12

Methods of the theory of hyperbolic differential equations

This chapter discusses the notions of the theory of hyperbolic differential equations and the existence theorems employed to construct solutions to the conformal Einstein field equations. Conformal methods allow, under suitable circumstances, the use of very general theorems of the theory of partial differential equations (PDEs) to obtain conclusions of a global nature about solutions to the Einstein field equations. The results presented in this chapter have been tailored to fit the general discussion of this book.

The basic result of the theory of hyperbolic PDEs that will be used in this book is Kato's existence, uniqueness and stability result for symmetric hyperbolic systems; see Theorem 12.4. In view of applications to the construction of anti-de Sitter-like spacetimes a basic existence and uniqueness result of the initial boundary value problem of symmetric hyperbolic equations is also discussed; see Theorem 12.6. The chapter concludes with an overview of the basic theory behind characteristic initial value problems; see Theorem 12.7.

12.1 Basic notions

As will be seen in Chapter 13, the conformal Einstein equations give rise to ***quasilinear evolution equations*** which, in local coordinates $x \equiv (x^\mu)$ on an open set $\mathcal{U} \subset \mathcal{M}$ of the spacetime manifold, take the form

$$\mathbf{A}^\mu(x, \mathbf{u})\partial_\mu \mathbf{u} = \mathbf{B}(x, \mathbf{u}) \tag{12.1}$$

where $\mathbf{u}$ is a $\mathbb{C}^N$-valued unknown for some positive integer N and $\mathbf{A}^\mu$, $\mu = 0, \ldots 3$, are $(N \times N)$ matrix-valued functions of the coordinates and of the vector-valued unknown $\mathbf{u}$; thus, one has as many equations as components in the vector $\mathbf{u}$. Finally, $\mathbf{B}(x, \mathbf{u})$ is a vector-valued function of x and $\mathbf{u}$. In what follows, *it will be assumed that the components of* $\mathbf{u}$ *are scalars*. The functions $\mathbf{A}^\mu(x, \mathbf{u})$ and $\mathbf{B}(x, \mathbf{u})$ are, in principle, non-linear functions of the entries of $\mathbf{u}$. If the matrices

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$\mathbf{A}^\mu$ do not depend on $\mathbf{u}$, one has a ***semilinear*** system. Without loss of generality, $\mathcal{U}$ can be regarded as some suitable subset of $\mathbb{R}^4$.

Following the terminology of Section 11.2 the term

$$\mathbf{A}^\mu(x,\mathbf{u})\partial_\mu \mathbf{u}$$

is known as the ***principal part*** of Equation (12.1). The ***symbol*** with respect to the unknown $\mathbf{u}$ at the point $p \in \mathcal{U}$ with coordinates $x = x(p)$ for a covector $\boldsymbol{\xi} \in T^*|_p(\mathcal{U})$ is given by the matrix

$$\boldsymbol{\sigma}(x,\mathbf{u},\boldsymbol{\xi}) \equiv \mathbf{A}^\mu(x,\mathbf{u})\xi_\mu.$$

Under a coordinate transformation $x' = x'(x)$, it follows from Equation (12.1) that

$$\mathbf{A}^{\mu\prime}(x',\mathbf{u})\partial_{\mu'}\mathbf{u} = \mathbf{B}(x',\mathbf{u}),$$

with

$$\mathbf{A}'^\mu(x',\mathbf{u}) = \frac{\partial x'^\mu}{\partial x^\nu}\mathbf{A}^\nu(x(x'),\mathbf{u}). \tag{12.2}$$

It then follows from the transformation law of covectors under coordinate transformations and Equation (12.2) that the symbol of the differential Equation (12.1) is an invariant.

12.1.1 Symmetric hyperbolic systems

The basic properties of the PDE (12.1) and of its solutions depend on the structure of its principal part. Given a matrix $\mathbf{A}$, the operation of taking the transpose of its complex conjugate will be denoted by $\mathbf{A}^*$. One has the following definition:

Definition 12.1 (***symmetric hyperbolic systems***) *Given a solution $\mathbf{u}(x)$, the system* (12.1) *is said to be **symmetric hyperbolic** at $(x,\mathbf{u})$ if:*

(i) the matrices $\mathbf{A}^\mu(x,\mathbf{u})$ are Hermitian; that is $(\mathbf{A}^\mu)^* = \mathbf{A}^\mu$
(ii) there exists a covector $\boldsymbol{\xi}$ such that $\boldsymbol{\sigma}(x,\mathbf{u},\boldsymbol{\xi}) = \mathbf{A}^\mu(x,\mathbf{u})\xi_\mu$ is a positive-definite matrix.

Given two vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^N$, their ***inner product*** is defined by

$$\langle \mathbf{u},\mathbf{v}\rangle \equiv \mathbf{u}^*\mathbf{v}.$$

It follows then that $\langle \mathbf{u},\mathbf{v}\rangle = \overline{\langle \mathbf{v},\mathbf{u}\rangle}$ with the overbar denoting the usual complex conjugation of scalars. Moreover, if $\mathbf{A}$ is a Hermitian $N\times N$ matrix, then

$$\langle \mathbf{u},\mathbf{A}\mathbf{v}\rangle = \langle \mathbf{A}\mathbf{u},\mathbf{v}\rangle, \qquad \langle \mathbf{u},\mathbf{A}\mathbf{u}\rangle = \overline{\langle \mathbf{u},\mathbf{A}\mathbf{u}\rangle}.$$

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Spacelike and timelike hypersurfaces with respect to a symmetric hyperbolic system

In what follows, let $\mathcal{S}$ denote a hypersurface on $\mathcal{U} \subset \mathcal{M}$ defined in terms of a smooth scalar $\phi \in \mathcal{X}(\mathcal{M})$ as

$$\mathcal{S} \equiv \{p \in \mathcal{U} \,|\, \phi(p) = 0\}, \tag{12.3}$$

where it is assumed that $\mathbf{d}\phi \neq 0$ so that $\mathcal{S}$ has everywhere a well-defined normal. The positivity condition (i) in Definition 12.1 allows one to define the ***causal nature*** of the hypersurface $\mathcal{S}$ with respect to solutions of Equation (12.1). More precisely, the hypersurface $\mathcal{S}$ is said to be ***spacelike with respect to a solution*** $\mathbf{u}$ ***to the symmetric hyperbolic system*** (12.1) if $\boldsymbol{\sigma}(x, \mathbf{u}, \mathbf{d}\phi)$ is positive definite for $p \in \mathcal{S}$. If $\boldsymbol{\sigma}(x, \mathbf{u}, \mathbf{d}\phi)$ has a non-vanishing determinant and is not positive definite, one says that $\mathcal{S}$ is ***timelike*** for the solution $\mathbf{u}$. Finally, if $\boldsymbol{\sigma}(x, \mathbf{u}, \mathbf{d}\phi)$ has a vanishing determinant, one says that $\mathcal{S}$ is ***null*** – this case is tied to the notion of *characteristics* to be discussed in the next section. These causal definitions are, in principle, independent of the homonymous notion defined in terms of a metric $\boldsymbol{g}$ on $\mathcal{M}$. However, as discussed in Chapter 14, for evolution equations arising from the Einstein field equations, the geometric and PDE notions agree; see Theorem 14.1.

12.1.2 Initial value problems and characteristics

Of particular relevance for a symmetric hyperbolic system of the form (12.1) is the so-called ***initial value problem*** whereby some initial data on a hypersurface $\mathcal{S}$ is prescribed and one purports to obtain the solution to the equation away from the initial hypersurface.

An ***initial data set*** for Equation (12.1) on a hypersurface $\mathcal{S}$ which is spacelike with respect to Equation (12.1) consists of a $\mathbb{C}^N$-valued function $\mathbf{u}_\star$ on $\mathcal{S}$ which is interpreted as the value of the solution $\mathbf{u}$ to Equation (12.1) on $\mathcal{S}$. A question which arises naturally in this context is whether all the components of the vector $\mathbf{u}_\star$ can be specified freely on $\mathcal{S}$.

It is convenient to introduce on $\mathcal{U}$ coordinates $x = (x^0, \underline{x}) = (x^0, x^1, x^2, x^3)$ adapted to $\mathcal{S}$ so that the hypersurface is represented by the condition $x^0 = 0$. Using these adapted coordinates and the initial data $\mathbf{u}_\star$ one can compute the *spatial derivatives* $\partial_\alpha \mathbf{u}_\star$ of $\mathbf{u}$ on $\mathcal{S}$. In order to determine the *time derivatives* $\partial_0 \mathbf{u}$ on $\mathcal{S}$ one substitutes the above into Equation (12.1) to obtain

$$\mathbf{A}^0(0, \underline{x}; \mathbf{u}_\star)(\partial_0 \mathbf{u})_\star + \mathbf{A}^\alpha(0, \underline{x}; \mathbf{u}_\star)\partial_\alpha \mathbf{u}_\star = \mathbf{B}(0, \underline{x}; \mathbf{u}_\star), \tag{12.4}$$

where it is observed that $(\partial_\alpha \mathbf{u})_\star = \partial_\alpha \mathbf{u}_\star$. This equation can be read as a linear algebraic system for $(\partial_0 \mathbf{u})_\star \equiv \partial_0 \mathbf{u}|_{\mathcal{S}}$ which can be solved if $\mathbf{A}^0(0, \underline{x}, \mathbf{u}_\star)$ can be inverted, that is, if

$$\det\big(\mathbf{A}^0(0, \underline{x}; \mathbf{u}_\star)\big) \neq 0.$$

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If $\det \mathbf{A}^0(0, \underline{x}; \mathbf{u}_\star) = 0$, then $M \equiv \operatorname{rank} \mathbf{A}^0(0, \underline{x}, \mathbf{u}_\star) < N$, and one can make linear combinations of the equations in (12.4) to obtain a new system on $\mathcal{S}$ of the form

$$\bar{\mathbf{A}}^0(0, \underline{x}; \mathbf{u}_\star)(\partial_0 \mathbf{u})_\star + \bar{\mathbf{A}}^\alpha(0, \underline{x}; \mathbf{u}_\star)\partial_\alpha \mathbf{u}_\star = \mathbf{B}(0, \underline{x}; \mathbf{u}_\star),$$

where

$$\bar{\mathbf{A}}^0(0, \underline{x}, \mathbf{u}_\star) = \begin{pmatrix} a^0_{11}(0, \underline{x}; \mathbf{u}_\star) & \cdots & a^0_{1N}(0, \underline{x}; \mathbf{u}_\star) \\ \vdots & \ddots & \vdots \\ a^0_{M1}(0, \underline{x}; \mathbf{u}_\star) & \cdots & a^0_{MN}(0, \underline{x}; \mathbf{u}_\star) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

has $N - M$ rows consisting of zeros. Hence, not all the derivatives $(\partial_0 \mathbf{u})_\star$ are determined by the initial data, and one has $N - M$ ***constraint equations*** which have to be satisfied by the initial data $\mathbf{u}_\star$.

The discussion of the previous paragraphs leads to the following general definition which also applies to evolution systems of the form (12.1) which are *not necessarily* symmetric hyperbolic:

Definition 12.2 (***characteristic surfaces of a first order PDE***) *A hypersurface $\mathcal{S}$ defined by a condition of the form* (12.3) *is said to be a* ***characteristic of a solution* u *of Equation (12.1)*** *if*

$$\det \big(\boldsymbol{\sigma}(x, \mathbf{u}, \mathbf{d}\phi)\big) = 0 \quad \textit{for } p \in \mathcal{S}. \tag{12.5}$$

If

$$\det \big(\boldsymbol{\sigma}(x, \mathbf{u}, \mathbf{d}\phi)\big) \neq 0 \quad \textit{for } p \in \mathcal{S},$$

then $\mathcal{S}$ is said to be ***nowhere characteristic*** *for the solution* $\mathbf{u}$ *of Equation (12.1).*

On a characteristic, the system (12.1) implies M ***transversal equations*** and $N - M$ ***interior equations*** on $\mathcal{S}$. If $M = 0$, so that the full system (12.1) reduces to interior equations, one says that $\mathcal{S}$ is a ***total characteristic*** of the system. More generally, given a point $p \in \mathcal{U}$, one defines its ***characteristic set*** (or ***Monge cone***) with respect to a solution $\mathbf{u}$ of Equation (12.1) as the subset $C^*_p \subset T^*|_p(\mathcal{U})$ defined by

$$C^*_p \equiv \big\{\boldsymbol{\xi} \in T^*|_p(\mathcal{U}) \;\big|\; \det(\boldsymbol{\sigma}(x, \mathbf{u}, \boldsymbol{\xi})) = 0,\ \boldsymbol{\xi} \neq 0\big\}.$$

That is, the elements of C^*_p are in the kernel of the symbol. The covectors $\boldsymbol{\xi}$ are sometimes called the ***null directions*** at p. The quantity $\det(\boldsymbol{\sigma}(x, \mathbf{u}, \boldsymbol{\xi}))$ can be read as a polynomial for the components of the covector $\boldsymbol{\xi}$ – the so-called ***characteristic polynomial***.

Well-posedness

An initial value problem for a system of the form (12.1) (not necessarily symmetric hyperbolic) with data prescribed on a hypersurface $\mathcal{S}$ which is *nowhere characteristic and timelike* with respect to the evolution system at the prescribed data $\mathbf{u}_\star$ will be called a ***Cauchy initial value problem***. If the initial data is prescribed on a hypersurface $\mathcal{N}$ which is characteristic, one speaks of a ***characteristic initial value problem***.

The definitions given in the previous paragraph are motivated by the notion of ***well-posedness***. In broad terms, an initial value problem is ***well posed*** if:

(i) There exist solutions to all initial data.
(ii) The solutions are uniquely determined by the initial data.
(iii) The solutions depend continuously on the initial data.

The first step in the analysis of the well-posedness of an initial value problem for a given class of PDEs is the formulation of the above requirements in a precise manner; see, for example, Rendall (2008) for further discussion on this. Initial value problems which are not well posed are said to be ***ill-posed***.

The Cauchy problem for a symmetric hyperbolic system of the form (12.1) is well-posed. By contrast, an initial value problem with data prescribed on a timelike hypersurface is ill-posed. A further example of an ill-posed problem is the Cauchy problem for elliptic equations. In the case of characteristic initial value problems the well-posedness of the problem depends on the causal relation between the region where one wants to obtain the solution and the initial characteristic surfaces; see Section 12.5.1. Although well-posed initial value problems are of natural importance in general relativity, ill-posed problems also arise in applications such as the uniqueness of stationary black holes; see, for example, Ionescu and Klainerman (2009a,b).

12.1.3 Some examples

The discussion of the previous paragraphs is best illuminated by a couple of explicit examples. Many of the features of these examples are generic and arise in the analysis of the evolution equations implied by the (conformal) Einstein field equations.

In what follows, let $(\mathcal{M}, \boldsymbol{g})$ denote a spacetime. On $\mathcal{U} \subset \mathcal{M}$ consider some local coordinates $x = (x^\mu)$ and a null frame $\{\boldsymbol{e}_{\boldsymbol{AA}'}\}$ with associated cobasis $\{\boldsymbol{\omega}^{\boldsymbol{AA}'}\}$. In terms of the local coordinates one writes $\boldsymbol{e}_{\boldsymbol{AA}'} = e_{\boldsymbol{AA}'}{}^\mu \boldsymbol{\partial}_\mu$ and $\boldsymbol{\omega}^{\boldsymbol{AA}'} = \omega^{\boldsymbol{AA}'}{}_\mu \mathbf{d}x^\mu$. Moreover, let $\{\epsilon_{\boldsymbol{A}}{}^A\}$ be a spinorial frame giving rise to the vector frame $\{\boldsymbol{e}_{\boldsymbol{AA}'}\}$; see the discussion in Section 3.1.9.

