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Order and Spectrum Preserving Maps on Positive Operators

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Abstract. We describe the general form of surjective maps on the cone of all positive operators that preserve order and spectrum. The result is optimal as shown by counterexamples. As an easy consequence, we characterize surjective order and spectrum preserving maps on the set of all self-adjoint operators.

1 Introduction and Statement of the Main Results

The problem of characterizing linear invertibility preserving maps was posed by Kaplansky in his influential lecture notes [8]. He was motivated by the famous Gleason–Kahane–Żelazko theorem [5,7]. Over the last few decades a lot of papers have been devoted to invertibility preserving maps. We refer the reader to [1,2] for more information on Kaplansky's problem. Another well-studied linear preserver problem in operator theory is the one dealing with positive maps. The structure of completely positive linear maps is understood quite well, while it is known that linear maps that are assumed to be merely positive may have quite complicated behaviour even in the low-dimensional matrix cases; see for example [17, Chapter 8]. In [3] the authors were interested in linear maps that are both positive and preserve invertibility. They proved that such maps between C^*-algebras are Jordan homomorphisms if they are assumed to be surjective, while this may be false otherwise.

Recall that a unital linear map $\phi: \mathcal{A} \to \mathcal{B}$ between two unital Banach algebras preserves invertibility if $\phi(a)$ is invertible for every invertible $a \in \mathcal{A}$. It follows easily that for every $a \in \mathcal{A}$ we have $\sigma(\phi(a)) \subset \sigma(a)$. Indeed, for a complex number λ such that $\lambda \notin \sigma(a)$ the element $\lambda - a$ is invertible, and hence, by unitality and invertibility preserving property, $\lambda - \phi(a)$ is invertible, or equivalently, $\lambda \notin \sigma(\phi(a))$. A unital linear map $\phi: \mathcal{A} \to \mathcal{B}$ preserves invertibility in both directions if for every $a \in \mathcal{A}$, the element $\phi(a)$ is invertible in $\mathcal{B}$ if and only if a is invertible. Clearly, ϕ preserves invertibility in both directions if and only if it preserves spectrum, that is, $\sigma(\phi(a)) = \sigma(a)$ for every $a \in \mathcal{A}$.

The theory of linear preservers is by now well developed. Problems concerning general preservers, that is, not necessarily linear maps having certain preserving properties, seem to be more difficult. The study of such maps has been initiated because of applications in mathematical physics; for some recent results, see papers [9,10,12,14]

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or the book [11] and the references therein. In view of these results it seems natural to ask if the above-mentioned result of Choi, Hadwin, Nordgren, Radjavi, and Rosenthal [3] can be treated in the more general non-linear setting as well.

Let H be a complex Hilbert space. We denote by $\mathcal{B}(H)$ and $\mathcal{S}(H)$ the algebra of all bounded linear operators on H and the real linear space of all bounded self-adjoint linear operators on H, respectively. The symbol $\mathcal{S}_+(H)$ stands for the cone of all positive operators:

$$\mathcal{S}_+(H) = \{A \in \mathcal{S}(H) : \langle Ax, x\rangle \geq 0 \text{ for all } x \in H\}.$$

Note that $\mathcal{B}(H)$ is partially ordered by the relation $\leq$ defined by $A \leq B \Leftrightarrow B - A \in \mathcal{S}_+(H)$.

Let $\mathcal{V} \in \{\mathcal{B}(H), \mathcal{S}(H), \mathcal{S}_+(H)\}$. A map $\phi\colon \mathcal{V} \to \mathcal{V}$ preserves order if $\phi(A) \leq \phi(B)$ for every pair $A, B \in \mathcal{V}$ satisfying $A \leq B$, and it preserves spectrum if $\sigma(\phi(A)) = \sigma(A)$ for every $A \in \mathcal{V}$. Obviously, linear maps on $\mathcal{B}(H)$ are positive if and only if they preserve order. The same is true for real linear maps on $\mathcal{S}(H)$.

Choi, Hadwin, Nordgren, Radjavi, and Rosenthal [3] studied linear positive invertibility preserving maps on C^* algebras. The study of order and spectrum preserving maps without linearity assumption is much more difficult, and at present understanding the structure of such maps for general C^*-algebras seems out of reach. Here, we are interested in the simplest case when the underlying C^*-algebra is the full operator algebra $\mathcal{B}(H)$. But even in this simplest case, the behaviour of order and spectrum preserving maps can be quite wild in the absence of the linearity assumption. To see this observe that for every $A \in \mathcal{B}(H)$, we have the unique decomposition $A = M + iN$ where $M, N \in \mathcal{S}(H)$. Choose any bijective map $\xi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ with the property that for each $N \in \mathcal{S}(H)$ there exists a unitary operator U_N such that $\xi(N) = U_N N U_N^*$ (note that unitary similarity is an equivalence relation and equivalence classes are unitary orbits of self-adjoint operators; a bijective map $\xi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ has the requested property if and only if each unitary orbit is invariant under ξ). For each $N \in \mathcal{S}(H)$ there are infinitely many unitary operators U satisfying $\xi(N) = UNU^*$, but we choose and fix one and denote it by U_N. We define $\phi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\phi(M + iN) = U_N(M + iN)U_N^*.$$

We first prove that ϕ is a bijective map. To prove surjectivity, take any $B = M + iN \in \mathcal{B}(H)$. Then there is a unique $L \in \mathcal{S}(H)$ such that $\xi(L) = N = U_L L U_L^*$. Set $A = U_L^* M U_L + iL$ and observe that $\phi(A) = B$ to complete the verification of surjectivity, checking that ϕ is injective is trivial as well. Since for each $A \in \mathcal{B}(H)$, $\phi(A)$ is unitarily similar to A, the map ϕ preserves spectrum. Finally, to see that it preserves order, assume that $A = M_1 + iN_1 \leq M_2 + iN_2 = B$ and note that $B - A \in \mathcal{S}_+(H)$ yields that $N_1 = N_2 = N$. It is then clear that $\phi(A) = U_N A U_N^* \leq U_N B U_N^* = \phi(B)$.

