(m, \frac{m(n-1)-a}{n-1-s}\right).$$

Then Q_2 is in the first quadrant. One can see, as before, with somewhat more computation, that at the vertices of Q_2 the inequalities (6.15) are satisfied. Hence they are satisfied throughout Q_2 and we can find a point (t_0, w_0) in Q_2 with integral coordinates at which the values of u and v given by (6.13) will be non-negative integers and for which $|t_0| \leqslant m$, $|w_0| \leqslant |m(n-1)|$.

Finally, suppose $a/s \geqslant m$ but $(k-a)/(n-s) \leqslant m$. We argue as above with the square Q_3 with vertices

$$\left(\frac{m(n-s)-(k-a)}{s-1}, 0\right), \quad \left(\frac{m(n-s)-(k-a)}{s-1}, m\right),$$
$$\left(\frac{m(n-1)-(k-a)}{s-1}, 0\right), \quad \left(\frac{m(n-1)-(k-a)}{s-1}, m\right),$$

and obtain a point (t_0, w_0) giving non-negative integral values for u and v from (6.13). In this case $|t_0| \leqslant m(n-1)/(s-1)$, $|w_0| \leqslant m$.

Thus equations (6.12) have a non-negative integral solution (t_0, u_0, v_0, w_0) with t_0 and w_0 bounded if n is fixed. It follows from (6.11) that

$$\sum x_\tau = nt_0 + \binom{n}{s} u_0 + \binom{n}{s+1} v_0 + w_0.$$

Eliminating u_0 and v_0 by (6.13), we have

$$\sum x_\tau = n\frac{s-1}{s+1}t_0 + \left(\frac{n(n-1-2s}{s(s+1)} + 1\right)w_0 + C^*(n, k, \lambda).$$

Since $C(n, k, \lambda) \leqslant \sum x_\tau$, the theorem follows.

A set of m points in a projective plane Π of order N is an m-arc if no three of its points are collinear. Clearly $m \leqslant N + 2$. For, suppose $m \geqslant N + 3$ and consider any point p on the arc. Then the $m - 1$ lines through p and the $m - 1$ other points of the arc are distinct. Since $m - 1 \geqslant N + 2$, we have a contradiction since there are only $N + 1$ lines through each point of Π. Qvist **(8)** has shown that if N is odd, $m \leqslant N + 1$. To complement this result, Bose **(2)** has shown that there are $(N + 2)$-arcs in desarguesian planes of even order.

We show how our theory of content provides an alternative proof of Qvist's theorem.

Let Σ be an $(N + 2)$-arc in a plane Π of odd order N. If p is any point of Σ, the lines joining p to the remaining $N + 1$ points of Σ are all distinct. These are all the lines through p. There are $\binom{N+2}{2}$ lines joining points of Σ. If we remove from Π the points of Σ and the lines joining them, we have a configuration Γ consisting of $N^2 - 1$ points and $\frac{1}{2}(N^2 - N)$ lines, with two lines of Γ intersecting in exactly one point. Hence $C(\frac{1}{2}(N^2 - N), N + 1, 1) \leqslant N^2 - 1$.

Let us calculate $C^*(\frac{1}{2}(N^2 - N), N + 1, 1)$ for odd N. By (6.8), we have $s = [\frac{1}{2}N] = \frac{1}{2}(N - 1)$, since N is odd. Theorem 6.2 then gives $C^*(\frac{1}{2}(N^2 - N), N + 1, 1) = N^2$. Since $C(\frac{1}{2}(N^2 - N), N + 1, 1) \geqslant C^*(\frac{1}{2}(N^2 - N), N+1, 1)$, we have a contradiction.

Observe that we do not obtain a contradiction when n is even. For in this case, $s = \frac{1}{2}N$ and $C^* = N^2 - 1$. (It is perhaps of some significance that our method, essentially based upon counting, is nevertheless able to exploit an arithmetical distinction.)

7. Representation of quadratic forms. If $D = (d_{ij})$ and $B = (b_{ij})$ are symmetric matrices over a field K of orders p and n respectively, where $p \geqslant n$, we say that D represents B over K if there exists an $n \times p$ matrix A over K such that $ADA^T = B$. Equivalently, one may say that the quadratic form $\sum\sum d_{ij}x_ix_j$ represents the form $\sum\sum v_{ij}y_iy_j$ over K. If B is a realizable matrix of order n and $C(B) = p \geqslant n$, then there is an $n \times p$ zero–one matrix A such that $AA^T = AI_pA^T = B$, so that I_p represents B over the rational field. Hasse **(7)**, basing himself upon earlier work of Minkowski, developed a theory of representation of quadratic forms over the rational field, and it was this theory that Bruck and Ryser **(3)** employed in their celebrated paper on the non-existence of finite projective planes of certain orders.