A spinorial curl equation

As a first example consider on $\mathcal{U} \subset \mathcal{M}$ a spinorial equation of the form

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{QA}\cdots\boldsymbol{D}} = F_{\boldsymbol{A}'\boldsymbol{A}\cdots\boldsymbol{D}}, \tag{12.6}$$

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for the components $\varphi_{\boldsymbol{QA}\cdots\boldsymbol{D}}$ of a spinor $\varphi_{QA\cdots D}$ with respect to the spin frame $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$. The spinor $\varphi_{QA\cdots D}$ is not assumed to have any particular symmetries and the field $F_{\boldsymbol{A}'\boldsymbol{A}\cdots\boldsymbol{D}}$ may depend on the coordinates or any other field. Notice that the unknowns of Equation (12.6) are scalars.

It is claimed that the combination

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{1}'}\varphi_{\boldsymbol{QA}\cdots\boldsymbol{D}} = F_{\boldsymbol{1}'\boldsymbol{A}\cdots\boldsymbol{D}}, \tag{12.7a}$$

$$-\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{0}'}\varphi_{\boldsymbol{QA}\cdots\boldsymbol{D}} = -F_{\boldsymbol{0}'\boldsymbol{A}\cdots\boldsymbol{D}}, \tag{12.7b}$$

is a symmetric hyperbolic system. In order to see this, observe that

$$\nabla^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{QA}\cdots\boldsymbol{D}} = \nabla_{\boldsymbol{1A}'}\varphi_{\boldsymbol{0A}\cdots\boldsymbol{D}} - \nabla_{\boldsymbol{0A}'}\varphi_{\boldsymbol{1A}\cdots\boldsymbol{D}}.$$

Thus, the principal part of the system (12.7a) and (12.7b) can be written in matricial form as

$$\mathbf{A}^{\mu}\partial_{\mu}\boldsymbol{\varphi} \equiv \begin{pmatrix} e_{\boldsymbol{11}'}{}^{\mu} & -e_{\boldsymbol{01}'}{}^{\mu} \\ -e_{\boldsymbol{10}'}{}^{\mu} & e_{\boldsymbol{00}'}{}^{\mu} \end{pmatrix} \partial_{\mu} \begin{pmatrix} \varphi_{\boldsymbol{0A}\cdots\boldsymbol{D}} \\ \varphi_{\boldsymbol{1A}\cdots\boldsymbol{D}} \end{pmatrix}.$$

The matrices $\mathbf{A}^{\mu}$ are Hermitian as $\boldsymbol{e}_{\boldsymbol{00}'}$ and $\boldsymbol{e}_{\boldsymbol{11}'}$ are real vectors and $\boldsymbol{e}_{\boldsymbol{01}'} = \overline{\boldsymbol{e}_{\boldsymbol{10}'}}$. Letting $\xi_{\mu} \equiv \omega^{\boldsymbol{00}'}{}_{\mu} + \omega^{\boldsymbol{11}'}{}_{\mu}$, a calculation shows that

$$\mathbf{A}^{\mu}\xi_{\mu} = \begin{pmatrix} e_{\boldsymbol{11}'}{}^{\mu}\omega^{\boldsymbol{00}'}{}_{\mu} + e_{\boldsymbol{11}'}{}^{\mu}\omega^{\boldsymbol{11}'}{}_{\mu} & -e_{\boldsymbol{01}'}{}^{\mu}\omega^{\boldsymbol{00}'}{}_{\mu} - e_{\boldsymbol{01}'}{}^{\mu}\omega^{\boldsymbol{11}'}{}_{\mu} \\ -e_{\boldsymbol{10}'}{}^{\mu}\omega^{\boldsymbol{00}'}{}_{\mu} - e_{\boldsymbol{10}'}{}^{\mu}\omega^{\boldsymbol{11}'}_{\mu} & e_{\boldsymbol{00}'}{}^{\mu}\omega^{\boldsymbol{00}'}{}_{\mu} + e_{\boldsymbol{00}'}\omega^{\boldsymbol{11}'}{}_{\mu} \end{pmatrix}.$$

Using $e_{\boldsymbol{AA}'}{}^{\mu}\omega^{\boldsymbol{BB}'}{}_{\mu} = \epsilon_{\boldsymbol{A}}{}^{\boldsymbol{B}}\epsilon_{\boldsymbol{A}'}{}^{\boldsymbol{B}'}$ it follows that

$$\mathbf{A}^{\mu}\xi_{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is clearly positive definite. Thus, the system (12.7a) and (12.7b) is symmetric hyperbolic as claimed. Given a generic covector $\boldsymbol{\xi}$, the characteristic polynomial is given by

$$\begin{aligned} \det(\mathbf{A}^{\mu}\xi_{\mu}) &= \det \begin{pmatrix} e_{\boldsymbol{11}'}{}^{\mu}\xi_{\mu} & -e_{\boldsymbol{01}'}{}^{\mu}\xi_{\mu} \\ -e_{\boldsymbol{10}'}{}^{\mu}\xi_{\mu} & e_{\boldsymbol{00}'}{}^{\mu}\xi_{\mu} \end{pmatrix} \\ &= \big(e^{\mu}_{\boldsymbol{11}'}e_{\boldsymbol{00}'}{}^{\nu} - e_{\boldsymbol{01}'}{}^{\mu}e_{\boldsymbol{10}'}{}^{\nu}\big)\xi_{\mu}\xi_{\nu} \\ &= \frac{1}{2}g^{\mu\nu}\xi_{\mu}\xi_{\nu}, \end{aligned}$$

where, in the last equality, Equation (3.30) relating the null frame and the metric has been used. Thus, the characteristics of Equation (12.6) are given by null hypersurfaces with respect to the metric $\boldsymbol{g}$. Furthermore, spacelike hypersurfaces with respect to solutions to the equation coincide with the $\boldsymbol{g}$-spacelike hypersurfaces so that the causal notions given by Equation (12.6) and the background metric $\boldsymbol{g}$ coincide.

The wave equation as a symmetric hyperbolic system

As a second example consider the wave equation

$$\nabla^a \nabla_a \phi = 0 \tag{12.8}$$

on a region $\mathcal{U} \subset \mathcal{M}$. In contrast to the previous example, this equation is second order, and thus, it does not fit into the scheme discussed so far. Nevertheless, the wave equation can be recast as a symmetric hyperbolic system for the scalar field ϕ and some further auxiliary fields.

The spinorial version of Equation (12.8) is given by

$$\nabla^{\boldsymbol{AA}'} \nabla_{\boldsymbol{AA}'} \phi = 0. \tag{12.9}$$

As a first step one introduces the auxiliary variable $\phi_{\boldsymbol{AA}'} \equiv \nabla_{\boldsymbol{AA}'}\phi$. Reading this definition as an equation for the scalar field ϕ and contracting with a spinor $\tau^{\boldsymbol{AA}'}$ representing a timelike vector τ^a, one obtains the evolution equation

$$\mathcal{P}\phi = \varphi, \tag{12.10}$$

where $\varphi \equiv \tau^{\boldsymbol{AA}'}\phi_{\boldsymbol{AA}'}$ and $\mathcal{P} \equiv \tau^{\boldsymbol{AA}'}\nabla_{\boldsymbol{AA}'}$ is the directional derivative along τ^a; see Section 4.3.1. Now, defining $\varphi_{\boldsymbol{AB}} \equiv \tau_{(\boldsymbol{B}}{}^{\boldsymbol{A}'}\phi_{\boldsymbol{A})\boldsymbol{A}'}$ one obtains the decomposition

$$\phi_{\boldsymbol{AA}'} = \frac{1}{2}\varphi\tau_{\boldsymbol{AA}'} - \tau^{\boldsymbol{Q}}{}_{\boldsymbol{A}'}\varphi_{\boldsymbol{AQ}}. \tag{12.11}$$

Having introduced the auxiliary variable $\phi_{\boldsymbol{AA}'}$ one needs to consider a suitable field equation for it. A convenient choice is given by the *no torsion condition*

$$\nabla_{\boldsymbol{AA}'}\nabla_{\boldsymbol{BB}'}\phi - \nabla_{\boldsymbol{BB}'}\nabla_{\boldsymbol{AA}'}\phi = 0,$$

which, in view of the definition of $\phi_{\boldsymbol{AA}'}$, can be rewritten as

$$\nabla_{\boldsymbol{AA}'}\phi_{\boldsymbol{BB}'} - \nabla_{\boldsymbol{BB}'}\phi_{\boldsymbol{AA}'} = 0. \tag{12.12}$$

Contracting the indices $_{\boldsymbol{A}'}$ and $_{\boldsymbol{B}'}$ and using the see-saw rule one obtains

$$\nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\phi_{\boldsymbol{B})\boldsymbol{Q}'} = 0,$$

which, as a result of the hermicity of $\phi_{\boldsymbol{AA}'}$ is completely equivalent to Equation (12.12). Finally, using the identity

$$\nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}\phi_{\boldsymbol{BQ}'} = \nabla_{(\boldsymbol{A}}{}^{\boldsymbol{Q}'}\phi_{\boldsymbol{B})\boldsymbol{Q}'} - \frac{1}{2}\epsilon_{\boldsymbol{AB}}\nabla^{\boldsymbol{QQ}'}\phi_{\boldsymbol{QQ}'}$$

and observing that from Equation (12.9) it follows that $\nabla^{\boldsymbol{QQ}'}\phi_{\boldsymbol{QQ}'} = 0$, one concludes that

$$\nabla_{\boldsymbol{A}}{}^{\boldsymbol{Q}'}\phi_{\boldsymbol{BQ}'} = 0.$$

Using Equation (12.11) one can perform a space spinor split of this equation. After some calculations one obtains the pair of equations

$$\mathcal{P}\varphi + 2\mathcal{D}^{\boldsymbol{AB}}\varphi_{\boldsymbol{AB}} = 0, \tag{12.13a}$$

$$\mathcal{P}\varphi_{\boldsymbol{AB}} - \mathcal{D}_{\boldsymbol{AB}}\varphi + 2\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\varphi_{\boldsymbol{B})\boldsymbol{Q}} = 0, \tag{12.13b}$$

where $\mathcal{D}_{\boldsymbol{AB}}$ denotes the directional derivative associated to the Sen connection relative to $\tau^{AA'}$; see Section 4.3.1. Equations (12.10), (12.13a) and (12.13b) are the basic evolution equations. For simplicity of presentation in Equations (12.13a) and (12.13b) the covariant derivatives of $\tau^{AA'}$ have been assumed to vanish. To obtain a system which is symmetric hyperbolic, some normalisation factors have to be added. Some experimentation renders

$$\begin{aligned}
&\mathcal{P}\phi = \varphi,\\
&\mathcal{P}\varphi + 2\mathcal{D}^{\boldsymbol{AB}}\varphi_{\boldsymbol{AB}} = 0,\\
&\frac{4}{(\boldsymbol{A}+\boldsymbol{B})!(2-\boldsymbol{A}-\boldsymbol{B})!}\left(\mathcal{P}\varphi_{\boldsymbol{AB}} - \mathcal{D}_{\boldsymbol{AB}}\varphi + 2\mathcal{D}_{(\boldsymbol{A}}{}^{\boldsymbol{Q}}\varphi_{\boldsymbol{B})\boldsymbol{Q}}\right) = 0,
\end{aligned}$$

which is claimed to be symmetric hyperbolic. From these equations, a calculation similar to the one carried out for the Maxwell equations yields the following matricial expression for the principal part:

$$\mathbf{A}^\mu\partial_\mu\boldsymbol{\phi} \equiv \begin{pmatrix} \tau^\mu & 0 & 0 & 0 & 0\\ 0 & \tau^\mu & 2e_{\boldsymbol{11}}{}^\mu & -4e_{\boldsymbol{01}}{}^\mu & 2e_{\boldsymbol{00}}{}^\mu\\ 0 & -2e_{\boldsymbol{00}}{}^\mu & 2\tau^\mu - 4e_{\boldsymbol{01}}{}^\mu & 4e_{\boldsymbol{00}}{}^\mu & 0\\ 0 & -4e_{\boldsymbol{01}}{}^\mu & -4e_{\boldsymbol{11}}{}^\mu & 4\tau^\mu & 4e_{\boldsymbol{00}}{}^\mu\\ 0 & -2e_{\boldsymbol{11}}{}^\mu & 0 & -4e_{\boldsymbol{11}}{}^\mu & 2\tau^\mu + 4e_{\boldsymbol{01}}{}^\mu \end{pmatrix}\partial_\mu\begin{pmatrix}\phi\\ \varphi\\ \varphi_0\\ \varphi_1\\ \varphi_2\end{pmatrix},$$

where $\varphi_0 \equiv \varphi_{\boldsymbol{00}}$, $\varphi_1 \equiv \varphi_{\boldsymbol{01}}$ and $\varphi_2 \equiv \varphi_{\boldsymbol{11}}$. Taking into account the reality conditions satisfied by the various frame coefficients one concludes that the matrices are Hermitian. Moreover, a short computation shows that $\mathbf{A}^\mu\tau_\mu$ is positive definite so that, indeed, one has obtained a symmetric hyperbolic system for the wave equation. Finally, a further computation shows that the characteristic polynomial of the system is given by

$$\det(\mathbf{A}^\mu\xi_\mu) = 8(\tau^\mu\xi_\mu)(g^{\nu\lambda}\xi_\nu\xi_\lambda)^2.$$

Accordingly, $\boldsymbol{g}$-null hypersurfaces are characteristics of the system.

12.2 Uniqueness and domains of dependence

An important property of the Cauchy initial value problem for symmetric hyperbolic systems is the ***uniqueness of solutions*** for a given prescription of initial data. The discussion of the uniqueness of solutions is naturally carried out in subsets of $\mathbb{R}^4$ known as lens-shaped domains. A ***lens-shaped domain***

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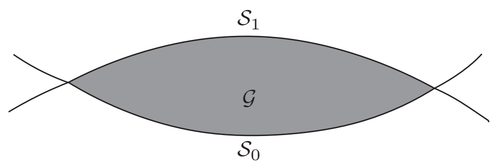

Figure 12.1 Schematic depiction of a lens-shaped domain $\mathcal{G}$. The hypersurfaces $\mathcal{S}_0$ and $\mathcal{S}_1$ are spacelike with respect to a solution $\mathbf{u}$ of a symmetric hyperbolic system of the form (12.1).

with respect to a solution $\mathbf{u}$ to a symmetric hyperbolic system of the form (12.1) is an open subset $\mathcal{G} \subset \mathbb{R}^4$ with compact closure and whose boundary is given by the union of two subsets $\mathcal{S}_0$ and $\mathcal{S}_1$ of hypersurfaces which are spacelike with respect to $\mathbf{u}$; see Figure 12.1. In terms of these domains one has the following result which exploits all the algebraic conditions in Definition 12.1:

Theorem 12.1 (***uniqueness of solutions of symmetric hyperbolic systems***) *Let $\mathcal{G}$ be a lens-shaped domain. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions to the initial value problem for the symmetric hyperbolic system*

$$\mathbf{A}^\mu(x,\mathbf{u})\partial_\mu \mathbf{u} = \mathbf{B}(x,\mathbf{u}), \qquad \mathbf{u}|_{\mathcal{S}_0} = \mathbf{u}_\star$$

then $\mathbf{u}_1 = \mathbf{u}_2$ on $\mathcal{G}$.