As we have seen, the behaviour of a bijective order and spectrum preserving map $\phi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ on two different cosets $A + \mathcal{S}(H)$ and $B + \mathcal{S}(H)$ can be completely non-related. The reason is that any two operators that are comparable with respect to $\leq$ belong to the same coset. And hence, when studying order and spectrum preserving maps in the absence of the linearity assumption we need to consider such maps on each coset separately. Thus, it is natural to study such maps acting on $\mathcal{S}(H)$. But we will do even a little bit better. Namely, each spectrum preserving map

$\phi: \mathcal{S}(H) \to \mathcal{S}(H)$ maps positive operators to positive operators. We will start by describing the general form of surjective order and spectrum preserving maps defined on the cone of positive operators only. The structural result for such maps acting on $\mathcal{S}(H)$ will follow easily.

Theorem 1.1 *Let H be a Hilbert space and let $\phi: \mathcal{S}_+(H) \to \mathcal{S}_+(H)$ be a surjective order and spectrum preserving map. Then there exists a unitary or anti-unitary operator $U: H \to H$ such that $\phi(A) = UAU^*$ for every $A \in \mathcal{S}_+(H)$.*

The converse is obviously true. Some further remarks should be added here. The first one is that almost all general (non-linear) preserver results, where the conclusion is similar to the above statement, can be solved by a unified approach based on the reduction to the problem of characterizing adjacency preserving maps [13–15]. We tried very hard to use this idea but eventually gave up and were forced to find a completely different approach. Recall that two operators are said to be adjacent if one is a rank one perturbation of the other. It is well known that the spectral analysis of rank one perturbations is far from being trivial and we believe that this is the reason that the approach via adjacency preservers is not suitable for our problem treating spectrum preserving maps.

The second remark is that if we replace the assumption of preserving order by a stronger assumption of preserving order in both directions, then under the surjectivity assumption we can remove the spectrum preserving property and still get the nice description of such maps; see [11, Theorem 2.5.1]. But we need to emphasize that there is a huge difference between preserving order in both directions and preserving it in one direction only. Even in the simplest case, that is, linear maps on matrices, it is rather easy to describe the general form of linear maps preserving order in both directions, while we do not understand the structure of linear maps preserving order in one direction only.

The last remark is that the above theorem is optimal. Indeed, we show now with counterexamples that all of the assumptions are indispensable. The theorem describes surjective maps having two preserving properties. It is rather easy to verify that assuming just one of them is not enough to get the same conclusion. Recall the well-known fact that $f(t) = \sqrt{t}$ is an operator monotone function on $[0, \infty)$ (this fact is a special case of the celebrated Löwner-Heinz inequality). Hence, $A \mapsto A^{\frac{1}{2}}$ is a bijective order preserving map from $\mathcal{S}_+(H)$ onto itself that is obviously not spectrum preserving. Define an equivalence relation on $\mathcal{S}_+(H)$ in the following way: $A \sim B \Leftrightarrow \sigma(A) = \sigma(B)$. Any bijective map from $\mathcal{S}_+(H)$ onto itself that maps each of equivalence classes onto itself is spectrum preserving. Of course, such maps can have quite a wild behaviour that is far from the nice form given in the conclusion of the above theorem. Finally, we will show that the surjectivity assumption is indispensable. Indeed, let H be infinite-dimensional. Then H can be identified with the direct orthogonal sum of two copies of H. Hence, maps on $\mathcal{S}_+(H)$ can be considered as maps from $\mathcal{S}_+(H)$ to $\mathcal{S}_+(H \oplus H)$. Assume that $\varphi: \mathcal{S}_+(H) \to \mathcal{S}_+(H)$ is any order preserving map satisfying

$$\sigma(\varphi(A)) \subset \sigma(A)$$

for every $A \in \mathcal{S}_+(H)$. Then clearly, the map $\phi(A) = A \oplus \varphi(A)$, $A \in \mathcal{S}_+(H)$, is an injective spectrum and order preserving map from $\mathcal{S}_+(H)$ into $\mathcal{S}_+(H \oplus H)$. Even more, it preserves order in both directions; that is,

$$A \le B \iff \phi(A) \le \phi(B)$$

for every pair $A, B \in \mathcal{S}_+(H)$. Let us give a few examples of a map φ with the required properties. The first one is $\varphi(A) = \sup\{\langle Ax, x\rangle : x \in H,\ \|x\| = 1\}I$. The next one is given by $\varphi(A) = 0$ if $A \in \mathcal{S}_+(H)$ is of finite rank and $\varphi(A) = A$ otherwise. And finally, let k be a positive integer. Then the map $\varphi\colon \mathcal{S}_+(H) \to \mathcal{S}_+(H)$ defined by $\varphi(A) = 0$ if $A \in \mathcal{S}_+(H)$ is of rank at most k, and $\varphi(A) = A$ otherwise, has the required properties, too. In the case where H is finite-dimensional, $\dim H = n$, we can identify $\mathcal{S}_+(H)$ with H_n^+, the set of all positive $n \times n$ matrices. For $A \in H_n^+$ we define $\operatorname{diag}(\lambda_1(A), \dots, \lambda_n(A))$ to be the diagonal $n \times n$ matrix whose diagonal entries are eigenvalues of A (counting their multiplicities) arranged in decreasing order. Then the map $A \mapsto \operatorname{diag}(\lambda_1(A), \dots, \lambda_n(A))$ from H_n^+ to itself is spectrum and order preserving [6, Corollary 7.7.4]. But non-surjective spectrum and order preserving maps can have even "wilder" behaviour in the finite-dimensional case. We will illustrate this in the two-dimensional case. If $\dim H = 2$, then $\mathcal{S}_+(H)$ can be identified with H_2^+, the space of all 2×2 hermitian positive matrices. We denote by $\mathcal{P}_1 \subset H_2^+ = \mathcal{S}_+(H)$ the set of all projections of rank one. If $P \in \mathcal{P}_1$, it is positive, and therefore its $(1,1)$-entry and $(2,2)$-entry are both nonnegative. The trace of projection of rank one is 1. Using the fact that its determinant is zero, we conclude that