If q is prime, there is determined a certain invariant $c_q(B)$, called the Hasse symbol, which has values $+1$ or -1. The Hasse–Minkowski theory implies that I_p represents B over the rationals if and only if the following conditions hold:

(7.1) $p \geqslant n,$

(7.2) B is positive semi-definite;

(7.3) If $p = n$, $c_q(B) = 1$ if q is odd; $c_2(B) = -1$.

(7.4) If $p = n + 1$, $c_q(B) = 1$ if q is odd; $c_2(B) = -1$.

(7.5) If $p = n + 2$, $c_q(B) = 1$ if q is odd and $-\det B$ is a q-adic square; $c_2(B) = -1$ if $-\det B$ is a 2-adic square.

(7.2) is a necessary condition for realizability, but clearly not sufficient. Counterexample:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

It would be interesting to have a direct proof of the fact that the realizability conditions of Theorems 5.3 and 5.4 imply that B is positive semi-definite.

Conditions (7.3), (7.4), and (7.5) are useful when one suspects that $C(B) = n$, $n + 1$, or $n + 2$. Note that (7.3) and (7.4) are identical. In case $\det B$ is an integral square, (7.5) may be replaced by

(7.5′) If $p = n + 2$, $c_q(B) = 1$ if $q \equiv 1 \pmod 4$.

If n is odd and if there exists a symmetric block design with parameters n, k, λ, it was demonstrated by Chowla and Ryser **(4)**, without using the Hasse–Minkowski theory, that the equation

$$z^2 = (k - 1)x^2 + (-1)^{\frac{1}{2}(n-1)}\lambda y^2 \tag{7.6}$$

has an integral solution with $x \neq 0$.

If (7.6) fails to have a solution, then there exists a prime q with

$$c_q((k - \lambda)I_r + \lambda J_n) \neq c_q(I_n).$$

Hence, not only will we have $C(n, k, \lambda) \geqslant n + 1$, as is implied by Theorem 6.3, but also, from (7.4), $C(n, k, \lambda) \geqslant n + 2$. Thus we have

THEOREM 7.1. *If* (7.6) *fails to have a solution with* $x \neq 0$, *then*

$$C(n, k, \lambda) \geqslant n + 2.$$

A slight strengthening of the Bruck–Ryser theorem is given by

COROLLARY 7.1. *If* $N \equiv 1$ *or* $2 \bmod 4$ *and if N is not a sum of two squares, then* $C(N^2 + N + 1, N + 1, 1) \geqslant N^2 + N + 3$.

In this case we cannot use (7.5′) even though $\det B$ is a square, because the prime q for which $c_q(B) = -1$ satisfies $q \equiv 3 \pmod 4$.

8. A non-symmetric analogue. Let $\{S_i\}$, $i = 1, 2, \ldots, n$, and $\{T_j\}$, $j = 1, \ldots, m$, be two families of subsets of the finite set P with p elements. Suppose that A_1 is the incidence matrix for the family $\{S_i\}$ and that A_2 is the transpose of the incidence matrix for the family $\{T_j\}$. Then A_1 is an $n \times p$ matrix and A_2 is a $p \times m$ matrix. If $B = (b_{ij}) = A_1 A_2$, then B is an $n \times m$ matrix and $b_{ij} = |S_i \cap T_j|$.

Suppose, conversely, that B is a given matrix with non-negative integral elements. We may ask whether there exists zero–one matrices A_1 and A_2 such that $B = A_1 A_2$. In contrast with the symmetric problem we have been discussing, the answer is always affirmative; hence no question of realizability arises.

THEOREM 8.1. *Let B be a matrix with non-negative integral elements. Then there exist zero–one matrices A_1 and A_2 such that $B = A_1 A_2$.*

Proof. We construct families $\{S_i\}$ and $\{T_j\}$ of subsets of a sufficiently large finite set P such that $|S_i \cap T_j| = b_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Let $\{R_{ij}\}$ be a family of mutually disjoint subsets of P such that $|R_{ij}| = b_{ij}$. Let

$$S_i = \bigcup_{j=1}^{m} R_{ij} \quad \text{and} \quad T_j = \bigcup_{i=1}^{n} R_{ij}.$$

Then $S_i \cap T_j = R_{ij}$ so that $|S_i \cap T_j| = b_{ij}$.

As before, we may ask for the smallest value of p for which $B = A_1 A_2$ where A_1 is an $n \times p$ zero–one matrix and A_2 is a $p \times m$ zero–one matrix. In combinatorial terms, we ask for the smallest set P with two families of subsets $\{S_i\}$ and $\{T_j\}$ having the intersection patterns presented by B. We call this minimum value of p the non-symmetric content of B and denote it by $\bar{C}(B)$. We conclude with some observations about $\bar{C}(B)$ which parallel the remarks and theorems in §§ 2, 3, and 4.