Proof This proof follows closely the discussion in Friedrich and Rendall (2000). Assume one has a symmetric hyperbolic system of the form (12.1) such that the matrices $\mathbf{A}^\mu$ and $\mathbf{B}$ are C^1 functions of their arguments. Moreover, let $\mathbf{u}_1$ and $\mathbf{u}_2$ denote two C^1 solutions. Let $\mathcal{G}$ denote a lens-shaped region with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$ whose boundary is given by the union of two hypersurfaces $\mathcal{S}_0$ and $\mathcal{S}_1$. Using a refined version of the ***mean value theorem*** (see the Appendix to this chapter for further discussion) it follows that there exist continuous functions $\mathbf{M}^\mu$ and $\mathbf{N}$ such that

$$\begin{aligned}\mathbf{A}^\mu(x,\mathbf{u}_1) - \mathbf{A}^\mu(x,\mathbf{u}_2) &= \mathbf{M}^\mu(x,\mathbf{u}_1,\mathbf{u}_2)(\mathbf{u}_1-\mathbf{u}_2),\\ \mathbf{B}(x,\mathbf{u}_1) - \mathbf{B}(x,\mathbf{u}_2) &= \mathbf{N}(x,\mathbf{u}_1,\mathbf{u}_2)(\mathbf{u}_1-\mathbf{u}_2).\end{aligned}$$

It follows then from Equation (12.1) that

$$\mathbf{A}^\mu(x,\mathbf{u}_1)\partial_\mu(\mathbf{u}_1-\mathbf{u}_2) + \Big(\mathbf{M}^\mu(x,\mathbf{u}_1,\mathbf{u}_2)\partial_\mu\mathbf{u}_2 + \mathbf{N}(x,\mathbf{u}_1,\mathbf{u}_2)\Big)(\mathbf{u}_1-\mathbf{u}_2) = 0.$$

This equation can be written in a more compact form as

$$\mathbf{A}^\mu(x,\mathbf{u}_1)\partial_\mu(\mathbf{u}_1-\mathbf{u}_2) = \mathbf{Q}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)(\mathbf{u}_1-\mathbf{u}_2)$$

with $\mathbf{Q}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)$ a continuous function of its arguments.

Now, choosing coordinates such that $x = (t, \underline{x})$ and using the evolution Equation (12.1) one can verify the identity

$$\partial_\mu\left(e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle\right) \\ = e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_2)\rangle, \qquad (12.14)$$

where

$$\mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2) \equiv -k\mathbf{A}^0(x,\mathbf{u}_1) + \partial_\mu\mathbf{A}^\mu(x,\mathbf{u}_1) \\ + \mathbf{Q}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2) + \mathbf{Q}^*(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2).$$

Integrating the identity (12.14) over the lens-shaped region $\mathcal{G}$ and using the Gauss theorem one has that

$$\int_{\mathcal{G}} \partial_\mu\big(e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle\big)\mathrm{d}^4x \\ = \int_{\mathcal{S}_1} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle\nu_\mu\mathrm{d}S \\ - \int_{\mathcal{S}_0} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle\nu_\mu\mathrm{d}S, \qquad (12.15)$$

where d^4x is the standard volume element in $\mathbb{R}^4$ and ν_μ denotes the outward pointing unit normal to $\partial\mathcal{G}$.

As $\mathcal{S}_1$ is spatial with respect to the symmetric hyperbolic system, it follows that $\mathbf{A}^\mu(x,\mathbf{u}_1)|_{\mathcal{S}_1}$ is positive definite. Hence, the integral over $\mathcal{S}_1$ in Equation (12.15) is non-negative. By assumption one has that $(\mathbf{u}_1 - \mathbf{u}_2)|_{\mathcal{S}_0} = \mathbf{0}$ so that the integral over $\mathcal{S}_0$ in (12.15) vanishes. Hence, one concludes that

$$\int_{\mathcal{G}} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_2)\rangle\mathrm{d}^4x \geq 0. \qquad (12.16)$$

Finally, as the matrix $\mathbf{A}^0(x,\mathbf{u}_1)$ is positive definite and $\mathcal{G}$ is compact, it follows that the constant $k > 0$ can be chosen so that $\mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)$ is negative definite uniformly on $\mathcal{G}$. In other words, there exists a positive constant C such that

$$0 > -C\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\rangle \geq \langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_2)\rangle.$$

Accordingly, the integral over $\mathcal{G}$ in inequality (12.16) can be made negative by a suitable choice of k. This is a contradiction unless $\mathbf{u}_1 = \mathbf{u}_2$ in $\mathcal{G}$. □

A corollary of the above theorem is the following:

Corollary 12.1 *If $\mathbf{u}|_{\mathcal{S}_0} = \mathbf{0}$ and $\mathbf{B}(x,\mathbf{u})$ is homogeneous in $\mathbf{u}$, then $\mathbf{u} = \mathbf{0}$ in $\mathcal{G}$.*

Proof The result follows directly from the previous theorem, observing that $\mathbf{u} = \mathbf{0}$ is a solution. □

The uniqueness Theorem 12.1 shows that, in a neighbourhood of an initial hypersurface $\mathcal{S}$, the solution of a symmetric hyperbolic system is determined by initial data on a compact subset of $\mathcal{S}$ as any point sufficiently close to $\mathcal{S}$ is contained in a lens-shaped region. This consideration leads to the notion of ***domain of dependence***.

Definition 12.3 (***domain of dependence***) *Let $\mathcal{R} \subset \mathcal{S}$. The **domain of dependence** $D(\mathcal{R})$ of $\mathcal{R}$ is the set of all points $p \in \mathcal{U} \subset \mathbb{R}^4$ such that the value of a solution $\mathbf{u}$ to Equation* (12.1) *at p is determined (uniquely) by the restriction of the initial data to $\mathcal{R}$.*

Remark. The term "domain of dependence" is sometimes used in the PDE literature to denote the set of points determining the value of a solution $\mathbf{u}$ at a given point. The notion of domain of dependence used in this book is then called *domain of influence*; see Rendall (2008) for further discussion.

The main property singled out by Definition 12.3 is that the solution of a symmetric hyperbolic system is determined at a given point by data on a proper subset of the initial hypersurface. Thus, the process of solving the Cauchy problem for the symmetric hyperbolic system (12.1) can be *localised in space*. This is a particular property of hyperbolic differential equations which distinguishes them from other types of PDEs. More precisely, if two initial data sets $\mathbf{u}_\star$ and $\bar{\mathbf{u}}_\star$ coincide on an open subset $\mathcal{R} \subset \mathcal{S}$, then the corresponding domains of influence and the solutions $\mathbf{u}$ and $\bar{\mathbf{u}}$ coincide as well. In other words, in the domain of influence $D(\mathcal{R})$ a solution $\mathbf{u}$ is independent of the behaviour of the data $\mathbf{u}_\star$ outside $\mathcal{R}$. In particular, there is no need to impose boundary or fall-off conditions away from $\mathcal{R}$. This observation is usually known as the ***localisability property*** of symmetric hyperbolic systems; that is, the theory does not depend on the global knowledge of the initial data in space. A related observation is that if on $\mathcal{S}$ one has two different intersecting coordinate patches $\mathcal{R}$ and $\mathcal{R}'$ such that on $\mathcal{R} \cap \mathcal{R}'$ one has $x' = x'(x)$, then, as a consequence the transformation rule of Equation (12.1) and the uniqueness of the solution on $D(\mathcal{R} \cap \mathcal{R}')$, one has that $\mathbf{u}'(x') = \mathbf{u}(x(x'))$.

Finite speed of propagation of solutions

A consequence of the existence of a domain of dependence for symmetric hyperbolic systems is the so-called ***finite speed of propagation of their solutions***. A rough estimate of this phenomenon can be constructed using an argument given in Rendall (2008).

As in previous sections, let $\mathbf{u}$ denote a solution to a symmetric hyperbolic system of the form (12.1) with initial data $\mathbf{u}_\star$ prescribed on the hypersurface

$$\mathcal{S}_0 \equiv \{p \in \mathcal{U} \,|\, t(p) = 0\}.$$

In what follows, assume that the support of $\mathbf{u}_\star$ is contained on a ball of radius $r_\star$ around the origin.

Now, given a point $p \in \mathcal{U}$ with coordinates $(t_\bullet, \underline{x}_\bullet) \equiv (t, x^\alpha_\bullet)$ and a constant $\beta > 0$ consider the paraboloidal hypersurface

$$\mathcal{S}_{\beta;(t_\bullet,\underline{x}_\bullet)} \equiv \big\{ p \in \mathcal{U} \,\big|\, t(p) = t_\bullet - \beta\, \delta_{\alpha\beta} \big(x^\alpha(p) - x^\alpha_\bullet\big)\big(x^\beta(p) - x^\beta_\bullet\big) \big\}.$$

The normal to these hypersurfaces is given by

$$\boldsymbol{\nu} = \mathbf{d}t - 2\beta\delta_{\alpha\beta}x^\alpha \mathbf{d}x^\beta.$$

Hence, assuming that $\mathbf{A}^0(x, \mathbf{u})$ is positive definite on $\mathcal{S}_0$ it follows that

$$\mathbf{A}^\mu(x, \mathbf{u})\nu_\mu = \mathbf{A}^0(x, \mathbf{u}) + 2\beta\delta_{\alpha\beta}x^\alpha \mathbf{A}^\beta(x, \mathbf{u})$$

can be made positive definite by choosing β sufficiently small, say, $\beta < \beta_0$, so that $\mathcal{S}_{\beta;(t_\bullet,\underline{x}_\bullet)}$ is spacelike with respect to Equation (12.1). For this choice of β the region $\mathcal{G}$ bounded by $\mathcal{S}_0$ and $\mathcal{S}_{\beta;(t_\bullet,\underline{x}_\bullet)}$ is a lens-shaped domain. Now, it can be verified that the intersection of $\mathcal{S}_{\beta;(t_\bullet,\underline{x}_\bullet)}$ with $\mathcal{S}_0$ lies outside a ball of radius

$$r \equiv |x_\bullet| - \sqrt{\frac{t_\bullet}{\beta}}, \qquad |x_\bullet|^2 \equiv \delta_{\alpha\beta}x^\alpha_\bullet x^\beta_\bullet.$$

Thus, if

$$|x_\bullet| - \sqrt{\frac{t_\bullet}{\beta}} > r_\star,$$

then the solution satisfies $\mathbf{u}(t_\bullet, \underline{x}_\bullet) = 0$ as $(t_\bullet, \underline{x}_\bullet)$ lies on the boundary of a lens-shaped region with trivial data. Accordingly, the support of $\mathbf{u}$ on the hypersurface

$$\mathcal{S}_{t_\bullet} = \big\{ p \in \mathcal{U} \,|\, t(p) = t_\bullet \big\}$$

must lie within a ball of radius $r_\star + \sqrt{t_\bullet/\beta}$; see Figure 12.2 for further details. Thus, the support of the solution gradually spreads in space at finite speed.

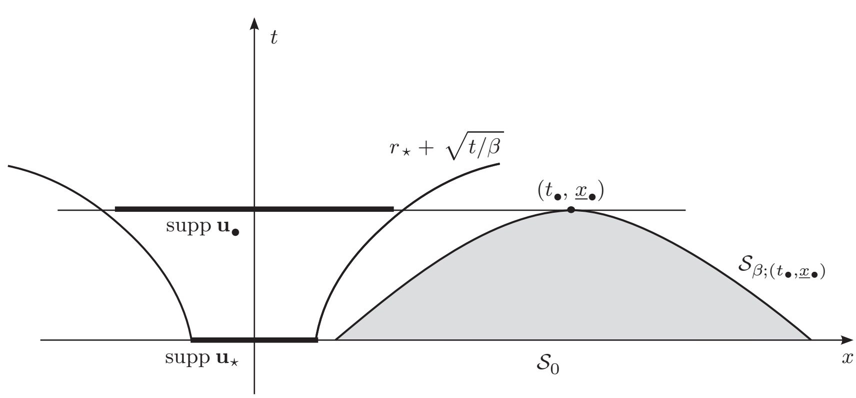

Figure 12.2 Schematic depiction of the rough estimate of the spread of the support of a solution to a symmetric hyperbolic system. The solution at $(t_\bullet, \underline{x}_\bullet)$ is determined by trivial data at the initial hypersurface $\mathcal{S}_0$; see the main text for further details.

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12.3 Local existence results for symmetric hyperbolic systems

The purpose of this section is to analyse the basic existence and stability results for symmetric hyperbolic systems of the form (12.1). The precise formulation of existence results is more technical than the one for uniqueness and requires a certain number of notions from the theory of functional analysis. These are discussed in the following subsection.

12.3.1 Sobolev spaces

The precise discussion of existence results for symmetric hyperbolic systems is carried out in terms of Sobolev spaces. The purpose of this section is to introduce some of the basic ideas concerning these function spaces. In a first step, the discussion will consider Sobolev spaces of functions over $\mathbb{R}^3$. These notions can be suitably extended to three-dimensional manifolds with a different topology.

In what follows, let $\underline{x} \equiv (x^\alpha)$ denote some particular choice of Cartesian coordinates and let d^3x be the standard volume element of $\mathbb{R}^3$. The discussion of solutions of symmetric hyperbolic systems of the form (12.1) leads to consider $\mathbb{C}^N$***-valued functions on*** $\mathbb{R}^3$; that is, $\mathbf{w}: \mathbb{R}^3 \to \mathbb{C}^N$. The space of smooth functions of this type will be denoted by $C^\infty(\mathbb{R}^3, \mathbb{C}^N)$. On $C^\infty(\mathbb{R}^3, \mathbb{C}^N)$ one can introduce, for $m \in \mathbb{N}$, a ***Sobolev norm*** via

$$||\mathbf{w}||_{\mathbb{R}^3,m} \equiv \left(\sum_{k=0}^{m} \sum_{\alpha_1,\ldots\alpha_k=1}^{3} \int_{\mathbb{R}^3} |\partial_{\alpha_k} \cdots \partial_{\alpha_1} \mathbf{w}|^2 \mathrm{d}^3x \right)^{1/2}, \tag{12.17}$$

for $\mathbf{w} \equiv (w_1, \ldots, w_N) \in C^\infty(\mathbb{R}^3, \mathbb{C}^N)$ where $|\mathbf{w}|^2 = \langle \mathbf{w}, \mathbf{w} \rangle$ is the standard norm in $\mathbb{C}^N$. For example, if $\mathbf{u} = (u)$ is a $\mathbb{C}$-valued function, one has that

$$||\mathbf{u}||^2_{\mathbb{R}^3,1} = \int_{\mathbb{R}^3} (u\,\overline{u} + \partial_1 u\,\overline{\partial_1 u} + \partial_2 u\,\overline{\partial_2 u} + \partial_3 u\,\overline{\partial_3 u})\mathrm{d}^3x.$$

Not all functions $\mathbf{w} \in C^\infty(\mathbb{R}^3, \mathbb{C}^N)$ satisfy $||\mathbf{w}||_{\mathbb{R}^3,m} < \infty$. For example, a constant function from $\mathbb{R}^3$ to $\mathbb{C}^N$ will have infinite Sobolev norm. In order for a function to have finite Sobolev norm, it must decay suitably at infinity. In view of the localisability property of hyperbolic equations discussed in Section 12.2 this restriction does not pose a problem in the subsequent considerations. Thus, in what follows, attention is restricted, for given $m \in \mathbb{N}$, to the space

$$\left\{ \mathbf{w} \in C^\infty(\mathbb{R}^3, \mathbb{C}^N) \,\middle|\, ||\mathbf{w}||_{\mathbb{R}^3,m} < \infty \right\}$$

of $\mathbb{C}^N$-valued functions over $\mathbb{R}^3$ with finite Sobolev norm of order m. This set is clearly a vector space, but not a ***Banach space***; that is, not all Cauchy sequences of functions in the set have a limit in the space. To obtain a Banach space one needs to *complete the space* by including the limit points of its Cauchy sequences. The completion of the space under the norm $||\ ||_{\mathbb{R}^3,m}$ defined by Equation (12.17) is called the ***Sobolev space*** $H^m(\mathbb{R}^3, \mathbb{C}^N)$. Given

$\mathbf{w}_\bullet \in H^m(\mathbb{R}^3, \mathbb{C}^N)$, the (open) ball of radius ε centred at $\mathbf{w}_\bullet$ with respect to the norm $|| \quad ||_{\mathbb{R}^3,m}$ is defined as the set

$$B_\varepsilon(\mathbf{w}_\bullet) \equiv \big\{\mathbf{w} \in H^m(\mathbb{R}^3, \mathbb{C}^N) \,\big|\, ||\mathbf{w} - \mathbf{w}_\bullet||_{\mathbb{R}^3,m} < \varepsilon\big\}.$$

When discussing symmetric hyperbolic systems of the form (12.1), it is convenient to consider their solutions $\mathbf{u}$ as $H^m(\mathbb{R}^3, \mathbb{C}^N)$-valued functions of the time coordinate t. This point of view is expressed by writing

$$\mathbf{u}(t, \cdot) : [0, T] \longrightarrow H^m(\mathbb{R}^3, \mathbb{C}^N).$$

If a $\mathbb{C}^N$-valued function $\mathbf{u}$ is such that for every $t \in [0, T]$, $\mathbf{u}(t, \cdot) \in H^m(\mathbb{R}^3, \mathbb{C}^N)$ with C^k-dependence on t, one writes

$$\mathbf{w} \in C^k\big([0, T]; H^m(\mathbb{R}^3, \mathbb{C}^N)\big).$$

For further details on Sobolev spaces, the reader is referred to Evans (1998).