$$P = \begin{bmatrix} a & \sqrt{a(1-a)}e^{it} \\ \sqrt{a(1-a)}e^{-it} & 1-a \end{bmatrix}$$

for some $a \in [0,1]$ and some $t \in \mathbb{R}$. Define a map

$$P = \begin{bmatrix} a & \sqrt{a(1-a)}e^{it} \\ \sqrt{a(1-a)}e^{-it} & 1-a \end{bmatrix} \longmapsto \widetilde{P}$$

from $\mathcal{P}_1$ to itself by $\widetilde{P} = P$ if $a \le \frac{1}{2}$ and

$$\widetilde{P} = \begin{bmatrix} 1-a & \sqrt{a(1-a)}e^{it} \\ \sqrt{a(1-a)}e^{-it} & a \end{bmatrix}$$

otherwise. Hence, the map $P \mapsto \widetilde{P}$ either maps a projection of rank one to itself, or it interchanges its diagonal entries. Further, for $P \in \mathcal{P}_1$ we will denote by $P^\perp$ the orthogonal rank one projection $P^\perp = I - P$. Each $A \in H_2^+$ with eigenvalues $\lambda_1(A) \ge \lambda_2(A) \ge 0$ can be written as

$$A = \lambda_1(A)P + \lambda_2(A)P^\perp \tag{1.1}$$

for some $P \in \mathcal{P}_1$. Clearly, if $\lambda_1(A) > \lambda_2(A)$, then the spectral projection P in the representation (1.1) is uniquely determined. In the case when $\lambda_1(A) = \lambda_2(A) = \lambda$, A is a scalar operator $A = \lambda I$, and then $A = \lambda Q + \lambda Q^\perp$ for every $Q \in \mathcal{P}_1$. We define a map $\phi\colon H_2^+ \to H_2^+$ by

$$\phi(A) = \phi(\lambda_1(A)P + \lambda_2(A)P^\perp) = \lambda_1(A)\widetilde{P} + \lambda_2(A)\widetilde{P}^\perp.$$

Clearly, ϕ preserves the spectrum. The following is not entirely obvious.

Proposition 1.2 *The map ϕ preserves order.*

Hence, the map ϕ is an example of a non-surjective spectrum and order preserving map that is not of the standard form as in our main result.

It is easy to deduce the structural theorem for surjective order and spectrum preserving maps on $\mathcal{S}(H)$ from our main result.

Corollary 1.3 *Let H be a Hilbert space and let $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ be a surjective order and spectrum preserving map. Then there exists a unitary or anti-unitary operator $U\colon H \to H$ such that $\phi(A) = UAU^*$ for every $A \in \mathcal{S}(H)$.*

Let us conclude the introduction by emphasizing another difference between the linear and non-linear settings. When studying unital linear invertibility preserving maps $\phi\colon \mathcal{A} \to \mathcal{B}$ on C^*-algebras, it is enough to assume that ϕ preserves involution; that is, $\phi(a^*) = (\phi(a))^*$, $a \in \mathcal{A}$, to conclude that ϕ is positive. Indeed, if a unital linear invertibility preserving map ϕ is $*$-preserving, then it maps self-adjoint elements of $\mathcal{A}$ into self-adjoint elements of $\mathcal{B}$ and applying the fact that a self-adjoint element is positive if and only if its spectrum is contained in $[0, \infty)$, we see that ϕ maps positive elements to positive elements. Moreover, if we restrict to real-linear maps acting between the real-linear spaces of all self-adjoint elements of C^*-algebras, then every spectrum preserving map is automatically positive, and hence it preserves order. Clearly, this is far from being true in the non-linear setting where none of the two preserving properties (preserving spectrum and preserving order) yields the other one.

In particular, a direct consequence of Corollary 1.3 is that every real-linear surjective spectrum preserving map $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ is of the form $\phi(A) = UAU^*$, $A \in \mathcal{S}(H)$, for some unitary or anti-unitary operator $U\colon H \to H$.

2 Preliminary Results

For a positive operator C on H we denote by $\operatorname{Im} C$ and $\operatorname{Ker} C$ the image and the null space of C, respectively. If C is of finite rank, then $\operatorname{tr} C$ stands for the trace of C. We start with a well-known and simple lemma that will be used quite often throughout the proof of our main result.

Lemma 2.1 *Assume that $C, D \in \mathcal{S}_+(H)$ satisfy $C \le D$. Then $\overline{\operatorname{Im} C} \subset \overline{\operatorname{Im} D}$.*

Proof With respect to the orthogonal direct sum decomposition $H = \overline{\operatorname{Im} D} \oplus \operatorname{Ker} D$ the operator D has the matrix representation

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

From $C \le D$ it follows trivially that

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

for some C_1, and hence, $\operatorname{Im} C \subset \overline{\operatorname{Im} D}$. Thus, $\overline{\operatorname{Im} C} \subset \overline{\operatorname{Im} D}$. ∎

We have included the above lemma and the proof to avoid possible confusion. Namely, the analogous statement for the images of operators (without taking closures) does not hold. There exist positive operators C and D such that $C \le D$, but $\operatorname{Im} C \subset \operatorname{Im} D$ is not true.

The next lemma is probably well known, and as it is very easy to prove, we will just formulate it without giving the proof.

Lemma 2.2 *Let $A, B, C \in \mathcal{S}(H)$, $t \in \mathbb{R}$, and $x \in H$. Assume that $A \le B \le C$ and $Ax = Cx = tx$. Then $Bx = tx$.*

Lemma 2.3 *Let $P, Q, R \in \mathcal{S}_+(H)$ be projections of rank one satisfying $P \perp Q$, that is, $PQ = QP = 0$ and $R \le P + Q$. Then*

$$\alpha(P,R) = \max\{t \in [0,\infty) : tR \le Q + 2P\} = \frac{2}{2 - \operatorname{tr}(PR)}.$$

A remark should be added here. If $P = R$, then obviously the quantity $\alpha(P,R) = 2$ is independent of the choice of a rank one projection Q that is orthogonal to P. Otherwise, the rank one projection Q satisfying both $P \perp Q$ and $R \le P + Q$ is uniquely determined.