The analogue of Theorem 2.1 holds and is proved in the same way; that is, $\bar{C}(B_1 + B_2) \leqslant \bar{C}(B_1) + \bar{C}(B_2)$. Again, it is obvious that $\bar{C}(B) \geqslant \operatorname{rank} B$ and that $\bar{C}(B) \geqslant \mu(B)$.

From our proof of Theorem 8.1 we can immediately deduce that

$$C(B) \leqslant \sum\sum b_{ij}.$$

A better upper bound is given by

THEOREM 8.2. *Let μ_j be the largest element in the jth column of B. Then*

$$\bar{C}(B) \leqslant \sum_{j=1}^{m} \mu_j(B).$$

Proof. From a set P with

$$p = \sum_{j=1}^{m} \mu_j(B)$$

objects, we may select a family of mutually disjoint subsets T_j, $j = 1, 2, \ldots, m$, with $|T_j| = \mu_j(B)$. Let R_{ij} be any subset of T_j with b_{ij} objects. Since $b_{ij} \leqslant \mu_j = |T_j|$, it is always possible to find such a subset. Let

$$S_i = \bigcup_{j=1}^{m} R_{ij}.$$

Then clearly $S_i \cap T_j = R_{ij}$ so that $|S_i \cap T_j| = b_{ij}$.

COROLLARY 8.1. *Let $\nu_i(B)$ be the largest element in the ith row of B. Then*

$$\bar{C}(B) \leqslant \sum_{i=1}^{n} \nu_i(B).$$

It is evident that the upper bound given by Theorem 8.2 is attained for diagonal matrices. As in the symmetric case, it is immediate that a row (or column) of zeros or one of two identical rows (or columns) may be removed from B without altering $\bar{C}(B)$. Also, if $\bar{B}$ is a submatrix of B, then

$$\bar{C}(\bar{B}) \leqslant \bar{C}(B).$$

The problem of determining $\bar{C}(B)$ may be formulated as a problem in integral linear programming. Let τ be an arbitrary non-empty subset of ω_n and let σ be an arbitrary non-empty subset of ω_m. Denote by $x_{\tau\sigma}$ the number of elements of P belonging to precisely those sets S_i for which $i \in \tau$ and precisely those sets T_j for which $j \in \sigma$. Then we have

$$b_{ij} = |S_i \cap T_j| = \sum_{i\in\tau,\ j\in\sigma} x_{\tau\sigma}, \tag{8.1}$$

the summation being over all pairs of subsets of ω_n and ω_m satisfying the stated conditions. $\bar{C}(B)$ is the minimum value of $\sum_{\tau,\sigma} x_{\tau\sigma}$ subject to the constraint (8.1) and the additional constraint $x_{\tau\sigma} \geqslant 0$, where, of course, we require that $x_{\tau\sigma}$ be integral.

Exactly as in the symmetric case we may formulate a dual problem. We are led to study admissible matrices, Z, which are now defined as matrices with n rows and m columns with the property that *all* submatrices have element-sum $\leqslant 1$. The inequality

$$\bar{C}(B) \geqslant Z \cdot B \qquad (Z \text{ admissible}) \tag{8.2}$$

is established by the same argument as before. Extreme matrices for the present problem are admissible matrices for which there is a set of mn independent equations stating that a certain set of mn submatrices have element sum 1.

For example, when $n = 2$ and $m = 2$, the extreme matrices are of the types

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

It follows easily that when $n = 2$ and $m = 2$,

$$\bar{C}(B) = \max(b_{11}, b_{12}, b_{21}, b_{22}, b_{11} + b_{22} - b_{12}, b_{11} + b_{22} - b_{21}, \\ b_{12} + b_{21} - b_{11}, b_{12} + b_{21} - b_{22})$$

so that $\bar{C}(B)$ is the largest of eight integers.

We may anticipate here that the sets of extreme matrices will have a more complicated structure than in the symmetric case, since we no longer restrict our attention to principal submatrices.

If B is symmetric and realizable, then clearly $\bar{C}(B) \leqslant C(B)$. Strict inequality may occur. Let

$$B_1 = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

It follows from Theorem 5.4 that B_1 is realizable and $C(B_1) = 8$. On the other hand, the intersection pattern is presented by B is obtained for the sets

$$S_1 = \{o_1, o_2, o_3, o_4\}, \qquad S_2 = \{o_1, o_2, o_5, o_6\},$$
$$S_3 = \{o_3, o_4, o_5, o_6\}, \qquad S_4 = \{o_3, o_5, o_7\},$$
$$T_1 = \{o_1, o_2, o_3, o_4, o_7\}, \qquad T_2 = \{o_1, o_2, o_5, o_6, o_7\},$$
$$T_3 = \{o_3, o_4, o_5, o_6\}, \qquad T_4 = \{o_1, o_3, o_6, o_7\},$$

so that $\bar{C}(B)_1 \leqslant 7$.

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