Embedding theorems

Functions in the Sobolev space $H^m(\mathbb{R}^3, \mathbb{C}^N)$ are not necessarily smooth. The reason for this is that by completing the space one has included functions with lower regularity. There is, nevertheless, a relation between functions in H^m and C^k spaces. This relation is expressed in terms of so-called ***embedding theorems***. For the particular case under consideration one has the following:

Proposition 12.1 (***Sobolev embedding theorem***) *If $m \geq 2 + k$, then $H^m(\mathbb{R}^3, \mathbb{C}^N) \subset C^k(\mathbb{R}^3, \mathbb{C}^N)$.*

In other words, if a function belongs to the H^m space, then it has at least $m - 2$ continuous derivatives. A proof of this result can be found in Taylor (1996a), chapter 4, section 1. It follows from Proposition 12.1 that a function over $\mathbb{R}^3$ is smooth (i.e. C^∞) if it belongs to H^m for every m.

Extensions of functions

To exploit the localisability property of hyperbolic equations it is often convenient to extend functions which are defined only on bounded subsets $\mathcal{R} \subset \mathbb{R}^3$ to functions with domain on the whole of $\mathbb{R}^3$. Defining in a natural way the norm $|| \quad ||_{\mathcal{R},m}$ and the Sobolev space $H^m(\mathcal{R}, \mathbb{C}^N)$ one has the following result:

Proposition 12.2 (***extension of functions on a compact domain***) *Assume that $\mathcal{R} \subset \mathbb{R}^3$ is bounded with smooth boundary $\partial\mathcal{R}$. Then there exists a linear operator*

$$\mathscr{E} : H^m(\mathcal{R}, \mathbb{C}^N) \longrightarrow H^m(\mathbb{R}^3, \mathbb{C}^N)$$

such that for each $\mathbf{u} \in H^m(\mathcal{R}, \mathbb{C}^N)$:

(i) $\mathscr{E}\mathbf{u} = \mathbf{u}$ *almost everywhere in* $\mathcal{R}$*.*
(ii) $\mathscr{E}\mathbf{u}$ *has support in an open bounded set* $\mathcal{R}' \supset \mathcal{R}$*.*
(iii) There exists a constant C *depending only on* $\mathcal{U}$ *and* $\mathcal{R}$ *such that*

$$||\mathscr{E}\mathbf{u}||_{\mathbb{R}^3,m} \leq C||\mathbf{u}||_{\mathcal{R},m}.$$

The $\mathbb{C}^N$*-valued function* $\mathscr{E}\mathbf{u}$ *is called an* ***extension of*** $\mathbf{u}$ ***to*** $\mathbb{R}^3$*.*

A discussion on how to prove this result can be found in Evans (1998).

12.3.2 Kato's existence and stability theorems

Using the terminology introduced in the previous subsections, it is now possible to discuss the basic existence and stability result for quasilinear symmetric hyperbolic systems of the form

$$\mathbf{A}^0(t, \underline{x}, \mathbf{u})\partial_t \mathbf{u} + \mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{B}(t, \underline{x}, \mathbf{u}). \tag{12.18}$$

In what follows, it will always be assumed that the matrices $\mathbf{A}^\mu$ are smooth functions of their arguments.

The basic local existence theorem

As it can be seen from the proof of the *Uniqueness Theorem* 12.1, the positive-definiteness of the matrix $\mathbf{A}^0(t, \underline{x}, \mathbf{u})$ plays a key role in determining the properties of solutions to the equation. On an initial hypersurface $\mathcal{S}$, this positivity can be set by fiat by choosing suitable initial data. However, in view of the quasilinearity of the equation, the positive-definiteness could be violated at some time as the solution evolves. Intuitively, one would expect this to lead to some sort of problems in the solution. For fixed $(t, \underline{x})$, and given a $\mathbb{C}^N$-valued function $\mathbf{w}$, one says that $\mathbf{A}^0(t, \underline{x}, \mathbf{w})$ is ***positive definite and bounded away from zero by*** $\delta > 0$ if

$$\langle \mathbf{z}, \mathbf{A}^0(t, \underline{x}, \mathbf{w})\mathbf{z}\rangle > \delta\langle \mathbf{z}, \mathbf{z}\rangle$$

for all $\mathbf{z} \in \mathbb{C}^N$.

The basic local existence result for the Cauchy problem of symmetric hyperbolic systems to be considered in this book is the following:

Theorem 12.2 (***local existence of solutions to symmetric hyperbolic systems***) *Consider the Cauchy problem*

$$\mathbf{A}^0(t, \underline{x}, \mathbf{u})\partial_t \mathbf{u} + \mathbf{A}^\alpha(t, \underline{x}, \mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{B}(t, \underline{x}, \mathbf{u}),$$
$$\mathbf{u}(0, \underline{x}) = \mathbf{u}_\star(\underline{x}) \in H^m(\mathbb{R}^3, \mathbb{C}^N) \qquad m \geq 4,$$

for a quasilinear symmetric hyperbolic system. If $\delta > 0$ *can be found such that* $\mathbf{A}^0(0, \underline{x}, \mathbf{u}_\star)$ *is positive definite with lower bound* δ *for all* $p \in \mathbb{R}^3$*, then there* *exists* $T > 0$ *and a unique solution* $\mathbf{u}$ *to the Cauchy problem defined on* $[0, T] \times \mathbb{R}^3$ *such that*

$$\mathbf{u} \in C^{m-2}\big([0, T] \times \mathbb{R}^3, \mathbb{C}^N\big).$$

Moreover, $\mathbf{A}^0(t, \underline{x}, \mathbf{u})$ *is positive definite with lower bound* δ *for* $(t, \underline{x}) \in [0, T] \times \mathbb{R}^3$.

This theorem is an adaptation of similar theorems given in Kato (1975a) and Friedrich (1986b). A proof of this result falls beyond the scope of this book. The interested reader is referred to references given above.

Remarks

(a) For convenience, the regularity of the solution has been stated in terms of C^k spaces. However, the conclusions of the theorem can be expressed in a more detailed manner. In particular, one has that the solution satisfies

$$\mathbf{u} \in C^1\big([0, T], H^{m-1}(\mathbb{R}^3, \mathbb{C}^N)\big).$$

The latter can be shown to imply $\mathbf{u} \in H^m([0, T] \times \mathbb{R}^3, \mathbb{C}^N)$ which, in turn, using a Sobolev embedding theorem in four-dimensions gives the regularity stated in the theorem.

(b) In most of the applications given in this book, the initial data $\mathbf{u}_\star$ will be assumed to be smooth, so that $\mathbf{u}_\star \in H^m(\mathbb{R}^3, \mathbb{C}^N)$ for all m. However, as $\mathbb{R}^3$ is an unbounded set, one cannot simply assume that $\mathbf{u}_\star \in C^\infty(\mathbb{R}^3, \mathbb{C}^N)$; compare the remark after Equation (12.17).

(c) As $\mathbf{A}^0$ is a smooth function of its arguments, it follows from the regularity of the solution $\mathbf{u}$ that $\langle \mathbf{z}, \mathbf{A}^0(t, \underline{x}, \mathbf{u})\mathbf{z}\rangle$ for $\mathbf{z} \in \mathbb{C}^N$ depends continuously on $(t, \underline{x})$.

(d) As $\mathbf{A}^0(t, \underline{x}, \mathbf{u})$ is positive definite for $(t, \underline{x}) \in [0, T] \times \mathbb{R}^3$, it follows that the hypersurfaces of constant t are spacelike with respect to the symmetric hyperbolic system (and the solution).

(e) The value of the lower bound δ can often be determined by inspection.

The basic stability result

Of great relevance is the notion of ***Cauchy stability*** – namely, the idea that, given a symmetric hyperbolic system, initial data which are close to each other should lead to solutions which are close in some sense and have a common existence time interval. In view of the inherent error in the physical process of measurement, Cauchy stability is fundamental for the applicability of differential equations to describe physical phenomena. In mathematical terms, the precise formulation of the *closeness* of initial data and solutions is expressed in terms of Sobolev norms.

In the remainder of this section let D be a bounded open subset of $H^m(\mathbb{R}^3, \mathbb{C}^N)$ such that for $\mathbf{w} \in D$ the matrix $\mathbf{A}^0(0, \underline{x}, \mathbf{w})$ is positive definite bounded away from zero by $\delta > 0$ for all $p \in \mathbb{R}^4$. The basic result describing the Cauchy stability of the symmetric hyperbolic system (12.18) is the following theorem, adapted from Kato (1975a):

Theorem 12.3 (***basic Cauchy stability for symmetric hyperbolic systems***) *Let $m \geq 4$. If $\mathbf{u}_\star \in D$ is given as an initial condition for the system* (12.18)*, then:*

(i) There exists $\varepsilon > 0$ such that a common existence time T can be chosen for all initial conditions in the open ball $B_\varepsilon(\mathbf{u}_\star) \subset D$.
(ii) If the solution $\mathbf{u}$ with initial data $\mathbf{u}_\star$ exists on $[0, T]$ for some $T > 0$, then the solutions to all initial conditions in $B_\varepsilon(\mathbf{u}_\star)$ exist on $[0, T]$ if $\varepsilon > 0$ is sufficiently small.
(iii) If ε and T are chosen as in (i) and one has a sequence $\mathbf{u}^n_\star \in B_\varepsilon(\mathbf{u}_\star)$ such that

$$||\mathbf{u}^n_\star - \mathbf{u}_\star||_{\mathbb{R}^3,m} \to 0 \qquad \text{as } n \to \infty,$$

then for the solutions $\mathbf{u}^n(t, \cdot)$ with $\mathbf{u}^n(0, \cdot) = \mathbf{u}^n_\star$ it holds that

$$||\mathbf{u}^n(t, \cdot) - \mathbf{u}(t, \cdot)||_{\mathbb{R}^3,m} \to 0, \qquad \text{as } n \to \infty,$$

uniformly for $t \in [0, T]$.

Remarks

(a) Point (i) in the previous theorem essentially states that, given a sufficiently small ball in the space of data on which the *Existence Theorem* 12.2 can be applied, then a common existence time for the solutions arising from this data can be found. Observe, however, that one has no control over the size of the common existence time; one only knows there is one.
(b) If the existence of a particular solution is known, then point (ii) states that, by shrinking the ball on the space of data, one can choose the known existence time as the common existence time.
(c) Point (iii) states that data close to certain reference data give rise to developments which are also close to the reference solution; this is the statement of *Cauchy stability.*
(d) The convergence stated in (iii) is uniform on $[0, T] \times \mathbb{R}^3$.

12.3.3 Localising solutions

The localisability property of hyperbolic equations allows one to apply the existence and stability results discussed in the previous sections to the case of an initial data problem where data are prescribed only on a compact region $\mathcal{R}$. Given smooth initial data $\mathbf{u}_\star$ for a symmetric hyperbolic equation of the form (12.1) on a region $\mathcal{R} \subset \mathbb{R}^3$, one can make use of Proposition 12.2 to extend the

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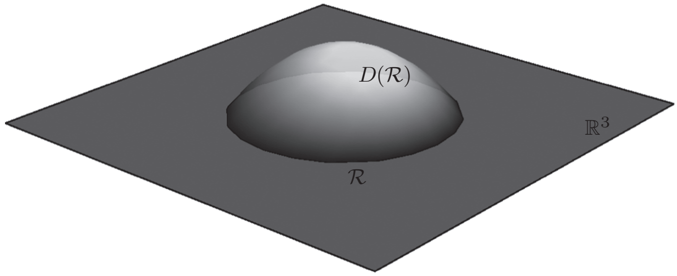

Figure 12.3 Localised solution arising from data prescribed on an open set $\mathcal{R} \subset \mathbb{R}^3$. The associated domain of dependence is denoted by $D(\mathcal{R})$.

initial data $\mathbf{u}_\star$ to the whole of $\mathbb{R}^3$ in a controlled manner. Denoting this extension by $\mathscr{E}\mathbf{u}_\star$, one has that by point (iii) of Proposition 12.2, $\mathscr{E}\mathbf{u}_\star \in H^m(\mathbb{R}^3, \mathbb{C}^N)$.

In order to make use of Theorems 12.2 and 12.3 it is necessary to assume that $\mathbf{A}^0(0, \underline{x}, \mathscr{E}\mathbf{u}_\star)$ is positive definite with some non-zero lower bound uniform on $\mathbb{R}^3$. Thus, one obtains a solution to Equation (12.1) with initial data on $\mathbb{R}^3$ given by $\mathbf{u}(0, \underline{x}) = \mathscr{E}\mathbf{u}_\star(\underline{x})$. As a consequence of the uniqueness of solutions on the domain of dependence, the solution $\mathbf{u}$ on $D(\mathcal{R})$ is independent of the particular extension of the initial data $\mathbf{u}_\star$ on $\mathcal{R}$ to $\mathbb{R}^3$; see Figure 12.3.

12.3.4 Existence and stability result on manifolds with compact spatial sections

The existence and stability Theorems 12.2 and 12.3 can be modified so as to apply to Cauchy problems where data is prescribed on compact, orientable three-dimensional manifolds. In what follows, the main ideas behind this construction are discussed.

Patching together solutions

In the remainder of this section let $\mathcal{S}$ denote an orientable, compact three-dimensional manifold – in most of the applications to be considered in this book one has $\mathcal{S} \approx \mathbb{S}^3$; however, any other compact, orientable topology will work as well. As a result of compactness, there exists a *finite cover* consisting of open sets $\mathcal{R}_1, \ldots, \mathcal{R}_M \subset \mathcal{S}$; that is, one has $\cup_{i=1}^M \mathcal{R}_i = \mathcal{S}$. On each of the open sets $\mathcal{R}_i$, $i = 1, \ldots, M$, one can introduce local coordinates $\underline{x}_i \equiv (x_i^\alpha)$ which allow one to identify $\mathcal{R}_i$ with open subsets $\mathcal{B}_i \subset \mathbb{R}^3$. As $\mathcal{S}$ is assumed to be a smooth manifold, the *coordinate patches* can be chosen so that the change of coordinates on intersecting sets is smooth.

Now, assume that a smooth function $\mathbf{u}_\star : \mathcal{S} \to \mathbb{C}^N$ has been prescribed on $\mathcal{S}$. In what follows, the restriction of $\mathbf{u}_\star$ to a particular open set $\mathcal{R}_i$ will be denoted by $\mathbf{u}_{i\star}$. Using the local coordinates x_i, the function $\mathbf{u}_{i\star}$ can be regarded as a function $\mathbf{u}_{i\star} : \mathcal{B}_i \to \mathbb{C}^N$.

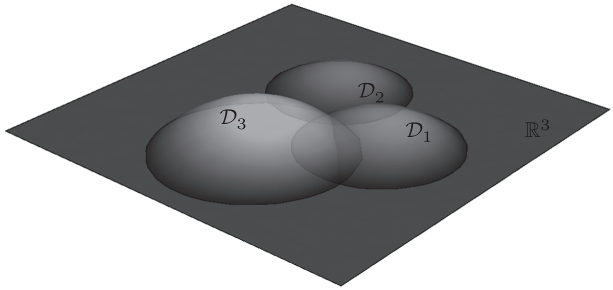

Figure 12.4 Construction of a solution by patching localised solutions to data prescribed on open sets $\mathcal{D}_1,\ \mathcal{D}_2,\ \mathcal{D}_3 \subset \mathbb{R}^3$.