Proof Let $e_1, e_2 \in H$ be orthonormal vectors that span the images of Q and P, respectively. With respect to the orthogonal direct sum decomposition $H = \operatorname{span}\{e_1, e_2\} \oplus \{e_1, e_2\}^\perp$ the rank one projections Q and P have the following matrix representations:

$$Q = \begin{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $R \le Q + P$, we have

$$R = \begin{bmatrix} \left[\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right] & 0 \\ 0 & 0 \end{bmatrix}.$$

In the rest of the proof we will forget the bordering zeroes; that is, we will identify operators Q, P, R with their upper-left 2×2 corners. Since R is a projection of rank one, it is of the form

$$R = \begin{bmatrix} 1-a & * \\ * & a \end{bmatrix}$$

for some $a \in [0,1]$. For a nonnegative real t, we have $tR \le Q + 2P$; that is,

$$t\begin{bmatrix} 1-a & * \\ * & a \end{bmatrix} \le \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

if and only if for every invertible 2×2 matrix T we have $tTRT^* \le T(Q+2P)T^*$. With a choice

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

we get that for a nonnegative real t we have $tR \le Q + 2P$ if and only if

$$\begin{bmatrix} t(1-a) & * \\ * & \frac{t}{2}a \end{bmatrix} \le \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 2×2 matrix on the left-hand side is a hermitian rank one matrix and is therefore unitarily similar to a diagonal matrix. Consequently, the above inequality holds if and only if the trace of the 2×2 matrix on the left-hand side is less than or equal 1, or equivalently,

$$t\left(1-\frac{a}{2}\right)\leq 1.$$

Applying the obvious fact that $a=\mathrm{tr}(PR)$, we conclude the proof. ■

Lemma 2.4 *Let $\mathcal{P}_1\subset H_2^+$ be the set of all projections of rank one. Assume that $R,Q,R_1,Q_1\in\mathcal{P}_1$ satisfy $\mathrm{tr}(RQ)\leq\mathrm{tr}(R_1Q_1)$. Let t,s_1,s_2 be nonnegative real numbers with $s_1\geq s_2$. Assume further that $tR\leq s_1Q+s_2Q^\perp$. Then $tR_1\leq s_1Q_1+s_2Q_1^\perp$.*

Proof The case when $t=0$ is trivial. So, assume that $t>0$. We first consider the case when $s_2=0$. Then clearly, $tR\leq s_1Q$ yields that $R=Q$ and $t\leq s_1$. From $R=Q$ and $\mathrm{tr}(RQ)\leq\mathrm{tr}(R_1Q_1)$ we get that $\mathrm{tr}(R_1Q_1)=1$, or equivalently, $R_1=Q_1$, which yields the desired inequality.

Hence, assume from now on that $s_2>0$. We will show that the condition

$$tR\leq s_1Q+s_2Q^\perp$$

is equivalent to

$$\mathrm{tr}(RQ)\,t\,\frac{s_1-s_2}{s_1s_2}\geq\frac{t-s_2}{s_2}.$$

Assume for a moment that we have already verified this. Then it is obvious that the conclusion of our lemma holds true.

So, assume that $tR\leq s_1Q+s_2Q^\perp$. After replacing the standard basis with an appropriate orthonormal basis of $\mathbb{C}^2$, we have

$$t\begin{bmatrix}a & *\\ * & 1-a\end{bmatrix}=tR\leq\begin{bmatrix}s_1 & 0\\ 0 & s_2\end{bmatrix},$$

which is, by the same argument as used in the previous lemma, equivalent to

$$t\begin{bmatrix}\frac{a}{s_1} & *\\ * & \frac{1-a}{s_2}\end{bmatrix}\leq\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

This is true if and only if the trace of the rank one matrix on the left-side is no larger than one. A straightforward computation shows that this happens if and only if

$$\mathrm{tr}(RQ)\,t\,\frac{s_1-s_2}{s_1s_2}\geq\frac{t-s_2}{s_2}.$$

The proof is completed. ■

Lemma 2.5 *Let $P\mapsto\widetilde{P}$ be the map from $\mathcal{P}_1$ to itself defined by*

$$P=\begin{bmatrix}a & \sqrt{a(1-a)}e^{it}\\ \sqrt{a(1-a)}e^{-it} & 1-a\end{bmatrix}\longmapsto\widetilde{P},$$

where $\widetilde{P}=P$ if $a\leq\frac{1}{2}$ and

$$\widetilde{P}=\begin{bmatrix}1-a & \sqrt{a(1-a)}e^{it}\\ \sqrt{a(1-a)}e^{-it} & a\end{bmatrix},$$

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otherwise.

Then for every pair $P, Q \in \mathcal{P}_1$, we have

$$\operatorname{tr}(\widetilde{P}\widetilde{Q}) \geq \operatorname{tr}(PQ).$$

Proof Let

$$P = \begin{bmatrix} a & \alpha \\ \overline{\alpha} & 1-a \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} b & \beta \\ \overline{\beta} & 1-b \end{bmatrix}$$

with $|\alpha|^2 = a(1-a)$ and $|\beta|^2 = b(1-b)$. Of course, if $P = \widetilde{P}$ and $Q = \widetilde{Q}$, then

$$\operatorname{tr}(\widetilde{P}\widetilde{Q}) = \operatorname{tr}(PQ).$$

If

$$\widetilde{P} = \begin{bmatrix} 1-a & \alpha \\ \overline{\alpha} & a \end{bmatrix} \quad \text{and} \quad \widetilde{Q} = \begin{bmatrix} 1-b & \beta \\ \overline{\beta} & b \end{bmatrix},$$

then again $\operatorname{tr}(\widetilde{P}\widetilde{Q}) = \operatorname{tr}(PQ)$, as can be verified by a straightforward computation. It remains to consider the case when $P = \widetilde{P}$ and $Q \neq \widetilde{Q}$, and the case when $P \neq \widetilde{P}$ and $Q = \widetilde{Q}$. Because of symmetry, we need to treat just one of the cases, say the first one. Thus, assume that $a \leq \frac{1}{2}$ and $b > \frac{1}{2}$ and

$$\widetilde{P} = P \quad \text{and} \quad \widetilde{Q} = \begin{bmatrix} 1-b & \beta \\ \overline{\beta} & b \end{bmatrix}.$$

