

NOTE ON A PARTITION THEOREM

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1. Introduction. In [1, p. 130] the following partition theorem was deduced from a general theorem concerning the limit of a recurrent sequence.

THEOREM. Let $r \geq 2$ be an integer. Let $P_1(n)$ denote the number of partitions of n into parts that are either even and not congruent to $4r-2 \pmod{4r}$ or odd and congruent to $2r-1, 4r-1 \pmod{4r}$. Let $P_2(n)$ denote the number of partitions of n of the form $n = b_1 + \dots + b_s$, where $b_i \geq b_{i+1}$, and for b_i odd, $b_i - b_{i+1} \geq 2r-1$ ($1 \leq i \leq s$, where $b_{s+1} = 0$). Then $P_1(n) = P_2(n)$.

Now considering the generating function for $P_1(n)$, we have

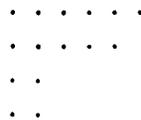
$$\begin{aligned} 1 + \sum_{n=1}^{\infty} P_1(n)q^n &= \prod_{j=1}^{\infty} (1 - q^{4rj-2})(1 - q^{2j})^{-1}(1 - q^{2rj-1})^{-1} \\ &= \prod_{j=1}^{\infty} (1 + q^{2rj-1})(1 - q^{2j})^{-1} \\ &\equiv 1 + \sum_{n=1}^{\infty} P_3(n)q^n, \end{aligned} \tag{1}$$

where $P_3(n)$ is the number of partitions of n into parts that are either even or else congruent to $2r-1 \pmod{2r}$ with the further restriction that only even parts may be repeated. Thus $P_1(n) = P_3(n)$.

The object of this note is to provide a simple combinatorial proof of the fact that $P_2(n) = P_3(n)$; equation (1) then yields the above theorem.

2. Proof that $P_2(n) = P_3(n)$. We provide a one-to-one correspondence between the sets of partitions to be counted.

Let π be a partition of the type enumerated by $P_2(n)$. Then represent π graphically with each even part $2m$ represented by two rows of m nodes and each odd part $2m+1$ represented by two rows of $m+1$ nodes and m nodes respectively. For example, $11+4$ becomes



Now read the graph vertically with the proviso that r columns are always to be grouped as a single part whenever the lowest node in the most right-hand column of the group comes from what was originally the right-hand-most node contributed by an odd part. Thus in our

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example with $r = 2$, we obtain in this manner the partition $4+4+2+2+3$. Now since the condition on partitions enumerated by $P_2(n)$ is $b_i - b_{i+1} \geq 2r - 1$ whenever b_i is odd, we see that our groupings of r columns always have one less node than a rectangle of $r \times 2v$ nodes; thus a part congruent to $2r - 1 \pmod{2r}$ is produced. Since originally odd parts were distinct, we see that now odd parts will be congruent to $2r - 1 \pmod{2r}$ and will not be repeated, and since originally all odd parts were greater or equal to $2r - 1$, we see that there will always be r columns available for each grouping. Thus in this way we have produced a partition of the type enumerated by $P_3(n)$. Clearly our correspondence is one-to-one into; however, the above process is reversible and thus the correspondence is onto. Hence $P_2(n) = P_3(n)$.

As an example we take $r = 2$, $n = 11$. The corresponding partitions are listed opposite each other in the following table

P_2	P_3
11	$2+2+2+2+3$
$9+2$	$4+2+2+3$
$8+3$	$7+2+2$
$7+4$	$4+4+3$
$7+2+2$	$6+2+3$
$6+5$	$7+4$
$5+2+2+2$	$8+3$
$4+4+3$	11

REFERENCES

1. G. E. Andrews, On Schur's second partition theorem, *Glasgow Math. J.* **8** (1967), 127–132.

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