The strategy is now to use the same procedure as described in Section 12.3.3 to ensure the existence of a solution on the domain of dependence of $\mathcal{B}_i$. Accordingly, one makes use of Proposition 12.2 to extend $\mathbf{u}_{i\star}$ to a function $\mathscr{E}\mathbf{u}_{i\star}$ defined on the whole of $\mathbb{R}^3$. Using the extended functions $\mathscr{E}\mathbf{u}_{i\star}$ one defines the norm

$$||\mathbf{u}_\star||_{\mathcal{S},m} \equiv \sum_{i=1}^{M} ||\mathbf{u}_{i\star}||_{\mathbb{R}^3,m}. \tag{12.19}$$

Assuming, as in Section 12.3.3, that $\mathbf{A}^0(0,\underline{x},\mathscr{E}\mathbf{u}_{i\star})$ is positive definite with lower bound $\delta_i > 0$, one obtains a unique solution $\mathbf{u}_i$ of Equation (12.1) with initial data $\mathbf{u}(0,\underline{x}) = \mathscr{E}\mathbf{u}_{i\star}(\underline{x})$ with existence interval $[0,T_i]$. The solution on $D(\mathcal{B}_i)$ is independent of the particular extension $\mathscr{E}\mathbf{u}_{i\star}$ being used, so that one can speak of a solution $\mathbf{u}_i$ on a domain $\mathcal{D}_i \subset [0,T_i] \times \mathcal{R}_i$; see Figure 12.4.

Now, given two solutions $\mathbf{u}_i$ and $\mathbf{u}_j$ defined, respectively, on intersecting domains $\mathcal{D}_i$ and $\mathcal{D}_j$ one has – following the discussion on the change of coordinates given in Section 12.1 and as a consequence of uniqueness – that $\mathbf{u}_i$ and $\mathbf{u}_j$ must coincide on $\mathcal{D}_i \cap \mathcal{D}_j$. Proceeding in the same manner over the whole finite cover of $\mathcal{S}$, one obtains a unique solution $\mathbf{u}$ on $[0,T] \times \mathcal{S}$ with $T \equiv \min_{i=1,\ldots M}\{T_i\}$ which is constructed by *patching together the localised solutions* $\mathbf{u}_1,\ldots,\mathbf{u}_M$ defined, respectively on the domains $\mathcal{D}_i,\ldots,\mathcal{D}_M$. Observe that the compactness of $\mathcal{S}$ ensures the existence of a minimum non-zero existence time for the whole of the domains $\mathcal{D}_i$.

A general existence and stability result

Using the ideas of the localisation of solutions discussed in the previous subsection, one can formulate a quite general existence and stability result for symmetric hyperbolic systems on manifolds whose spatial sections are given by orientable, compact three-dimensional manifolds. The hypotheses of this theorem are very similar to the ones in Theorems 12.2 and 12.3.

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Theorem 12.4 (***existence and stability result for symmetric hyperbolic systems on compact spatial sections***) *Given an orientable, compact, three-dimensional manifold $\mathcal{S}$, consider the Cauchy problem*

$$\begin{aligned}
&\mathbf{A}^0(t,\underline{x},\mathbf{u})\partial_t\mathbf{u} + \mathbf{A}^\alpha(t,\underline{x},\mathbf{u})\partial_\alpha\mathbf{u} = \mathbf{B}(t,\underline{x},\mathbf{u}),\\
&\mathbf{u}(0,\underline{x}) = \mathbf{u}_\star(\underline{x}) \in H^m(\mathcal{S},\mathbb{C}^N) \qquad \text{for } m \geq 4,
\end{aligned}$$

for a quasilinear symmetric hyperbolic system. If $\delta > 0$ can be found such that $\mathbf{A}^0(0,\underline{x},\mathbf{u}_\star)$ is positive definite with lower bound δ for all $x \in \mathcal{S}$, then:

(i) There exists $T > 0$ and a unique solution $\mathbf{u}$ to the Cauchy problem defined on $[0,T]\times\mathcal{S}$ such that

$$\mathbf{u} \in C^{m-2}\big([0,T]\times\mathcal{S},\mathbb{C}^N\big).$$

Moreover, $\mathbf{A}^0(t,\underline{x},\mathbf{u})$ is positive definite with lower bound δ for $(t,\underline{x}) \in [0,T]\times\mathcal{S}$.

(ii) There exists $\varepsilon > 0$ such that one common existence time T can be chosen for all initial conditions in the open ball $B_\varepsilon(\mathbf{u}_\star)$ and such that $B_\varepsilon(\mathbf{u}_\star) \subset D$.

(iii) If the solution $\mathbf{u}$ with initial data $\mathbf{u}_\star$ exists on $[0,T]$ for some $T > 0$, then the solutions to all initial conditions in $B_\varepsilon(\mathbf{u}_\star)$ exist on $[0,T]$ if $\varepsilon > 0$ is sufficiently small.

(iv) If ε and T are chosen as in (ii) and one has a sequence $\mathbf{u}^n_\star \in B_\varepsilon(\boldsymbol{u}_\star)$ such that

$$||\mathbf{u}^n_\star - \mathbf{u}_\star||_{\mathcal{S},m} \to 0, \qquad \text{as } n \to \infty,$$

then for the solutions $\mathbf{u}^n(t,\cdot)$ with $\mathbf{u}^n(0,\cdot) \equiv \mathbf{u}^n_\star$ it holds that

$$||\mathbf{u}^n(t,\cdot) - \mathbf{u}(t,\cdot)||_{\mathcal{S},m} \to 0, \qquad \text{as } n \to \infty$$

uniformly in $t \in [0,T]$.

Remarks similar to the ones after Theorems 12.2 and 12.3 apply to this result. Further discussion and details can be found in Friedrich (1991).

12.4 Local existence for boundary value problems

As will be seen in Chapter 17, the construction of anti-de Sitter-like spacetimes leads one to consider initial boundary value problems for symmetric hyperbolic systems of the form (12.1). In this type of problem one prescribes initial data on a spacelike hypersurface $\mathcal{S}$ and boundary data on a timelike hypersurface $\mathcal{T}$. These two hypersurfaces intersect on a two-dimensional hypersurface $\mathcal{E} \equiv \mathcal{S}\cap\mathcal{T}$, the ***edge***, on which the initial and the boundary conditions need to satisfy some compatibility conditions; see Figure 12.5. In view of the localisation property of

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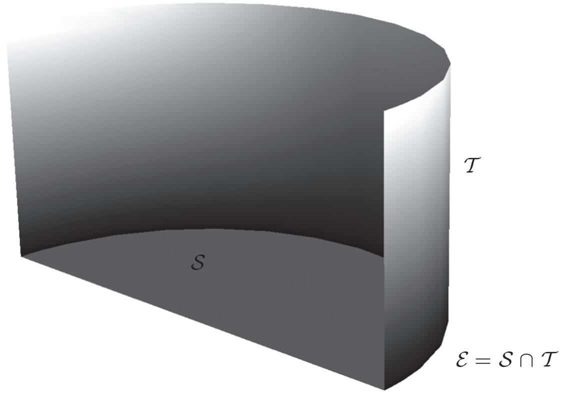

Figure 12.5 Geometric setting of the initial boundary value problem for symmetric hyperbolic systems. The initial data are prescribed on the three-dimensional spacelike hypersurface $\mathcal{S}$; boundary data are prescribed on the three-dimensional timelike hypersurface $\mathcal{T}$. The initial and boundary data must satisfy certain compatibility conditions (corner conditions) on the edge $\mathcal{E} = \mathcal{S} \cap \mathcal{T}$.

symmetric hyperbolic systems, it is sufficient to analyse the problem close to the edge. The solution away from the boundary is obtained by patching domains of dependence.

12.4.1 Basic setting

In a neighbourhood of a point $p \in \mathcal{E}$, one can introduce coordinates $x = (x^\mu)$ such that the domain $\mathcal{U}$ in which the solution to the boundary value problem takes the form

$$\mathcal{U} = \{x \in \mathbb{R}^4 \,|\, x^0 \geq 0,\ x^3 \geq 0\},$$

while the initial hypersurface and the boundary are given, respectively, by

$$\mathcal{S} \equiv \{x \in \mathcal{U} \,|\, x^0 = 0\},$$
$$\mathcal{T} \equiv \{x \in \mathcal{U} \,|\, x^3 = 0\}.$$

The ***normal matrix*** $\mathbf{A}^3(x, \mathbf{u})$ in a symmetric hyperbolic system of the form (12.1) plays a crucial role in the specification of admissible boundary conditions leading to a well-posed initial boundary value problem. Due to the use of coordinates adapted to the boundary, the properties of the matrix $\mathbf{A}^3$ determine the relation between the timelike boundary $\mathcal{T}$ and the characteristics of the hyperbolic evolution equation.

In what follows let $\mathbf{T}(x)$ denote a smooth map from $\mathcal{T}$ to the vector subspaces of $\mathbb{C}^N$ and require as boundary condition that

$$\mathbf{u}(x) \in \mathbf{T}(x), \qquad x \in \mathcal{T}.$$

The map $\mathbf{T}$ is restricted by the requirements:

(i) The set $\mathcal{T}$ is a characteristic of (12.1) of constant multiplicity; that is,

$$\dim \operatorname{Ker}\big(\mathbf{A}^3\big) = \text{constant} > 0, \qquad x \in \mathcal{T}.$$

(ii) The map $\mathbf{T}$ satisfies the ***non-positivity condition***

$$\langle \mathbf{u}, \mathbf{A}^3(x,\mathbf{u})\mathbf{u}\rangle \leq 0, \qquad \mathbf{u} \in \mathbf{T}(x),\ x \in \mathcal{T}.$$

(iii) The dimension of the subspace $\mathbf{T}(x)$, $x \in \mathcal{T}$, is equal to the number of non-positive eigenvalues of $\mathbf{A}^3(x,\mathbf{u})$ counting multiplicities.

An important property of Hermitian matrices is that they can be diagonalised by unitary matrices and that all their eigenvalues are real. Accordingly, after a redefinition of the dependent variables one can assume that, at a given point $x \in \mathcal{T}$, the normal matrix $\mathbf{A}^3(x,\mathbf{u})$ has the form

$$\mathbf{A}^3(x,\mathbf{u}) = \kappa \begin{pmatrix} -\mathbf{I}_{j\times j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{k\times k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{l\times l} \end{pmatrix}, \qquad \kappa > 0,$$

where $\mathbf{I}_{j\times j}$ and $\mathbf{I}_{l\times l}$ are, respectively, $j \times j$ and $l \times l$ unit matrices and $\mathbf{0}_{k\times k}$ is the $k \times k$ zero matrix. Moreover, one has that $j + k + l = N$. Writing

$$\mathbf{u}(x) = \begin{pmatrix} \mathbf{a}(x) \\ \mathbf{b}(x) \\ \mathbf{c}(x) \end{pmatrix} \in \mathbb{C}^j \times \mathbb{C}^k \times \mathbb{C}^l,$$

one finds that the linear subspaces admitted by condition (ii) are of the form

$$\mathbf{c} - \mathbf{H}\mathbf{a} = \mathbf{0}$$

with $\mathbf{H} = \mathbf{H}(x)$ an $l \times j$ matrix satisfying

$$-\langle \mathbf{a}, \mathbf{a}\rangle + \langle \mathbf{H}\mathbf{a}, \mathbf{H}\mathbf{a}\rangle \leq 0, \qquad \mathbf{a} \in \mathbb{C}^j.$$

This condition can be reexpressed, alternatively, as $\mathbf{H}^*\mathbf{H} \leq \mathbf{I}_{j\times j}$. The key observation is that the above procedure gives no freedom to prescribe data for the component $\mathbf{b}$ of $\mathbf{u}$ associated with the kernel of the normal matrix $\mathbf{A}^3(x,\mathbf{u})$. In particular, if $\mathbf{A}^3(x,\mathbf{u}) = 0$, one has that the boundary is a total characteristic (see Section 12.1.2) and no boundary conditions can be specified on $\mathcal{T}$ – the solution $\mathbf{u}$ on $\mathcal{T}$ is directly determined by the initial conditions on the edge $\mathcal{E}$. More generally, by a further redefinition of the dependent variables one obtains the ***inhomogeneous maximally dissipative boundary conditions***

$$\mathbf{q}(x) = \mathbf{c}(x) - \mathbf{H}(x)\mathbf{a}(x), \qquad x \in \mathcal{T},$$

with $\mathbf{q}(x)$ a $\mathbb{C}^l$-valued function representing the free boundary data on $\mathcal{T}$.

Corner conditions

To obtain a smooth solution to an initial boundary value problem for a symmetric hyperbolic system of the form (12.18), the initial data prescribed on $\mathcal{S}$ and the boundary data at $\mathcal{T}$ must satisfy certain compatibility conditions at the edge $\mathcal{E} = \partial\mathcal{S} = \mathcal{S} \cap \mathcal{T}$ – the so-called ***corner conditions***. More precisely, if one has initial data of the form

$$\mathbf{u}(0, \underline{x}) = \mathbf{u}_\star(\underline{x}) \quad \text{on } \mathcal{S},$$

with $\mathbf{u}_\star$ smooth and maximally dissipative boundary conditions of the form

$$\mathbf{T}(t, \underline{x})\mathbf{u}(t, \underline{x}) = \mathbf{q}(t, \underline{x}) \quad \text{on } \mathcal{T}, \tag{12.20}$$

then one requires that

$$\mathbf{T}(0, \underline{x})\mathbf{u}_\star|_\mathcal{E} = \mathbf{q}(0, \underline{x}).$$

Higher order corner conditions can be obtained by considering the system (12.18). Evaluating at $\mathcal{E}$ one obtains

$$\mathbf{A}^0(\mathbf{u}_\star)|_\mathcal{E}(\partial_t \mathbf{u})|_\mathcal{E} + \mathbf{A}^\alpha(\mathbf{u}_\star)|_\mathcal{E}(\partial_\alpha \mathbf{u}_\star)|_\mathcal{E} = \mathbf{B}(\mathbf{u}_\star)|_\mathcal{E}.$$

As $\mathbf{A}^0(\mathbf{u}_\star)|_\mathcal{E}$ is positive definite, the above equation can be used to solve for $(\partial_t \mathbf{u})|_\mathcal{E}$. The result should be consistent, upon substitution, with what is obtained from differentiating the boundary condition (12.20). Namely,

$$(\partial_t \mathbf{T})|_\mathcal{E}\mathbf{u}|_\mathcal{E} + \mathbf{T}|_\mathcal{E}(\partial_t \mathbf{u})|_\mathcal{E} = (\partial_t \mathbf{q})|_\mathcal{E}.$$

Further higher order boundary conditions are obtained in an analogous manner by differentiating (12.18) successively with respect to t.

12.4.2 Uniqueness of the solutions to the boundary value problem

Insight into the role of the maximally dissipative boundary conditions can be obtained from the analysis of the uniqueness of solutions to the boundary value problem. The argument follows a strategy similar to the one employed in Theorem 12.1 with a domain $\mathcal{G}$ whose boundary consists of portions of the initial hypersurface $\mathcal{S}_0$, the boundary $\mathcal{T}$ and a hypersurface $\mathcal{S}_1$ which is spacelike with respect to the symmetric hyperbolic system; see Figure 12.6. Set $\mathcal{M} = [0, \infty) \times \mathcal{S}$ such that $\mathcal{S}$ and $\mathcal{T}$ can be identified, in a natural way, as the boundary of $\mathcal{M}$. Define the coordinate $x^0 \equiv t$ in such a way that $\mathcal{S}_0 = \{p \in \mathcal{M} \,|\, t = 0\}$.