Then

$$\operatorname{tr}(\widetilde{P}\widetilde{Q}) = a(1-b) + (1-a)b + \alpha\overline{\beta} + \overline{\alpha}\beta,$$

while

$$\operatorname{tr}(PQ) = ab + (1-a)(1-b) + \alpha\overline{\beta} + \overline{\alpha}\beta.$$

But then

$$\begin{aligned}\operatorname{tr}(\widetilde{P}\widetilde{Q}) - \operatorname{tr}(PQ) &= a(1-b) + (1-a)b - ab - (1-a)(1-b) \\ &= (2a-1)(1-2b) \geq 0,\end{aligned}$$

as desired. ∎

3 Proofs of Main Results

Proof of Theorem 1.1 There is nothing to prove if $\dim H = 1$. So, assume from now on that $\dim H \geq 2$. For $A \in \mathcal{S}_+(H)$ and $t \in [0, \infty)$ we have by the spectral theorem for self-adjoint operators that $\sigma(A) = \{t\}$ if and only if $A = tI$. Hence, by the spectrum preserving property we conclude that $\phi(tI) = tI$ for every non-negative real number t. We denote by $\mathcal{P}$ the set of all non-trivial projections on H, $\mathcal{P} = \{P \in \mathcal{S}_+(H) : P^2 = P \text{ and } P \neq 0, I\}$. Clearly, for every $A \in \mathcal{S}_+(H)$ we have the equivalence: $\sigma(A) = \{0,1\} \Leftrightarrow A \in \mathcal{P}$. Moreover, ϕ is surjective, and therefore $\phi(\mathcal{P}) = \mathcal{P}$. As always, $\operatorname{rank} A$ stands for the dimension of the image of A, $\operatorname{rank} A \in \mathbb{N} \cup \{0, \infty\}$. Our first claim is that $\operatorname{rank} \phi(A) \geq \operatorname{rank} A$ for every $A \in \mathcal{S}_+(H)$. Indeed, it is enough to prove that for every positive integer n and every $A \in \mathcal{S}_+(H)$, the inequality $\operatorname{rank} A \geq n$ yields that $\operatorname{rank} \phi(A) \geq n$. So, assume that $\operatorname{rank} A \geq n$. Then, by the spectral theorem for bounded self-adjoint operators we can find $B \in \mathcal{S}_+(H)$ such that $B \leq A$ and the spectrum of B contains n different positive real numbers. It follows that $\phi(B) \leq \phi(A)$ and the spectrum of $\phi(B)$ contains n different positive real numbers. Thus, $\operatorname{rank} \phi(B) \geq n$, and by Lemma 2.1, $\operatorname{rank} \phi(A) \geq n$.

Our next claim is that if A and B are positive operators both having a finite spectrum, $\operatorname{Im} A \cap \operatorname{Im} B = \{0\}$, and A_1 and B_1 are positive operators such that $\phi(A_1) = A$ and $\phi(B_1) = B$, then $\operatorname{Im} A_1 \cap \operatorname{Im} B_1 = \{0\}$. Assume on the contrary that $\operatorname{Im} A_1 \cap \operatorname{Im} B_1 \neq \{0\}$. Let $x \in \operatorname{Im} A_1 \cap \operatorname{Im} B_1$ be a nonzero vector and R the rank one projection on its linear span. Since A and B are both with finite spectra, the same is true for A_1 and B_1. Therefore, we have

$$A_1 = \sum_{j=1}^{m} t_j P_j \quad \text{and} \quad B_1 = \sum_{k=1}^{n} s_k Q_k$$

for some positive real numbers $t_1, \dots, t_m, s_1, \dots, s_n$, some pairwise orthogonal non-zero projections $P_1, \dots, P_m$, and some pairwise orthogonal non-zero projections $Q_1, \dots, Q_n$. Set $t = \min\{t_1, \dots, t_m, s_1, \dots, s_n\}$ and use the fact that $tR \leq A_1, B_1$ to conclude that $\phi(tR) \leq A$ and $\phi(tR) \leq B$. Clearly, $\phi(tR) \neq 0$. Because A and B have finite spectra, their images are closed, and hence, by Lemma 2.1, we have $\operatorname{Im} \phi(tR) \subset \operatorname{Im} A \cap \operatorname{Im} B$, a contradiction.

In the next step we will prove that if Q_α, $\alpha \in J$, is a family of projections such that $\overline{\sum_{\alpha \in J} \operatorname{Im} Q_\alpha} = H$, and P_α, $\alpha \in J$, are projections such that $\phi(P_\alpha) = Q_\alpha$, $\alpha \in J$, then

$$\overline{\textstyle\sum_{\alpha \in J} \operatorname{Im} P_\alpha} = H. \tag{3.1}$$

Indeed, if (3.1) does not hold, then let R be the projection whose image is $\overline{\sum_{\alpha \in J} \operatorname{Im} P_\alpha}$. Since $R \neq 0, I$, we have $\sigma(R) = \{0, 1\}$. Moreover, for every $\alpha \in J$, we have $P_\alpha \leq R$. Consequently, $Q_\alpha \leq \phi(R)$ and because $\sigma(\phi(R)) = \{0, 1\}$, $\phi(R)$ is a projection with a non-trivial null space. But using Lemma 2.1 once more, $H = \overline{\sum_{\alpha \in J} \operatorname{Im} Q_\alpha} \subset \operatorname{Im} \phi(R)$, a contradiction. From now on we will use Lemma 2.1 without even mentioning it.

