NOTE ON A PARTITION THEOREM

by GEORGE E. ANDREWS†

(Received 17 February, 1969)

1. Introduction. In [**1**, p. 130] the following partition theorem was deduced from a general theorem concerning the limit of a recurrent sequence.

THEOREM. *Let* $r \geqq 2$ *be an integer. Let* $P_1(n)$ *denote the number of partitions of n into parts that are either even and not congruent to* $4r-2 \pmod{4r}$ *or odd and congruent to* $2r-1$, $4r-1 \pmod{4r}$. *Let* $P_2(n)$ *denote the number of partitions of n of the form* $n = b_1 + \ldots + b_s$, *where* $b_i \geqq b_{i+1}$, *and for* b_i *odd,* $b_i - b_{i+1} \geqq 2r-1$ $(1 \leqq i \leqq s$, *where* $b_{s+1} = 0)$. *Then* $P_1(n) = P_2(n)$.

Now considering the generating function for $P_1(n)$, we have

$$1 + \sum_{n=1}^{\infty} P_1(n) q^n = \prod_{j=1}^{\infty} (1-q^{4rj-2})(1-q^{2j})^{-1}(1-q^{2rj-1})^{-1} \tag{1}$$

$$= \prod_{j=1}^{\infty} (1+q^{2rj-1})(1-q^{2j})^{-1}$$

$$\equiv 1 + \sum_{n=1}^{\infty} P_3(n) q^n,$$

where $P_3(n)$ is the number of partitions of n into parts that are either even or else congruent to $2r-1 \pmod{2r}$ with the further restriction that only even parts may be repeated. Thus $P_1(n) = P_3(n)$.

The object of this note is to provide a simple combinatorial proof of the fact that $P_2(n) = P_3(n)$; equation (1) then yields the above theorem.

2. Proof that $P_2(n) = P_3(n)$. We provide a one-to-one correspondence between the sets of partitions to be counted.

Let π be a partition of the type enumerated by $P_2(n)$. Then represent π graphically with each even part $2m$ represented by two rows of m nodes and each odd part $2m+1$ represented by two rows of $m+1$ nodes and m nodes respectively. For example, $11+4$ becomes

```
. . . . . .
. . . . .
. .
. .
```

Now read the graph vertically with the proviso that r columns are always to be grouped as a single part whenever the lowest node in the most right-hand column of the group comes from what was originally the right-hand-most node contributed by an odd part. Thus in our

† Partially supported by NSF Grant GP 8075.

https://doi.org/10.1017/S0017089500000938

example with $r = 2$, we obtain in this manner the partition $4+4+2+2+3$. Now since the condition on partitions enumerated by $P_2(n)$ is $b_i - b_{i+1} \geqq 2r-1$ whenever b_i is odd, we see that our groupings of r columns always have one less node than a rectangle of $r \times 2v$ nodes; thus a part congruent to $2r-1 \pmod{2r}$ is produced. Since originally odd parts were distinct, we see that now odd parts will be congruent to $2r-1 \pmod{2r}$ and will not be repeated, and since originally all odd parts were greater or equal to $2r-1$, we see that there will always be r columns available for each grouping. Thus in this way we have produced a partition of the type enumerated by $P_3(n)$. Clearly our correspondence is one-to-one into; however, the above process is reversible and thus the correspondence is onto. Hence $P_2(n) = P_3(n)$.

As an example we take $r = 2$, $n = 11$. The corresponding partitions are listed opposite each other in the following table

P_2	P_3
11	2+2+2+2+3
9+2	4+2+2+3
8+3	7+2+2
7+4	4+4+3
7+2+2	6+2+3
6+5	7+4
5+2+2+2	8+3
4+4+3	11

REFERENCES

1. G. E. Andrews, On Schur's second partition theorem, *Glasgow Math. J.* **8** (1967), 127–132.

THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802