Theorem 12.5 (***uniqueness of solutions of the initial boundary problem with maximally dissipative boundary conditions***) *Let $\mathcal{G}$ be a domain as given above. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions to the initial value problem for the symmetric hyperbolic system*

$$\mathbf{A}^\mu(x, \mathbf{u})\partial_\mu \mathbf{u} = \mathbf{B}(x, \mathbf{u}), \qquad \mathbf{u}|_{\mathcal{S}_0} = \mathbf{u}_\star,$$

with the same maximally dissipative boundary conditions, then $\mathbf{u}_1 = \mathbf{u}_2$ on $\mathcal{G}$.

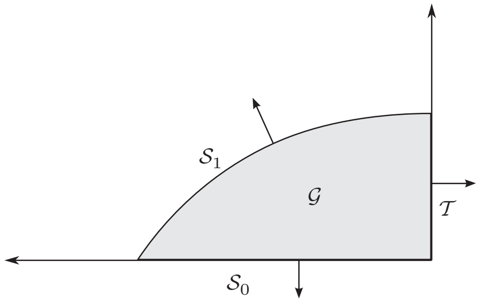

Figure 12.6 Integration domain for the uniqueness argument for initial boundary value problems. The boundary $\partial\mathcal{G}$ consists of portions of the initial hypersurface $\mathcal{S}_0$, the timelike boundary $\mathcal{T}$ and a spacelike hypersurface $\mathcal{S}_1$.

Proof Starting from the identity (12.14) one integrates over a domain $\mathcal{G}$ as depicted in Figure 12.6, where $\mathcal{S}_1$ is spacelike with respect to the symmetric hyperbolic system. Applying the Gauss identity one obtains

$$\begin{aligned}
&\int_{\mathcal{S}_1} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \nu_\mu \mathrm{d}S \\
&\quad - \int_{\mathcal{S}_0} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \nu_\mu \mathrm{d}S \\
&\quad - \int_{\mathcal{T}} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \nu_\mu \mathrm{d}S \\
&\qquad = \int_{\mathcal{G}} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \mathrm{d}^4x,
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{P}(x,\mathbf{u}_1,\mathbf{u}_2) \equiv{}& -k\mathbf{A}^0(x,\mathbf{u}_1) + \partial_\mu \mathbf{A}^\mu(x,\mathbf{u}_1) \\
&+ \mathbf{Q}(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2) + \mathbf{Q}^*(x,\mathbf{u}_1,\mathbf{u}_2,\partial\mathbf{u}_2)
\end{aligned}$$

and $\mathbf{Q}$ obtained as in Theorem 12.1 using the mean value theorem. Exploiting the positive definiteness of $\mathbf{A}^0(x,\mathbf{u}_1)$, one can make the volume integral over $\mathcal{G}$ negative. Moreover, as $\mathbf{u}_1$ and $\mathbf{u}_2$ coincide on $\mathcal{S}_0$ one obtains

$$\begin{aligned}
&\int_{\mathcal{S}_1} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \nu_\mu \mathrm{d}S \\
&\qquad \leq \int_{\mathcal{T}} e^{-kt}\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{A}^\mu(x,\mathbf{u}_1)(\mathbf{u}_1 - \mathbf{u}_2)\rangle \nu_\mu \mathrm{d}S \leq 0,
\end{aligned}$$

where the last inequality follows from the negative definiteness of the maximally dissipative boundary conditions. Thus, one obtains a contradiction with the fact that the surface integral over the spacelike hypersurface $\mathcal{S}_1$ is positive unless $\mathbf{u}_1 = \mathbf{u}_2$. □

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12.4.3 The basic existence result for the initial boundary value problem of symmetric hyperbolic systems

One has the following basic local existence theorem for the initial boundary value problem with maximally dissipative boundary conditions:

Theorem 12.6 (***local existence for initial boundary value problems***) *Given the initial boundary value problem*

$$\mathbf{A}^0(t,\underline{x},\mathbf{u})\partial_t\mathbf{u} + \mathbf{A}^\alpha(t,\underline{x},\mathbf{u})\partial_\alpha\mathbf{u} = \mathbf{B}(t,\underline{x},\mathbf{u}), \tag{12.21a}$$

$$\mathbf{T}(t,\underline{x})\mathbf{u} = \mathbf{q}(t,\underline{x}) \quad on\ \mathcal{T}, \tag{12.21b}$$

$$\mathbf{u}(0,\underline{x}) = \mathbf{u}_\star(\underline{x}), \quad on\ \mathcal{S}, \tag{12.21c}$$

with (12.21a) *symmetric hyperbolic,* $\mathbf{A}^0(0,\underline{x},\mathbf{u}_\star)$ *positive definite and* $\mathbf{q}$*,* $\mathbf{u}_\star$ *smooth, assume that the boundary condition* (12.21b) *is maximally dissipative with respect to the normal matrix* $\mathbf{A}^3(t,\underline{x},\mathbf{u})$ *and that the boundary data satisfy corner conditions at* $\mathcal{E} = \mathcal{S}\cap\mathcal{T}$ *to all orders. Then, the initial boundary value problem has a unique smooth solution* $\mathbf{u}(t,\underline{x})$ *defined on*

$$\mathcal{M}_T = \{p \in [0,\infty)\times\mathcal{S} \,|\, 0 \le t(p) < T\},$$

for some $T > 0$.

The reader is refereed to Guès (1990), Friedrich (1995) and Friedrich and Nagy (1999) for details and remarks concerning the proof. As a consequence of the localisability property of hyperbolic equations, the problem can be split into two parts: an interior one away from the boundary in which the standard

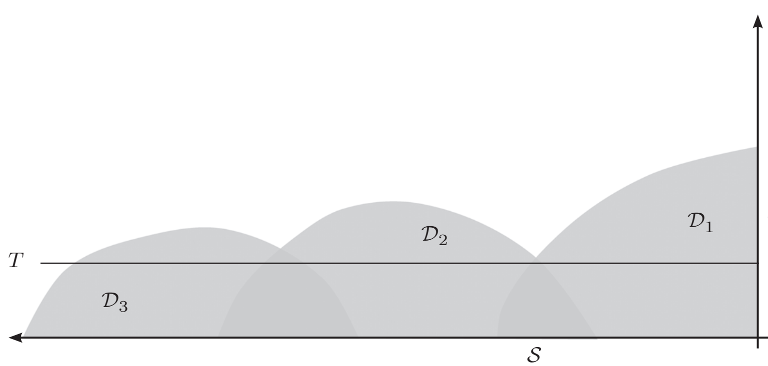

Figure 12.7 Construction of a solution to an initial boundary value problem which is global in space by patching domains. The solution patch $\mathcal{D}_1$ near the boundary $\mathcal{T}$ is obtained using Theorem 12.6. The existence on the domains $\mathcal{D}_2$ and $\mathcal{D}_3$ away from the boundary are obtained by means of Theorem 12.3. The uniqueness of solutions ensures that the solution on the intersections "match together". Due to the compactness of the initial hypersurface it is possible to obtain an existence time T common to all domains.

local existence for the Cauchy problem (as described in Section 12.3) is used, and a boundary part in which the boundary and edge conditions play a role; see Figure 12.7. The local solutions are then patched together to obtain the solution on the whole of $\mathcal{M}_T$.

Remark. The question of the stability of solutions to the initial boundary value problem will not be analysed here. Stability questions for initial boundary value problems are much more complicated than for the Cauchy case. At the time of writing, there are no applications of stability results for boundary value problems involving the conformal field equations.

12.5 Local existence for characteristic initial value problems

Characteristic initial value problems arise naturally in applications to general relativity; see Chapter 18. The purpose of this section is to discuss a method to analyse the local existence of solutions to the characteristic initial value problem for symmetric hyperbolic equations due to Rendall (1990). The idea behind this method is to reduce the characteristic problem to a standard Cauchy problem where the *standard* theory of Section 12.3 can be applied.

12.5.1 General remarks on the characteristic problem

In what follows, consider a quasilinear symmetric hyperbolic system of the form given by Equation (12.1) on $\mathbb{R}^4$. In contrast to the analysis of the Cauchy problem where it is convenient to single out one of the coordinates as a time coordinate, in the characteristic problem it is convenient to make use of coordinates adapted to the characteristic hypersurfaces.

As discussed in Section 12.1.2, for quasilinear equations like (12.1), the notion of characteristic hypersurfaces depends on the solution $\mathbf{u}$. Thus, it is, in principle, unclear on which hypersurfaces one can prescribe the characteristic initial data. There are two approaches to get around this difficulty:

(i) ***Fix the data first, then look for the hypersurface.*** Choose a smooth function $\mathbf{v}$ on $\mathcal{U} \subset \mathbb{R}^4$ such that the matrices $\mathbf{A}^\mu(x, \mathbf{v})$ are defined at each point of $\mathbb{R}^4$, and choose a smooth function $\phi \in \mathcal{X}(\mathcal{U})$, $\mathbf{d}\phi \neq 0$ in $\mathcal{U}$, such that the hypersurface

$$\mathcal{N} \equiv \{x \in \mathcal{U} \mid \phi(x) = 0\} \tag{12.22}$$

is characteristic with respect to $\mathbf{A}^\mu(x, \mathbf{v})$, that is, such that

$$\det(\mathbf{A}^\mu(x, \mathbf{v})\partial_\mu \phi) = 0.$$

The *characteristic data* on $\mathcal{N}$ is then given as the restriction of $\mathbf{v}$ to $\mathcal{N}$; that is, $\mathbf{u}_\star = \mathbf{v}|_{\mathcal{N}}$.

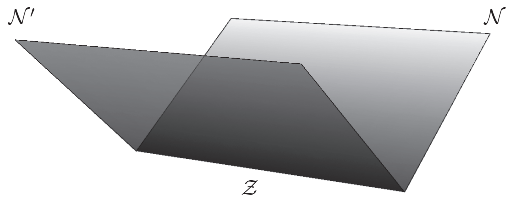

Figure 12.8 Initial hypersurfaces $\mathcal{N}$ and $\mathcal{N}'$ on a characteristic initial value problem. The set $\mathcal{Z} \neq \emptyset$ is the intersection of $\mathcal{N}$ and $\mathcal{N}'$.

(ii) ***Choose the hypersurface first, then look for suitable data.*** Alternatively, one can choose some hypersurface $\mathcal{N}$ in $\mathcal{U} \subset \mathbb{R}^4$ defined as in (12.22), and then consider only those smooth functions $\mathbf{u}_\star$ such that

$$\det(\mathbf{A}^\mu(x, \mathbf{u}_\star)\partial_\mu \phi) = 0.$$

Approach (ii) is more natural in applications where geometric information of the initial hypersurface is available. *This point of view will be adopted in the rest of this section.*

A peculiarity of characteristic initial value problems for the system (12.1) is that data need to be prescribed on *two intersecting characteristic hypersurfaces* $\mathcal{N}$ and $\mathcal{N}'$; see Figure 12.8. Intuitively, this is a consequence of the existence of a subsystem of equations in (12.1) which is intrinsic to the hypersurface $\mathcal{N}$, so that one does not have enough evolution equations transverse to the hypersurface for all the components of $\mathbf{u}$. Alternatively, one can formulate characteristic initial value problems by prescribing initial data on a *cone*. This is a more technically involved problem and will not be discussed here. The interested reader is referred to Cagnac (1981) and Dossa (1997) for further details.

Well- and ill-posed characteristic problems

In what follows, let $\mathcal{N}$ and $\mathcal{N}'$ denote two hypersurfaces on $\mathcal{U} \subset \mathbb{R}^4$ with non-empty intersection $\mathcal{Z} \equiv \mathcal{N} \cap \mathcal{N}'$. One can introduce coordinates u and v such that, at least in a neighbourhood of $\mathcal{N} \cap \mathcal{N}'$, one can write

$$\mathcal{N} \equiv \{p \in \mathcal{U} \,|\, u(p) = 0\}, \qquad \mathcal{N}' \equiv \{p \in \mathcal{U} \,|\, v(p) = 0\}. \tag{12.23}$$

Given suitable initial data on $\mathcal{N} \cup \mathcal{N}'$ one would like to make some statement about the existence and the uniqueness of solutions to Equation (12.1) on some open set

$$\mathcal{V} \subset \{p \in \mathcal{U} \,|\, u(p) \geq 0,\ v(p) \geq 0\}.$$

By symmetry, one could also look for a solution in the region

$$\{p \in \mathcal{U} \,|\, u(p) \leq 0,\ v(p) \leq 0\};$$

see Figure 12.9 (a).

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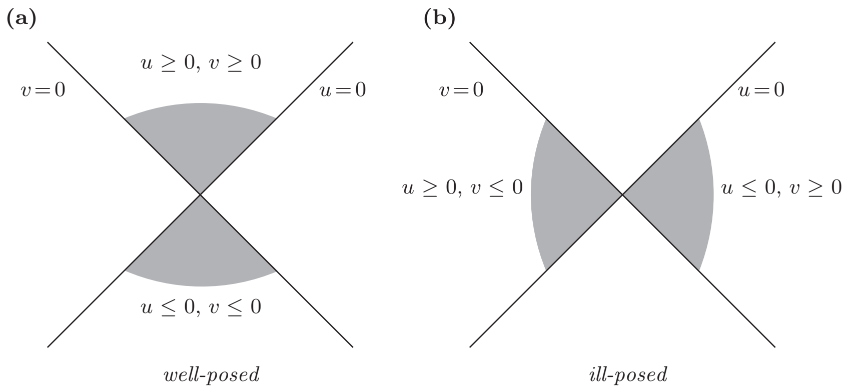

Figure 12.9 Schematic representation of well-posed (a) and ill-posed (b) characteristic initial value problems.

The problem of looking for solutions in domains of the form

$$\bar{\mathcal{V}} \subset \{p \in \mathcal{U} \,|\, u(p) \leq 0,\ v(p) \geq 0\}$$

or

$$\bar{\bar{\mathcal{V}}} \subset \{p \in \mathcal{U} \,|\, u(p) \geq 0,\ v(p) \leq 0\}$$

is *ill-posed* – the reason will become clear once the Rendall's reduction procedure to a Cauchy problem is discussed in Section 12.5.3. Under suitable circumstances, it may be possible to establish uniqueness of a solution – but not existence – for this ill-posed problem. These ideas have been used by Ionescu and Klainerman (2009a,b) to obtain a new strategy to prove the uniqueness of stationary black holes.