Our next goal is to prove that for every finite rank projection Q there exists a projection P such that $\phi(P) = Q$ and $\operatorname{rank} Q = \operatorname{rank} P$. We only need to consider the case where $\operatorname{rank} Q = k \neq 0$. Then we can write

$$Q = Q_1 + \cdots + Q_k$$

where $Q_1, \dots, Q_k$ are pairwise orthogonal rank one projections. Set $B = Q_1 + 2Q_2 + \cdots + kQ_k$. By surjectivity, there exists $A \in \mathcal{S}_+(H)$ such that $\phi(A) = B$, and by the spectrum-preserving property and spectral theorem for self-adjoint operators, we have

$$A = P_1 + 2P_2 + \cdots + kP_k$$

for some non-zero pairwise orthogonal projections $P_1, \dots, P_k$. Define $P = P_1 + \cdots + P_k$ and note that P is a projection satisfying $P \leq A$. It follows that $\phi(P) \leq B$. We know that

$$k \leq \operatorname{rank} P \leq \operatorname{rank} \phi(P) \leq \operatorname{rank} B = k,$$

and therefore, $\operatorname{rank} P = \operatorname{rank} \phi(P) = k$. Hence, both $\phi(P)$ and Q are rank k projections whose images are subspaces of the image of B, and since $\operatorname{rank} B = k$, we have $\phi(P) = Q$, as desired.

Now we will show that every finite rank projection is mapped by ϕ into a projection of the same rank. So, let P be a finite rank projection of rank m, $m \neq 0$. There is nothing to prove if $\dim H = m$, that is, $P = I$. Hence, we assume that $\dim H > m$.

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We denote $\phi(P) = Q$ and we already know that Q is a non-trivial projection with $\dim \operatorname{Im} Q \geq m$. The surjectivity yields the existence of a non-trivial projection P_1 such that $\phi(P_1) = I - Q$. By what we have already proved, we know that

$$\operatorname{Im} P \cap \operatorname{Im} P_1 = \{0\} \quad \text{and} \quad \overline{\operatorname{Im} P + \operatorname{Im} P_1} = H.$$

Since $\operatorname{Im} P$ is finite-dimensional, the sum of subspaces $\operatorname{Im} P + \operatorname{Im} P_1$ is closed, and therefore,

$$H = \operatorname{Im} P \oplus \operatorname{Im} P_1.$$

In particular, the codimension of $\operatorname{Im} P_1$ in H is m.

Assume now that $\dim \operatorname{Im} Q > m$ (of course, this includes the possibility that Q is not of a finite rank). Then we can find a projection Q_1 of rank $m+1$ such that $Q_1 \leq Q$. By the previous statement, we can find a projection R_1 of rank $m+1$ such that $\phi(R_1) = Q_1$. Because $\operatorname{Im} Q_1 \cap \operatorname{Im}(I - Q) = \{0\}$, we have $\operatorname{Im} R_1 \cap \operatorname{Im} P_1 = \{0\}$, contradicting the fact that the dimension of $\operatorname{Im} R_1$ is $m+1$ and the codimension of $\operatorname{Im} P_1$ in H is m.

We denote by $\mathcal{P}_k$ the set of all projections of rank k, $k = 1, 2, \ldots$ The last two claims yield that

$$\phi(\mathcal{P}_k) = \mathcal{P}_k, \quad k = 1, 2, \ldots$$

Let t be any positive real number and let P be any finite rank projection. We claim that $\phi(tP) = tQ$ for some projection Q with $\operatorname{rank} Q = \operatorname{rank} P$. To prove this we consider the map $\phi_t : \mathcal{S}_+(H) \to \mathcal{S}_+(H)$ defined by $A \mapsto \frac{1}{t}\phi(tA)$, $A \in \mathcal{S}_+(H)$. This is obviously a surjective order and spectrum preserving map, and hence, everything that has been proved so far for the map ϕ holds for ϕ_t as well. In particular, $\phi_t(P)$ is a projection having the same rank as P. This is exactly what we wanted to prove.

Actually, more is true. Namely, for any positive real number t and any finite rank projection P, we have $\phi(tP) = t\phi(P)$. Clearly, this is true when $t = 1$. We will verify only the case when $t > 1$, since the case when $t < 1$ goes through in an almost identical way. We know that $\phi(tP) = tQ$ for some projection Q with $\operatorname{rank} Q = \operatorname{rank} P = \operatorname{rank} \phi(P)$. From $P \leq tP$, it follows that $\phi(P) \leq tQ$, and consequently, $\operatorname{Im} \phi(P) \subset \operatorname{Im} Q$, which further yields that $\phi(P) = Q$, as desired.

We are now ready to verify that $\operatorname{rank} \phi(A) = \operatorname{rank} A$ for every finite rank operator $A \in \mathcal{S}_+(H)$. It has already been proved that $\operatorname{rank} \phi(A) \geq \operatorname{rank} A$. To prove the opposite inequality we apply the fact $A \leq \|A\| P$, where P is the projection on the image of A. Then $\phi(A) \leq \phi(\|A\| P)$, and since $\operatorname{rank} \phi(\|A\| P) = \operatorname{rank} P = \operatorname{rank} A$, we are done.

Our next claim is that for every pair $P, R \in \mathcal{P}_1$, we have

$$\operatorname{tr}\big(\phi(P)\phi(R)\big) \geq \operatorname{tr}(PR). \tag{3.2}$$

There is nothing to prove if $P = R$. So assume that $P \neq R$. Then there is a unique rank two projection S such that $P, R \leq S$. Set $Q = S - P$. Then Q is a projection of rank one such that $P \perp Q$ and $R \leq P + Q$.

Set $A = \phi(Q + 2P)$. Then A is of rank two, and if $\dim H \geq 3$, we have $\sigma(A) = \sigma(Q + 2P) = \{0, 1, 2\}$. If $\dim H = 2$, then $\sigma(A) = \sigma(Q + 2P) = \{1, 2\}$. In both cases it follows that $A = T_1 + 2T_2$, where T_1 and T_2 is an orthogonal pair of projections of rank one. Furthermore, we have

$$2\phi(P) = \phi(2P) \leq \phi(Q + 2P) = A = T_1 + 2T_2 \leq 2I,$$

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and using Lemma 2.2 we conclude that $T_2 = \phi(P)$. Hence, $A = T_1 + 2\phi(P)$, where T_1 is a rank one projection orthogonal to the rank one projection $\phi(P)$.