12.5.2 Interior equations on the characteristic hypersurfaces

As seen in Section 12.1.2, on a characteristic surface, a system of the form (12.1) implies a subsystem of interior equations on the hypersurface. Assuming that the freely specifiable part of $\mathbf{u}$ is smooth on the characteristic hypersurface, these interior equations can be used to compute the remaining components of $\mathbf{u}$ and their derivatives to any arbitrary order. For conciseness, the subsequent analysis is restricted to the characteristic $\mathcal{N}$ as given by (12.23). The situation on $\mathcal{N}'$ is completely analogous. Letting $x \equiv (u, \underline{y})$ with $\underline{y} \equiv (y^\alpha) = (v, x^2, x^3)$ and using the chain rule, Equation (12.1) can be rewritten as

$$\boldsymbol{\sigma}(u, \underline{y}; \mathbf{u}, \mathbf{d}u)\partial_u \mathbf{u} + \mathbf{A}^\alpha(u, \underline{y}; \mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{B}(u, \underline{y}; \mathbf{u}) \qquad (12.24)$$

with

$$\boldsymbol{\sigma}(u, \underline{y}; \mathbf{u}, \mathbf{d}u) \equiv \mathbf{A}^\mu(u, \underline{y}; \mathbf{u})\frac{\partial u}{\partial x^\mu}.$$

If $\mathcal{N}$ is a characteristic hypersurface for some function $\mathbf{u}_\star$, then one has that

$$\det\big(\boldsymbol{\sigma}(0,\underline{y};\mathbf{u}_\star,\mathbf{d}u)\big)=0.$$

Let $m=\dim\operatorname{Ker}\boldsymbol{\sigma}(0,y;\mathbf{u}_\star,\mathbf{d}u)$. It follows that there exist m vectors $\mathbf{k}_{(i)}$, $i=1,\ldots,m$ such that

$$\boldsymbol{\sigma}(0,\underline{y};\mathbf{u}_\star,\mathbf{d}u)\mathbf{k}_{(i)}=0.$$

That is, the $\mathbf{k}_{(i)}$ are eigenvectors of $\boldsymbol{\sigma}(0,\underline{y};\mathbf{u}_\star,\mathbf{d}u)$ with zero eigenvalue. Thus, one has that

$$\langle\mathbf{k}_{(i)},\boldsymbol{\sigma}(0,\underline{y};\mathbf{u}_\star,\mathbf{d}u)\partial_u\mathbf{u}\rangle=\langle\boldsymbol{\sigma}(0,\underline{y};\mathbf{u}_\star,\mathbf{d}u)\mathbf{k}_{(i)},\partial_u\mathbf{u}\rangle=0,$$

since $\boldsymbol{\sigma}(u,\underline{y};\mathbf{u},\mathbf{d}u)$ is Hermitian as a consequence of the symmetric hyperbolicity of (12.1). Thus, from Equation (12.24) one obtains

$$\langle\mathbf{k}_{(i)},\mathbf{A}^\alpha(0,\underline{y};\mathbf{u}_\star)\partial_\alpha\mathbf{u}_\star\rangle=\langle\mathbf{k}_{(i)},\mathbf{B}(0,\underline{y};\mathbf{u}_\star)\rangle,\qquad i=1,\ldots m,\tag{12.25}$$

a system of m (scalar) interior equations for the components of $\mathbf{u}$. In what follows, it will be assumed that the components of $\mathbf{u}$ have been chosen such that the free data on $\mathcal{N}$ consist of $N-m$ variables $\breve{\mathbf{u}}_\star$ – the so-called ***u-data***. The remaining m variables, $\bar{\mathbf{u}}_\star$, constrained by the equations in (12.25), are called the ***u-variables***. Thus, one obtains the split

$$\mathbf{u}_\star=(\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star)\qquad\text{on }\mathcal{N}.\tag{12.26}$$

In terms of this split, the scalar intrinsic equations (12.25) can be rewritten in matricial form as

$$\bar{\mathbf{A}}^\alpha(\underline{y},\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star)\partial_\alpha\bar{\mathbf{u}}=\bar{\mathbf{B}}(\underline{y},\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star),\tag{12.27}$$

for some $(m\times m)$-matrix valued smooth functions $\bar{\mathbf{A}}^\alpha$ and an m-vector valued function $\bar{\mathbf{B}}$. For simplicity, *it will assumed that the system* (12.27) *is a symmetric hyperbolic system on* $\mathcal{N}$ which can be solved, at least locally, in a neighbourhood $\mathcal{W}\subset\mathcal{N}$ of the two-dimensional surface $\mathcal{Z}$ where initial data for the u-variables $\bar{\mathbf{u}}_\star$ is prescribed. In this way, one obtains the value of the whole components of $\mathbf{u}_\star$ on $\mathcal{W}$. Assuming that $\breve{\mathbf{u}}_\star$ is smooth on $\mathcal{N}$, higher intrinsic derivatives can be obtained in a similar manner by formally differentiating Equation (12.27) with respect to ∂_α an arbitrary number of times, say, M. In this manner, one obtains a system of the form

$$\bar{\mathbf{A}}^\alpha(\underline{y},\partial_{\boldsymbol{\alpha}}\breve{\mathbf{u}}_\star,\partial_{\boldsymbol{\alpha}}\bar{\mathbf{u}}_\star)\partial_\alpha\bar{\mathbf{u}}_{\boldsymbol{\alpha}}=\bar{\mathbf{B}}(\underline{y},\partial_{\boldsymbol{\alpha}}\breve{\mathbf{u}}_\star,\partial_{\boldsymbol{\alpha}}\bar{\mathbf{u}}_\star,\bar{\mathbf{u}}_{\boldsymbol{\alpha}}),\tag{12.28}$$

where multi-index notation has been used so that

$$\partial_{\boldsymbol{\alpha}}\breve{\mathbf{u}}_\star\equiv\big(\breve{\mathbf{u}}_\star,\partial_\alpha\breve{\mathbf{u}}_\star,\partial_{\alpha_1}\partial_{\alpha_2}\breve{\mathbf{u}}_\star,\ldots,\partial_{\alpha_1}\cdots\partial_{\alpha_{M-1}}\breve{\mathbf{u}}_\star\big),$$
$$\partial_{\boldsymbol{\alpha}}\bar{\mathbf{u}}_\star\equiv\big(\bar{\mathbf{u}}_\star,\partial_\alpha\bar{\mathbf{u}}_\star,\partial_{\alpha_1}\partial_{\alpha_2}\bar{\mathbf{u}}_\star,\ldots,\partial_{\alpha_1}\cdots\partial_{\alpha_{M-1}}\bar{\mathbf{u}}_\star\big)$$

and

$$\bar{\mathbf{u}}_{\boldsymbol{\alpha}} \equiv \partial_{\alpha_1} \cdots \partial_{\alpha_M} \bar{\mathbf{u}}_\star.$$

By assumption, Equation (12.28) is a symmetric hyperbolic system on $\mathcal{N}$ so that by prescribing initial data for $\bar{\mathbf{u}}_{\boldsymbol{\alpha}}$ on $\mathcal{Z}$ and assuming that the lower order intrinsic derivatives $\partial_{\boldsymbol{\alpha}}\bar{\mathbf{u}}$ have been solved for, one obtains a solution in a neighbourhood of $\mathcal{Z}$ on $\mathcal{N}$. Thus, one can obtain, recursively, the interior partial derivatives

$$\mathbf{u}_\star,\, \partial_\alpha \mathbf{u}_\star, \partial_{\alpha_1}\partial_{\alpha_2}\mathbf{u}_\star, \ldots, \partial_{\alpha_1}\cdots\partial_{\alpha_M}\mathbf{u}_\star \qquad \text{on } \mathcal{W} \subset \mathcal{N},$$

with $\mathcal{W} \supset \mathcal{Z}$.

Now, not only the interior derivatives on $\mathcal{N}$ can be computed. Using the split (12.26), the subset of $N-m$ equations in (12.1) which are transversal to $\mathcal{N}$ can be written as

$$\mathbf{C}^u(0,\underline{y},\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star)\partial_u\breve{\mathbf{u}}_\star + \mathbf{C}^\alpha(0,\underline{y},\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star)\partial_\alpha\breve{\mathbf{u}}_\star = \mathbf{D}(0,\underline{y},\bar{\mathbf{u}}_\star,\bar{\mathbf{u}}), \tag{12.29}$$

with $\mathbf{C}^\mu$ smooth $(N-m)\times(N-m)$-matrix valued functions and $\mathbf{D}$ an $(N-m)$-vector valued function of their arguments. For clarity of the presentation it is convenient to write $\partial_u\breve{\mathbf{u}}_\star \equiv (\partial_u\breve{\mathbf{u}})_\star$. By construction, the matrix $\mathbf{C}^\mu$ is invertible, so that Equation (12.29) can be regarded as an algebraic linear system of equations determining the transversal derivatives $\partial_u\breve{\mathbf{u}}$ on $\mathcal{N}$ in terms of $\mathbf{u}_\star$ and $\partial_\alpha\mathbf{u}_\star$. To compute the transversal derivatives of the u-variables $\bar{u}_\star$, one differentiates the interior system (12.27) to obtain a system of the form

$$\bar{\mathbf{A}}^\alpha(\underline{y},\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star)\partial_\alpha(\partial_u\bar{\mathbf{u}}_\star) = \bar{\mathbf{B}}(\underline{y},\breve{\mathbf{u}}_\star,\partial_u\breve{\mathbf{u}}_\star,\bar{\mathbf{u}}_\star,\partial_u\bar{\mathbf{u}}_\star).$$

As in the case of the system (12.27), the above system can be solved in some neighbourhood of $\mathcal{Z}$ on $\mathcal{N}$ if initial data for $\partial_u\bar{\mathbf{u}}_\star$ are given on $\mathcal{Z}$. This procedure can be repeated to obtain higher order transversal derivatives.

The procedure described in the previous paragraphs can also be implemented on the characteristic hypersurface $\mathcal{N}'$. By analogy to the case of $\mathcal{N}$, one can split the unknown $\mathbf{u}$ as

$$\mathbf{u} = (\breve{\mathbf{u}}'_\star, \bar{\mathbf{u}}'_\star), \qquad \text{on } \mathcal{N}',$$

where $\breve{\mathbf{u}}'_\star$ are v-***data*** which can be specified freely on $\mathcal{N}'$ and $\bar{\mathbf{u}}'_\star$ are v-***variables*** constrained by interior equations analogous to (12.28). In what follows, these interior equations are assumed to be symmetric hyperbolic on $\mathcal{N}'$. Applying a procedure similar to that used on $\mathcal{N}$, all the derivatives of $\mathbf{u}$ on $\mathcal{N}'$ to any desired order can be computed if $\breve{\mathbf{u}}_\star$ is suitably smooth, and the required initial data are supplied on $\mathcal{Z}$.

The discussion described in the previous paragraphs is summarised in the following proposition:

Proposition 12.3 (***evaluation of derivatives on the initial characteristic surface***) *Let $\mathcal{N}$ and $\mathcal{N}'$ denote two characteristic hypersurfaces for the symmetric hyperbolic system* (12.1) *having a non-empty two-dimensional intersection $\mathcal{Z} = \mathcal{N} \cap \mathcal{N}'$. If smooth u-data and v-data are prescribed, respectively, on $\mathcal{N}$ and $\mathcal{N}'$ and the values of the u-variables and v-variables are prescribed on $\mathcal{Z}$ in such a way that the freely specifiable data are smooth on $\mathcal{N} \cup \mathcal{N}'$, then all derivatives of $\mathbf{u}$ on $\mathcal{N} \cup \mathcal{N}'$ to any desired order can be computed in a neighbourhood $\mathcal{W} \subset \mathcal{N}$ of $\mathcal{Z}$.*

12.5.3 Reduction to a standard Cauchy problem

The observations summarised in Proposition 12.3 are the cornerstone of a reduction procedure of the characteristic problem on $\mathcal{N} \cup \mathcal{N}'$ to a standard Cauchy problem for which the theory discussed in Section 12.3 is applicable. This approach to analysing the characteristic initial value problem for hyperbolic equations was originally introduced by Rendall (1990).

In what follows, suppose that characteristic initial data have been prescribed on $\mathcal{N} \cup \mathcal{N}'$ in a manner consistent with Proposition 12.3 so that the values of $\mathbf{u}$ and its derivatives to any order are known in a neighbourhood $\mathcal{W}$ of $\mathcal{Z}$ on $\mathcal{N} \cup \mathcal{N}'$. Rendall's reduction proceeds first by constructing an extension of $\mathbf{u}$ to a neighbourhood $\mathcal{U}$ of $\mathcal{Z}$ in $\mathbb{R}^4$. This type of extension of functions is different from the one discussed in Section 12.3.1 where functions defined on open subsets of a certain space are extended to functions on the whole space. In the present case one needs to extend a function defined on a closed set of $\mathbb{R}^4$. There exists a general result, ***Whitney's extension theorem***, which allows one to obtain the required extension; see the Appendix to this chapter for more details.

To apply Whitney's extension theorem to the collection of fields

$$\{\mathbf{u}_\star, (\partial_\mu \mathbf{u})_\star, (\partial_{\mu_1}\partial_{\mu_2}\mathbf{u})_\star, \ldots, (\partial_{\mu_1}\cdots\partial_{\mu_M}\mathbf{u})_\star\} \tag{12.30}$$

on $\mathcal{W} \subset \mathcal{N} \cup \mathcal{N}'$ for some non-negative integer M, one has to verify that the various fields in this collection are related to each other in the same way as the derivatives of a function are related to each other in a Taylor expansion. The key condition on these *Taylor-like expansions* ensuring the existence of an extension is a requirement on the vanishing rate of the remainder of the expansions. Given two points on $\mathcal{N}$ away from $\mathcal{Z}$ this vanishing of the remainder follows from smoothness of the free data on the characteristic hypersurface, and the fact that the ***derivative candidates*** in (12.30) have been obtained solving hyperbolic differential equations on $\mathcal{N}$ and algebraic equations. The functions thus obtained are smooth on $\mathcal{N}$ and admit a standard Taylor expansion on the characteristic hypersurface. This also holds for two points on $\mathcal{N}'$ away from $\mathcal{Z}$. Thus, the difficulty is to verify Whitney's condition for two points, respectively, on $\mathcal{N}$ and $\mathcal{N}'$, so that one writes

$$x = (0, v, x^2, x^3), \qquad x' = (u, 0, x'^2, x'^3).$$

The complication arises from the fact that the characteristic initial hypersurface $\mathcal{N} \cup \mathcal{N}'$ is continuous only at $\mathcal{Z}$. It is convenient to define a point $x_* \in \mathcal{Z}$ as

$$x_* \equiv \left(0, 0, \tfrac{1}{2}(x^2 + x'^2), \tfrac{1}{2}(x^3 + x'^3)\right).$$

Using the cosines law, it follows that there exists a constant $C > 0$ such that

$$|x - x_*|^2 + |x' - x_*|^2 \leq C|x - x'|^2.$$

To apply Whitney's extension theorem it is necessary to establish that the remainder of the Taylor-like expansion about x vanishes at x' as fast as it would do if the function had an extension as x and x' tend to a common point, say, x_*. The inequality above shows that the points x and x' cannot get closer to each other without getting close to a point on $\mathcal{Z}$. This idea, together with the Cauchy stability of solutions to the interior equations which determine the constrained components of the data on $\mathcal{N} \cup \mathcal{N}'$ yields the required vanishing rate.

Applying Whitney's extension theorem to the collection of derivative candidates (12.30) one obtains a smooth function $\hat{\mathbf{u}}$ in a neighbourhood $\mathcal{U}$ of $\mathcal{Z}$ on $\mathbb{R}^4$. The function $\hat{\mathbf{u}}$ satisfies

$$\hat{\mathbf{u}} = \mathbf{u}_\star, \quad \partial_\mu \hat{\mathbf{u}} = (\partial_\mu \mathbf{u})_\star, \quad \partial_{\mu_2}\partial_{\mu_1}\hat{\mathbf{u}} = (\partial_{\mu_2}\partial_{\mu_1}\mathbf{u})_\star, \ldots,$$

on $\mathcal{W} \subset \mathcal{N} \cup \mathcal{N}'$. In general, $\hat{\mathbf{u}}$ is not a solution to Equation (12.1) away from $\mathcal{N} \cup \mathcal{N}'$. Nevertheless,

$$\Delta \equiv \mathbf{A}^\mu(x, \hat{\mathbf{u}})\partial_\mu \hat{\mathbf{u}} - \mathbf{B}(x, \hat{\mathbf{u}})$$

vanishes to all orders on $\mathcal{W} \subset \mathcal{N} \cup \mathcal{N}'$ and

$$\delta \equiv \begin{cases} 0 & u > 0, \quad v > 0, \\ \Delta & \text{elsewhere,} \end{cases}$$

is smooth in a neighbourhood of $\mathcal{N} \cap \mathcal{N}'$ where $\hat{\mathbf{u}}$ exists.