Denote $a = \operatorname{tr}(PR) \in [0,1]$ and set $b = \frac{2}{2-a}$. By Lemma 2.3 we have $bR \leq Q + 2P$, and therefore, $b\phi(R) \leq T_1 + 2\phi(P)$. In particular,

$$\operatorname{Im}\phi(R) = \operatorname{Im} b\phi(R) \subset \operatorname{Im}\big(T_1 + 2\phi(P)\big) = \operatorname{Im}\big(T_1 + \phi(P)\big),$$

or equivalently, $\phi(R) \leq T_1 + \phi(P)$. Applying Lemma 2.3 once more we conclude that

$$\frac{2}{2 - \operatorname{tr}(PR)} = b \leq \alpha\big(\phi(P), \phi(R)\big) = \frac{2}{2 - \operatorname{tr}(\phi(P)\phi(R))}$$

from which we immediately get (3.2).

We will now restrict our attention to the case where $\dim H \geq 3$ and introduce a new map $\varphi\colon \mathcal{P}_1 \to \mathcal{P}_1$. For each $Q \in \mathcal{P}_1$ there exists $P \in \mathcal{P}_1$ such that $\phi(P) = Q$. Choose such a P and define $\varphi(Q) = P$ (note that we already know that such a P exists, but so far we have not verified that it is uniquely determined). We claim that if $Q_1, Q_2 \in \mathcal{P}_1$ are orthogonal rank one projections, then $\varphi(Q_1) = P_1$ and $\varphi(Q_2) = P_2$ must be orthogonal, too. Indeed, by (3.2) we have

$$0 \leq \operatorname{tr}(P_1P_2) \leq \operatorname{tr}\big(\phi(P_1)\phi(P_2)\big) = \operatorname{tr}(Q_1Q_2) = 0,$$

and therefore $\operatorname{tr}(P_1P_2) = 0$ which yields that $P_1 \perp P_2$, as desired.

The following slight extension of Wigner-Uhlhorn theorem can be found in [16, Proposition 2.6]. Let $\xi\colon \mathcal{P}_1 \to \mathcal{P}_1$ be an injective map. Assume that for every maximal orthogonal subset $\mathcal{S}$ of $\mathcal{P}_1$, the set $\xi(\mathcal{S})$ is a maximal orthogonal set of rank one projections. Then there exists an either unitary or antiunitary operator $U\colon H \to H$ such that $\xi(P) = UPU^*$ for every $P \in \mathcal{P}_1$. Note that the assumption $\dim H \geq 3$ is essential for this proposition.

We will now verify that φ satisfies the assumptions of this statement. Clearly, it is injective, and as it preserves orthogonality, it maps every maximal orthogonal subset $\mathcal{S}$ of $\mathcal{P}_1$ in an orthogonal set of projections of rank one. Using (3.1) we see that $\varphi(\mathcal{S})$ is maximal as well.

Hence, we can apply the above statement for the map φ. In particular, we see that ϕ maps $\mathcal{P}_1$ bijectively onto itself, and after composing ϕ with a unitary or anti-unitary congruence, we can assume with no loss of generality that we have $\phi(P) = P$ for every $P \in \mathcal{P}_1$. In order to complete the proof, we need to show that $\phi(A) = A$ for every $A \in \mathcal{S}_+(H)$.

We already know that this is true when $A = tP$, where t is any nonnegative real number and P any projection of rank one. Next, let P be a projection of rank one, t any positive real number, and c any real number $c \in [0,t]$. We will show that $\phi(A) = A$ for $A = tI - cP$. Let Q be any rank one projection orthogonal to P. Then $tQ \leq tI - cP \leq tI$, and consequently, $tQ \leq \phi(tI - cP) \leq tI$. It follows from Lemma 2.2 that $\phi(tI - cP) = t(I-P) + dP$ for some real number d. But

$$\{t, t-c\} = \sigma(tI - cP) = \sigma(t(I-P) + dP) = \{t, d\},$$

and therefore $d = t - c$, as desired.

Now take an arbitrary nonzero $A \in \mathcal{S}_+(H)$ with a finite spectrum. Then $A = \sum_{j=1}^k t_jP_j$ for some positive pairwise distinct real numbers $t_1, \dots, t_k$ and some nonzero pairwise orthogonal projections $P_1, \dots, P_k$. We need to show that for every $j \in$

$\{1,\dots,k\}$, and for every nonzero $x \in \operatorname{Im} P_j$ we have $\phi(A)x = t_j x$, while for every $x \in H$ that is orthogonal to $\operatorname{Im} P$ we have $\phi(A)x = 0$. Here, $P = P_1 + \cdots + P_k$. To prove the first part we take the rank one projection Q whose image is spanned by x, set $t = \max t_j$, observe that

$$t_j Q \le A \le tI - (t - t_j)Q,$$

and then we get the desired conclusion by using known facts $\phi(t_j Q) = t_j Q$ and $\phi(tI - (t - t_j)Q) = tI - (t - t_j)Q$ together with Lemma 2.2. To prove the second assertion we use $A \le t(I - R)$ for every rank one projection R that is orthogonal to P.

For every $A \in \mathcal{S}_+(H)$ and every $\varepsilon > 0$ we can find positive operators A_1, A_2 such that both A_1 and A_2 have finite spectra, $A_1 \le A \le A_2$, and $A_2 - A_1 \le \varepsilon I$. From

$$A_1 = \phi(A_1) \le \phi(A) \le \phi(A_2) = A_2$$

and from the fact that ε was an arbitrary positive real number, we finally conclude that $\phi(A) = A$ for every $A \in \mathcal{S}_+(H)$. This concludes the proof in the case where $\dim H \ge 3$.

Hence, it remains to consider the two-dimensional case.

Thus, we assume that $\dim H = 2$. But then $\mathcal{S}_+(H)$ can be identified with H_2^+, the set of all positive 2×2 matrices. Hence, ϕ is a surjective order and spectrum preserving map from H_2^+ onto itself. As before, we denote by $\mathcal{P}_1$ the set of all 2×2 projections of rank one. We know that $\phi(\mathcal{P}_1) = \mathcal{P}_1$ and that (3.2) holds. We will prove even more, namely, that for every pair $P, R \in \mathcal{P}_1$ we have

$$\operatorname{tr}\big(\phi(P)\phi(R)\big) = \operatorname{tr}(PR). \tag{3.3}$$

Assume on the contrary that $\operatorname{tr}(\phi(P)\phi(R)) > \operatorname{tr}(PR)$ for some $P, R \in \mathcal{P}_1$. Without loss of generality we may, and we will assume that

$$\phi(P) = P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have

$$R = \begin{bmatrix} a & * \\ * & 1-a \end{bmatrix}$$

for some $a \in [0,1]$ and because of $\operatorname{tr}(\phi(P)\phi(R)) > \operatorname{tr}(PR)$,

$$\phi(R) = \begin{bmatrix} b & * \\ * & 1-b \end{bmatrix}$$

with $b > a$. By surjectivity, there exists $Q \in \mathcal{P}_1$ such that $\phi(Q) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and since $\operatorname{tr}(\phi(P)\phi(Q)) = 0$, the inequality (3.2) yields that $\operatorname{tr}(PQ) = 0$ which further yields that $Q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. But then

$$\operatorname{tr}\big(\phi(Q)\phi(R)\big) = 1 - b < 1 - a = \operatorname{tr}(QR),$$

a contradiction.

It would be now possible to deduce from (3.3) that ϕ maps $\mathcal{P}_1$ bijectively onto itself and then we could apply classical Wigner's theorem. However, a non-bijective version of Wigner's theorem is now available (for a very short proof, see [4]), and due to this theorem, there exists a 2×2 unitary matrix U such that either for every $P \in \mathcal{P}_1$ we have $\phi(P) = UPU^*$, or for every $P \in \mathcal{P}_1$ we have $\phi(P) = UP^tU^*$. Here P^t stands for the transpose of P.

One can now complete the proof as in the case when $\dim H \geq 3$. In fact, in the two-dimensional case the rest of the proof is even shorter and simpler. ∎

Remark The elaborate proof we have presented could be substantially shortened if we assume that the map ϕ is injective.

Proof of Proposition 1.2 Let $A = \lambda_1(A)P + \lambda_2(A)P^\perp$ and $B = \lambda_1(B)Q + \lambda_2(B)Q^\perp$ be 2×2 hermitian positive matrices satisfying $A \leq B$. Then, by [6, Corollary 7.7.4], we know that

$$0 \leq \lambda_2(A) \leq \lambda_1(A) \leq \lambda_1(B) \quad \text{and} \quad \lambda_2(A) \leq \lambda_2(B).$$

Our assumption that

$$\lambda_1(A)P + \lambda_2(A)P^\perp \leq \lambda_1(B)Q + \lambda_2(B)Q^\perp$$

is equivalent to

$$\big(\lambda_1(A) - \lambda_2(A)\big)P \leq \big(\lambda_1(B) - \lambda_2(A)\big)Q + \big(\lambda_2(B) - \lambda_2(A)\big)Q^\perp.$$

Indeed, we have obtained the second inequality by subtracting $\lambda_2(A)I$ on both sides of the first one.

We need to prove that

$$\lambda_1(A)\widetilde{P} + \lambda_2(A)\widetilde{P}^\perp \leq \lambda_1(B)\widetilde{Q} + \lambda_2(B)\widetilde{Q}^\perp,$$

or equivalently,

$$\big(\lambda_1(A) - \lambda_2(A)\big)\widetilde{P} \leq \big(\lambda_1(B) - \lambda_2(A)\big)\widetilde{Q} + \big(\lambda_2(B) - \lambda_2(A)\big)\widetilde{Q}^\perp.$$

The last inequality follows directly from our assumption and Lemmas 2.4 and 2.5. ∎

Proof of Corollary 1.3 Let a be any nonnegative real number. We define $\mathcal{S}_a(H) = \{A \in \mathcal{S}(H) : A \geq -aI\} = \{A \in \mathcal{S}(H) : \sigma(A) \subset [-a, \infty)\}$. In particular, $\mathcal{S}_0(H) = \mathcal{S}_+(H)$. Because $\phi\colon \mathcal{S}(H) \to \mathcal{S}(H)$ is a surjective spectrum preserving map, it maps every $\mathcal{S}_a(H)$ onto itself. The map $X \mapsto X - aI$ is a bijective order preserving map from $\mathcal{S}_+(H)$ onto $\mathcal{S}_a(H)$, whose inverse $X \mapsto X + aI$ preserves order as well. Using the spectral mapping theorem, we see that for every $a \geq 0$ the map $\phi_a\colon \mathcal{S}_+(H) \to \mathcal{S}_+(H)$ defined by

$$\phi_a(A) = \phi(A - aI) + aI, \quad A \in \mathcal{S}_+(H),$$

is a surjective order and spectrum preserving map. Hence, by Theorem 1.1, for each $a \geq 0$ there exists a unitary or anti-unitary operator $U_a\colon H \to H$ such that

$$\phi_a(A) = U_a A U_a^*$$

for every $A \in \mathcal{S}_+(H)$. After composing ϕ with a unitary or anti-unitary similarity we can assume that $U_0 = I$, that is, $\phi(A) = A$ for every $A \in \mathcal{S}_+(H)$.

Let $A \in \mathcal{S}(H)$ be any operator satisfying $A \geq -aI$. Then

$$U_a(A + aI)U_a^* = \phi_a(A + aI) = \phi(A) + aI$$

yields that $\phi(A) = U_a A U_a^*$. In particular, $B = \phi(B) = U_a B U_a^*$ for every $B \in \mathcal{S}_+(H)$. Using standard arguments we conclude that $U_a = zI$ for some complex number z of modulus one, which further implies that ϕ_a is the identity. Hence, $\phi(A) = A$ for every $A \in \mathcal{S}(H)$, as desired. ∎

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