The desired reduction to a Cauchy problem is now obtained by considering the equation

$$\mathbf{A}^\mu(x, \hat{\mathbf{u}} + \mathbf{v})\partial_\mu(\hat{\mathbf{u}} + \mathbf{v}) - \mathbf{B}(x, \hat{\mathbf{u}} + \mathbf{v}) = \delta, \tag{12.31}$$

for the unknown $\mathbf{v}$ together with the initial data

$$\mathbf{v}_\star = \mathbf{0}, \quad \text{on} \quad \mathcal{S} \equiv \{p \in \mathbb{R}^4 \,|\, u(p) + v(p) = 0\}. \tag{12.32}$$

By assumption, the hypersurface $\mathcal{S}$ has a neighbourhood around $\mathcal{N} \cap \mathcal{N}'$ which is spacelike with respect to Equation (12.31) so that the Cauchy problem given by (12.31) together with the initial data (12.32) is well posed and the theory of Section 12.3 is readily applicable. In particular, one obtains a unique solution $\mathbf{v}$ in a neighbourhood $\mathcal{V}$ of $\mathcal{Z}$ on $\mathbb{R}^4$; see Figure 12.10.

Outside the intersection of $\mathcal{V}$ with the quadrant

$$\{p \in \mathbb{R}^4 \,|\, u(p) \geq 0,\ v(p) \geq 0\},$$

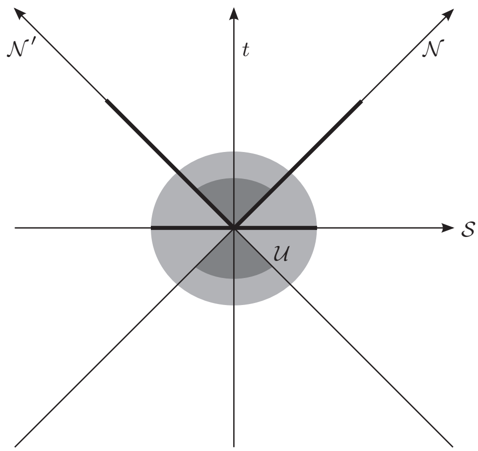

Figure 12.10 Schematic reduction of a characteristic initial value problem to a Cauchy problem. The data on $\mathcal{N} \cup \mathcal{N}'$ (thick line) are extended by means of Whitney's extension theorem to a neighbourhood $\mathcal{U}$ (light gray region) of $\mathcal{Z} \equiv \mathcal{N} \cap \mathcal{N}'$. This extension implies data for an auxiliary initial value problem on the spacelike hypersurface $\mathcal{S}$. The solution to the characteristic problem is found in the upper and lower quadrants (dark gray areas).

Equation (12.31) takes the form

$$\mathbf{A}^\mu(x, \hat{\mathbf{u}} + \mathbf{v})\partial_\mu(\hat{\mathbf{u}} + \mathbf{v}) - \mathbf{B}(x, \hat{\mathbf{u}} + \mathbf{v}) = \mathbf{A}^\mu(x, \hat{\mathbf{u}})\partial_\mu \hat{\mathbf{u}} - \mathbf{B}(x, \hat{\mathbf{u}})$$

so that $\mathbf{v} = 0$ is clearly a solution – by uniqueness, it is the only solution. In contrast, on $\mathcal{V} \cap \{p \in \mathbb{R}^4 \,|\, u(p) \geq 0,\ v(p) \geq 0\}$ one has the equation

$$\mathbf{A}^\mu(x, \hat{\mathbf{u}} + \mathbf{v})\partial_\mu(\hat{\mathbf{u}} + \mathbf{v}) = \mathbf{B}(x, \hat{\mathbf{u}} + \mathbf{v}).$$

As $\hat{\mathbf{u}} + \mathbf{v}$ coincides with $\mathbf{u}_\star$ on $\mathcal{N} \cup \mathcal{N}'$ one concludes, again by uniqueness of the solution of the reduced Cauchy problem, that

$$\mathbf{u} \equiv \hat{\mathbf{u}} + \mathbf{v}$$

is the required solution to the posed characteristic initial value problem.

The discussion in this section is summarised in the following theorem:

Theorem 12.7 (***local existence for the standard characteristic problem***) *Let $\mathcal{N}$ and $\mathcal{N}'$ denote two characteristic hypersurfaces for the symmetric hyperbolic system* (12.1) *with smooth, freely specifiable data on $\mathcal{N}$ and $\mathcal{N}'$ as given in Proposition 12.3. Then there exists a unique solution* $\mathbf{u}$ *to the characteristic initial value problem in a neighbourhood $\mathcal{V}$ of $\mathcal{Z}$ with $u \geq 0$, $v \geq 0$.*

Remark. If one were to attempt a similar reduction procedure to construct a solution on the regions for which either $u \geq 0$, $v \leq 0$ or $u \leq 0$, $v \geq 0$,

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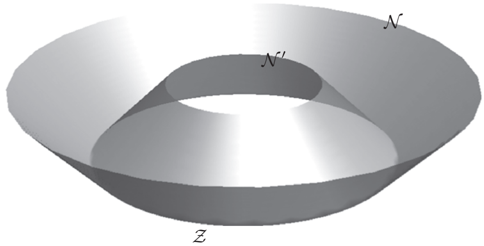

Figure 12.11 Characteristic cones $\mathcal{N}$ and $\mathcal{N}'$ intersecting on a two-dimensional hypersurface $\mathcal{Z}$ which is diffeomorphic to $\mathbb{S}^2$.

one would end up with an initial value problem with data prescribed on a timelike hypersurface. This is an ill-posed problem. Accordingly, the original characteristic problems are, themselves, also ill-posed.

The case $\mathcal{Z} \approx \mathbb{S}^2$

A case that occurs naturally in applications of conformal methods in general relativity is an initial characteristic problem where the intersection $\mathcal{Z} = \mathcal{N} \cap \mathcal{N}'$ is diffeomorphic to the 2-sphere $\mathbb{S}^2$. This is the case, for example, of the intersection of two light cones; see Figure 12.11. The method discussed in the previous section can be adapted to this case; see Kánnár (1996b).

Assuming in what follows that $\mathcal{Z} \approx \mathbb{S}^2$, consider an atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ of $\mathcal{Z}$ and closed sets $V_1 \subset U_1$ and $V_2 \subset U_2$ which also cover $\mathcal{Z}$; that is, $\mathcal{Z} = V_1 \cup V_2$. Furthermore, define two smooth functions η_1 and η_2 with compact support on $\mathbb{R}^2$ by

$$\eta_1(x) \equiv \begin{cases} 1 & x \in \phi_1(V_1) \\ 0 & x \in \mathbb{R}^2 \setminus \phi_1(U_1), \end{cases} \qquad \eta_2(x) \equiv \begin{cases} 1 & x \in \phi_2(V_2) \\ 0 & x \in \mathbb{R}^2 \setminus \phi_2(U_2). \end{cases}$$

In what follows, denote by $\breve{\mathbf{u}}_{1\star}$ and $\breve{\mathbf{u}}_{2\star}$, respectively, the restriction to U_1 and U_2 of the freely specifiable data on $\mathcal{N} \cup \mathcal{N}'$. It follows that the functions $\eta_1 \breve{\mathbf{u}}_{1\star}$ and $\eta_2 \breve{\mathbf{u}}_{2\star}$ define a smooth initial value data set on the initial hypersurfaces $\mathcal{N}_1 \equiv \mathbb{R}^+ \times \{0\} \times \mathbb{R}^2$ and $\mathcal{N}_2 \equiv \{0\} \times \mathbb{R}^+ \times \mathbb{R}^2$ which coincides with $\mathbf{u}_{1\star}$ and $\mathbf{u}_{2\star}$ on $\mathbb{R}^+ \times \{0\} \times \phi_1(V_1)$ and $\{0\} \times \mathbb{R}^+ \times \phi_2(V_2)$.

The intrinsic equations on $\mathcal{N}_1$ and $\mathcal{N}_2$ can now be solved in a manner similar to what was done in Section 12.5.2. In this way, one obtains two complete characteristic initial data sets $\mathbf{u}_{1\star}$ and $\mathbf{u}_{2\star}$ on $\mathcal{N}_1 \cup \mathcal{N}_2$. The (interior and transversal) derivatives of these data can be computed to any desired order. Using Theorem 12.7 one obtains two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ in a neighbourhood $\mathcal{V}$ of $\mathcal{N}_1 \cap \mathcal{N}_2$. Their restrictions to the Cauchy development of $\mathbb{R}^+ \times \mathbb{R}^+ \times \phi_1(V_1)$ and $\mathbb{R}^+ \times \mathbb{R}^+ \times \phi_2(V_2)$ are local solutions to the original problem. The solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ can be glued together to obtain a global solution on $\mathcal{Z} \approx \mathbb{R}^2$. As a consequence of the uniqueness of the local solutions $\mathbf{u}_1$ and $\mathbf{u}_2$, it follows that their restriction to a part of the Cauchy development of $\mathbb{R}^+ \times \mathbb{R}^+ \times \phi_1(V_1 \cap V_2)$ and $\mathbb{R}^+ \times \mathbb{R}^+ \times \phi_2(V_1 \cap V_2)$ – where both solutions exist – must be related by a coordinate transformation. In this manner, one obtains a smooth function in a neighbourhood of $\mathscr{Z}$.

12.6 Concluding remarks

This chapter has provided a succinct discussion of the theory of the local existence and uniqueness of quasilinear symmetric hyperbolic evolution equations. Of course, this is not the only way the subject can be approached. Nor are the issues raised the only relevant ones in the analysis of the evolution problem in general relativity. Thus, it is important to make some remarks concerning some ideas and approaches which have been omitted.

12.6.1 Wave equations

The analysis of Section 12.1.3 gives a hint on how the theory of second-order hyperbolic equations (wave equations) can be reduced to the analysis of symmetric hyperbolic equations. There is, however, a well-developed theory for the local existence and stability of systems of quasilinear equations of the form

$$g^{\mu\nu}(x, \mathbf{u})\partial_\mu\partial_\nu\mathbf{u} = \mathbf{B}(x, \mathbf{u}, \partial\mathbf{u}), \tag{12.33}$$

which does not rely on the reduction to a first order system; see Hughes et al. (1977). Equation (12.33) is quasilinear in the sense that $g^{\mu\nu}$ is the contravariant version of a Lorentzian metric which is allowed to depend not only on the coordinates but also on the unknowns. This "stand alone" theory relaxes the differentiability assumptions made on the equation and data; see, for example, Rendall (2008) for more details.

The results of Hughes et al. (1977) are similar, in spirit, to the results given in Theorems 12.2 and 12.3: given suitably smooth initial data for Equation (12.33) one obtains a unique solution for some existence time T; moreover one also has a Cauchy stability result. It should be pointed out that this theory applies, in fact, to a class of second-order equations more general than that given in Equation (12.33).

Systems of quasilinear wave equations of the form (12.33) arise naturally in the reduction procedure for the Einstein field equations based on the use of wave coordinates; see the Appendix to Chapter 14. Historically, this was the first approach to the Cauchy problem in general relativity; see Fourès-Bruhat (1952).

12.6.2 Global existence of solutions

Conformal methods allow the reformulation of several questions on the global existence of solutions to the Einstein field equations as a local existence question

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for symmetric hyperbolic systems. Accordingly, the issue of global existence of solutions to symmetric hyperbolic equations has not been addressed in this chapter. Nevertheless, this question is at the heart of current research work in the area; see, for example, Klainerman (2008) for a discussion.

As already pointed out, the local theory of solutions to hyperbolic equations depends solely on the properties of the principal part of the equations. To construct a theory of global existence one has to include the lower order terms of the equations into the analysis. Under certain circumstances, the analysis of the eigenvalues of the matrix arising from the linearisation of the lower order terms of a quasilinear system gives a strong indication of whether one can expect global existence and stability of solutions; see, for example, Kreiss and Lorenz (1998). More generally, one can identify structures in the evolution equations which allow one to prove global existence. One of these structures is the so-called ***null condition***; see, for example, Klainerman (1984).

12.7 Further reading

The theory of hyperbolic differential equations, in general, and their application to the analysis of solutions of the Einstein field equations, in particular, is an extensive area of research so that any list of references can provide only a partial impression of the field. For an overview of the whole field of the theory of PDEs and the interconnection between the various types of equations the reader is referred to Klainerman (2008).

Readers interested in further details of the basic aspects of the theory of PDEs are referred to the classical references by Garabedian (1986) and Courant and Hilbert (1962). A modern introduction to the subject is given in Evans (1998). A comprehensive exposition of the subject is given in the three-volume treatise of Taylor (1996a,b,c). Detailed accounts of the theory of the Cauchy problem for symmetric hyperbolic systems are discussed in the original references by Kato (1975a); Fischer and Marsden (1972) for first-order symmetric hyperbolic systems and Hughes et al. (1977) for second-order equations. A review of the ideas contained in these works can be found in Kato (1975b).

A comprehensive discussion of the role of PDEs in general relativity is given in Rendall (2008). A more compact review is Friedrich and Rendall (2000). Complementary discussions on the topics covered in these references can be found in Rendall (2006) and Reula (1998).

Appendix

A generalised mean value theorem

In the proofs of the uniqueness of solutions for symmetric hyperbolic systems, Theorems 12.2 and 12.5, the following generalisation of the mean value theorem has been used; see Hamilton (1982). In the following result, $M(N\times N,\mathbb{C})$ denotes the set of $N\times N$ matrices with complex entries.

Lemma 12.1 *Let $\mathcal{U}\subset\mathbb{R}^N$, and let $\mathbf{F}:\mathcal{U}\to\mathbb{C}^N$ be a C^1 map. Then there exists a continuous map $\mathbf{M}:\mathcal{U}\times\mathcal{U}\to M(N\times N,\mathbb{C})$ such that*

$$\mathbf{F}(\mathbf{u})-\mathbf{F}(\mathbf{v})=\mathbf{M}(\mathbf{u},\mathbf{v})(\mathbf{u}-\mathbf{v}).$$

The proof is an application of the *fundamental theorem of calculus.*

Whitney's extension theorem

In what follows, let $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$ denote multi-indices. The factorial $\boldsymbol{\alpha}!$ is defined as $\boldsymbol{\alpha}!\equiv\alpha_1!\cdots\alpha_n!$. Moreover, let

$$|\boldsymbol{\beta}|\equiv\beta_1+\cdots+\beta_n.$$

In terms of this notation one has the following:

Theorem 12.8 (***Whitney's extension theorem***) *Given a non-negative integer k, suppose $\{f_{\boldsymbol{\alpha}}\}$, $|\boldsymbol{\alpha}|<k$ is a collection of real valued functions defined on a closed set $\mathcal{A}\subset\mathbb{R}^n$ satisfying*

$$f_{\boldsymbol{\alpha}}(x)=\sum_{|\boldsymbol{\beta}|\leq k-|\boldsymbol{\alpha}|}\frac{1}{\boldsymbol{\beta}!}f_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(x')(x-x')^{\boldsymbol{\beta}}+R_{\boldsymbol{\alpha}}(x,x'),$$

for every x, $x'\in\mathcal{A}$ and each multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}|\leq k$ such that for every $x_0\in\mathcal{A}$

$$R_{\boldsymbol{\alpha}}=o\big(|x-x'|^{k-|\boldsymbol{\alpha}|}\big),\qquad \text{as}\qquad x,\,x'\to x_0.$$

Then there exists a C^k function $g:\mathbb{R}^n\to\mathbb{R}$ such that

$$g=f_{\mathbf{0}},\qquad \partial_{\boldsymbol{\alpha}}g=f_{\boldsymbol{\alpha}}\qquad \text{on } \mathcal{A}.$$

In other words, for a closed set $\mathcal{A}$, if one is given a function f and candidates $f_{\boldsymbol{\alpha}}$ for its partial derivaties on $\mathcal{A}$, then f can be extended to all of $\mathbb{R}^n$ in such a way that the candidates are indeed the derivatives of f as long as the remainder has a suitable behaviour. A priori, it is not possible to identify the functions $f_{\boldsymbol{\alpha}}$ with the derivatives of f as $\mathcal{A}$ is a closed set and transversal derivatives f to $\partial\mathcal{A}$ may not be defined. For further details on Theorem 12.8 and its proof, see, for example, Abraham and Robbin (1967).